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ON THE NUMBER OF ZEROES OF EXPONENTIAL SYSTEMS*[1)]

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Abstract

A system $E : C^n \to C^n$ is said to be an exponential one if its terms are $ae^{im_1 Z_1} \cdots e^{im_n Z_n}$. This paper proves that for almost every exponential system $E : C^n \to C^n$ with degree $(q_1, \cdots, q_n)$, E has exactly $\Pi_{j=1}^n (2q_j)$ zeroes in the domain

$$D = \{(Z_1, \cdots, Z_n) \in C^n : \ Z_j = x_j + iy_j, \ x_j, \ y_j \in R, \ 0 \le x_j < 2\pi, \ j = 1, \cdots, n\},$$

and all these zeroes can be located with the homotopy method.

§1. Introduction

Let E: $C^n \to C^n$ be an exponential system, where C^n is the n-dimensional complex space. By an exponential system, we mean that each term in every equation is of the form

$$ae^{im_1 Z_1} \cdots e^{im_n Z_n}, \tag{1.1}$$

where $i = \sqrt{-1}$, a is a complex number, Z_j a complex variable, and m_j an integer. For each term in each equation, consider the sum $|m_1| + \cdots + |m_n|$. Let q_j be the maximum sum in equation j. We assume $q_j > 0$ for all j. We call q_j the degree of E_j, and $(q_1, \cdots, q_n)$ the degree of the system E. In this paper, let

$$D = \{(Z_1, \cdots, Z_n) \in C^n : \ Z_j = x_j + iy_j, \ x_j, \ y_j \in R, \ 0 \le x_j < 2\pi, \ j = 1, \cdots, n\},$$

Let E: $C^n \to C^n$ be given as above. Now, we distinguish certain coefficients of E. Let a_{kj} be the coefficient of term $e^{iq_k Z_j}$ in E_k, and b_{kj} the coefficient of the term $e^{-iq_k Z_j}$ in E_k for $k, j = 1, \cdots, n$. Let $A = ((a_{kj})|(b_{kj})) \in C^{2n^2}$. Define B to be the other coefficients of the terms with degree q_k in E_k for all $k = 1, \cdots, n$. Let a_i be the constant term of E_i for $i = 1, \cdots, n$ and $a = (a_1, \cdots, a_n) \in C^n$. Let b be all coefficients of E other than a, A and B. Then (a, A, b, B) uniquely defines E. We write E as $E(\cdot, a, A, b, B)$.

Utilizing homotopy methods, this paper studies zero distribution of exponential systems. Section 2 discusses numbers of the zeroes of the systems. Section 3 applies the results to triangular polynomial systems. Section 4 explores the relationship between exponential systems and polynomial systems, and points out that it is unreasonable to transform exponential systems into corresponding polynomial systems for the purpose of locating all isolated zeroes. Section 5 contains several numerical examples.

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§2. Main result

Lemma 1[1]. *Let $H : R^n \times R^m \to R^p$ be a smooth mapping. Suppose 0 is a regular value of H. Then for almost all $a \in R^m$, 0 is a regular value of $H(\cdot, a) : R^n \to R^p$.*

Lemma 2[1]. *Suppose $F : C^n \to C^n$ is an analytic mapping. Regard F as a real mapping $F : R^{2n} \to R^{2n}$ in the way of identifying $(Z_1, \cdots, Z_n)$ with $(x_1, y_1, \cdots, x_n, y_n)$, where $Z_j = x_j + iy_j$, $i = \sqrt{-1}$, x_j, $y_j \in R$, for $j = 1, \cdots, n$. Then the real Jacobian determinant $\det\partial F/\partial(x_1, y_1, \cdots, x_n, y_n)$ is nonnegative everywhere. Furthermore, if 0 is a regular value of F, then the determinant is positive in $F^{-1}(0)$.*

Lemma 3[2]. *Let $H : R^n \times [0, 1] \to R^n$ be a smooth mapping. Suppose 0 is a regular value of H. Then for any curve $\lambda(s) = (x(s), t(s))$ in $H^{-1}(0)$,*

$$\text{sgn}\ \dot{t}(s) = \text{sgn}\ \det\ \frac{\partial H}{\partial x}(\lambda(s)) \quad \textit{for all}\ \ s\ ,$$

or

$$\text{sgn}\ \dot{t}(s) = -\text{sgn}\ \det\ \frac{\partial H}{\partial x}(\lambda(s)) \quad \textit{for all}\ \ s\ ,$$

where s is the arc length.

Let E be an exponential system with degree $(q_1, \cdots, q_n)$, and define an auxiliary mapping $E_0 = (E_{01}, \cdots, E_{0n}) : C^n \to C^n$ by

$$E_{0j}(Z) = e^{iq_jZ_j} + e^{-iq_jZ_j}, \ \text{for} \quad j = 1, \cdots, n\ .$$

It is clear that E_0 has exactly $\Pi_{j=1}^n(2q_j)$ zeroes in D and 0 is a regular value of E_0.

Define homotopy $H : C^n \times [0, 1] \to C^n$ by

$$H(Z, t) = tE(Z) + (1 - t)E_0(Z)\ . \tag{2.1}$$

Then $H(\cdot, 0) = E_0(\cdot)$ and $H(\cdot, 1) = E(\cdot)$. The following lemma is direct from Lemma 1.

Lemma 4. *Assume H as in (2.1). Then for all A,b and B, and for almost all $a \in C^n$, 0 is a regular value of H.*

We say H is regular if 0 is a regular value of H. Fix $a \in C^n$ such that H is regular. Then $H^{-1}(0)$ is a one-dimensional manifold. By Lemmas 2 and 3, for any curve $\lambda(s) = (Z(s), t(s))$ of $H^{-1}(0)$, $t(s)$ is a monotone function of s. So we can write $\lambda(s)$ as $\lambda(t) = (Z(t), t), 0 \le t \le 1$. Hence, we have

Lemma 5. *Assume H as above. Then $H^{-1}(0)$ consists of four kinds of curves as follows (shown in* Fig.1*):*

(1) *curves of finite lengths starting at $C^n \times \{0\}$ and ending at $C^n \times \{1\}$;*
(2) *unbounded curves with only one boundary point in $C^n \times \{0\}$;*
(3) *unbounded curves with only one boundary point in $C^n \times \{1\}$;*
(4) *unbounded curves in $C^n \times (0, 1)$.*

Now, we prove that for almost all $A \in C^{2n^2}$, $H^{-1}(0)$ is bounded. First, we give some definitions. Let E be an exponential system with degree $(q_1, \cdots, q_n)$. Let $s = (s_1, \cdots, s_n) \in \{1, -1\}^n$. Define $\bar{Z}_j = e^{is_jZ_j}$ for $j = 1, \cdots, n$. Then $E(Z, a, A, b, B)$ becomes a mapping $E_s(\bar{Z}, a, A, b, B)$ that consists of the terms like $a\bar{Z}_1^{m_1} \cdots \bar{Z}_n^{m_n}$. Let $PE_s = (PE_{s1}, \cdots, PE_{sn})$ be the polynomial part of $E_s(\cdot, a, A, b, B)$. That is, $PE_{sk}(\bar{Z})$ consists of all polynomial terms like $a\bar{Z}_1^{m_1} \cdots \bar{Z}_n^{m_n} (m_j \ge 0$ for $j = 1, \cdots, n)$ in $PE_{sk}(\bar{Z})$, $k = 1, \cdots, n$. We call PE_s the polynomial system of E with respect to s. It is clear that the degree of PE_s is $(q_1, \cdots, q_n)$. Since the number of the elements of $\{1, -1\}^n$ is 2^n, we have 2^n different polynomial systems PE_s, each of which has a different s.

Let PH_s be the polynomial system of H with respect to some $s \in \{1,-1\}^n$. That is,

$$PH_s(Z,t) = tPE_s(Z) + (1-t)PE_{0s}(Z)\ ,$$

where PE_{0s} and PE_s are respectively the polynomial systems of E_0 and E with respect to s. It is clear that the degree of PH_s in Z is $(q_1,\cdots,q_n)$. Let $\bar{P}\bar{H}_s$ be the homogeneous polynomial system of PH_s in Z with highest degree. Then (A,B) and s uniquely define $\bar{P}\bar{H}_s$. The next lemma is direct from [3].

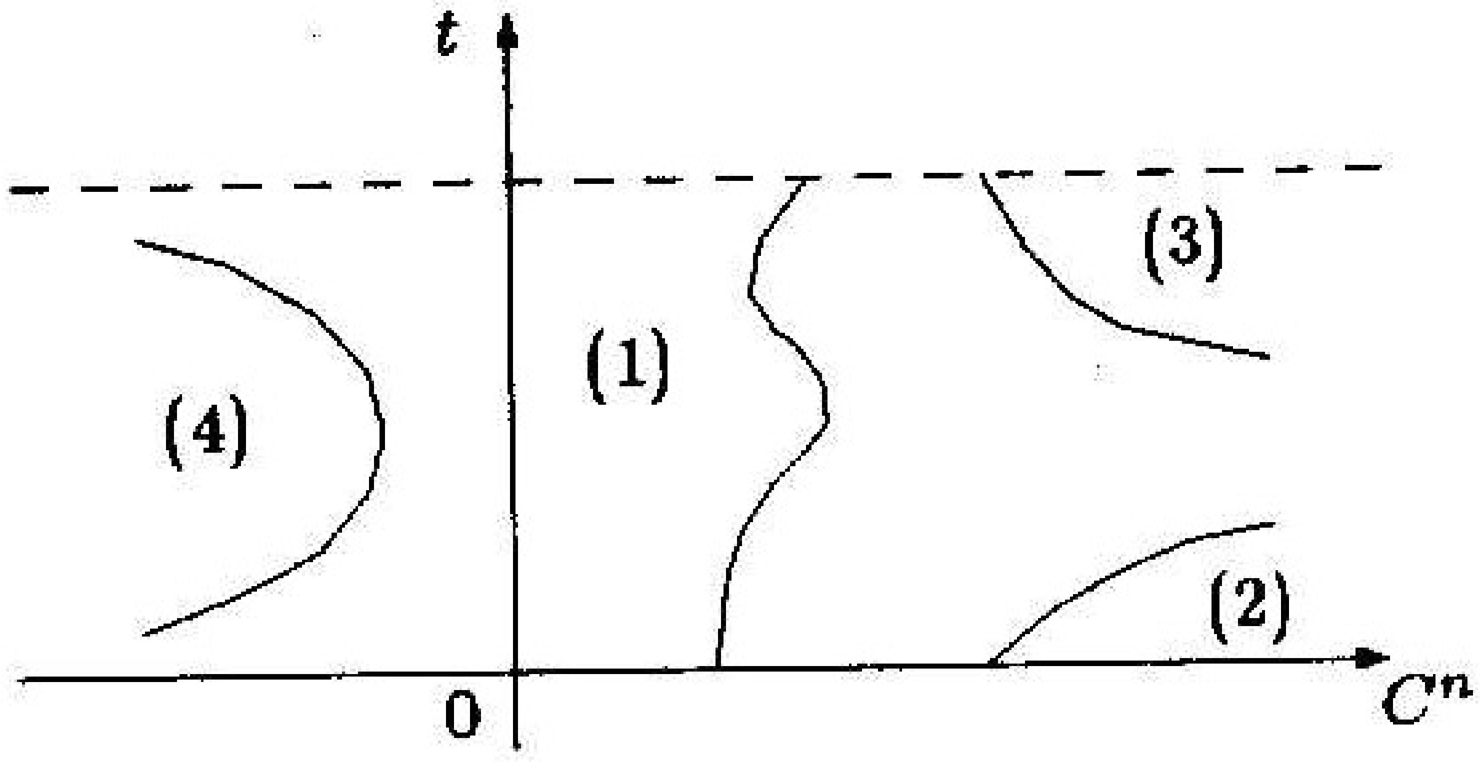

Fig. 1

Lemma 6. *Assume $\bar{P}\bar{H}_s$ as above. Then for almost all $A \in C^{2n^2}$ and for all B, 0 is a regular value of $\bar{P}\bar{H}_s$ in the domain $(C^n \setminus \{0\}) \times [0,1]$.*

By the homogeneity of $\bar{P}\bar{H}_s$ in Z, if 0 is a regular value of $\bar{P}\bar{H}_s$ in the domain $(C^n \setminus \{0\}) \times [0,1]$, it is easy to know that for any $t \in [0,1]$, $\bar{P}\bar{H}_s(.,t)$ has only a trivial zero.

Lemma 7. *Let $E(\cdot,a,A,b,B)$ be an exponential system with degree $(q_1,\cdots,q_n)$. Assume H as in (2.1). Then for almost all $A \in C^{2n^2}$ and $a \in C^n$, for arbitrary b and B, $H^{-1}(0)$ is bounded.*

Proof. Suppose $H^{-1}(0)$ is unbounded. Choose $\{(Z(k),t(k))\}_{k=1}^{\infty} \subset H^{-1}(0)$ such that $Z(k) \to \infty$ as $k \to \infty$ and $t(k) \in [0,1]$. First, if $y_j \to \infty$ as $k \to \infty$ for some j, without loss of generality, we assume that $y_j(k) \to +\infty$ or $y_j(k) \to -\infty$ as $k \to \infty$. Define $s = (s_1,\cdots,s_n)$ as follows. For $j = 1,\cdots,n$,

$$s_j = \begin{cases} 1, & \text{if } y_j(k) \to -\infty \text{ as } k \to \infty\ , \\ -1, & \text{otherwise}\ . \end{cases}$$

Let PH_s be the polynomial system of H with respect to s. Let $\eta(k) = (\eta_1(k),\cdots,\eta_n(k))$ and $\eta_j(k) = e^{is_jZ_j(k)}$ for $j = 1,\cdots,n$. With the fact $e^{iZ_j} = e^{-y_j}$ (cos x_j + $i$sin x_j) and the definition of s, we have

$$\begin{aligned} \bar{P}\bar{H}_{sj}(\eta(k)/\|\eta(k)\|,\ t(k)) &= \|\eta(k)\|^{-q_j}\bar{P}\bar{H}_{s_j}(\eta(k),t(k)) \\ &= \|\eta(k)\|^{-q_j}(PH_{s_j}(\eta(k),t(k)) - H_j(Z(k),t(k))) \\ &\to 0 \text{ as } k \to \infty\ . \end{aligned}$$

Hence, any cluster point (Z^0,t^0) of $\{(\eta(k)/\|\eta(k)\|,\ t(k))\}_{k=1}^{\infty}$ is a nontrivial zero since $\bar{P}\bar{H}_s(Z^0,t^0) = 0$ and $\|Z^0\| = 1$. It contradicts Lemma 6. Thus, for almost all $A \in C^{2n^2}$, $H^{-1}(0)$ is bounded in directions $y_1,\cdots,y_n$.

Now, suppose $x_j(k) \to \infty$ as $k \to \infty$ for some j. By the periodicity of H and the boundedness of $H^{-1}(0)$ in directions $y_1,\cdots,y_n$, there is a bounded set of $D \times [0,1]$ such that the set contains infinitely many curves of $H^{-1}(0)$. This contradicts the regularity of H. So $H^{-1}(0)$ is bounded in directions $x_1,\cdots,x_n$.

Hence, $H^{-1}(0)$ is bounded for almost all $A \in C^{2n^2}$

Now, we are ready to prove our main result.

Theorem 8. *Let $E(\cdot, a, A, b, B)$ be an exponential system with degree $(q_1, \cdots, q_n)$. Then for all b and B, and for almost all $a \in C^n$ and $A \in C^{2n^2}$, $E(., a, A, b, B)$ has exactly $\Pi_{j=1}^n(2q_j)$ zeroes in D.*

Proof. Define H as in (2.1). By Lemmas 5 and 7, any component of $H^{-1}(0)$ has two boundary points; one is in $C^n \times \{0\}$ and the other in $C^n \times \{1\}$.

Now, we prove thst any two curves $\lambda_1(t)$ and $\lambda_2(t)$ in $H^{-1}(0)$ starting at different zeroes $(Z^1, 0)$ and $(Z^2, 0)$ (that is, $Z^1 - Z^2 \neq 2k\pi$ for all integers k) of E_0 intersect $C^n \times \{1\}$ at different zeroes $(Z^{1*}, 1)$ and $(Z^{2*}, 1)$ of E respectively. Otherwise, suppose $Z^{1*} - Z^{2*} = 2k_0\pi$ for some integer k_0. Then, by the periodicity of H, the curve in $H^{-1}(0)$ starting at $(Z^1 - 2k_0\pi,\ 0)$ must intersect $C^n \times \{1\}$ at $(Z^{2*}, 1)$. This contradicts the regularity of H.

Similarly, any two curves in $H^{-1}(0)$ starting at different zeroes of E must intersect $C^n \times \{0\}$ at different zeroes of E_0.

Since E_0 has exactly $\Pi_{j=1}^n(2q_j)$ zeroes in domain D, the number of zeroes of E is exactly $\Pi_{j=1}^n(2q_j)$ in D.

§3. The Number of Zeroes of Triangular Polynomial Systems

Consider $P = (P_1, \cdots, P_n) : C^n \to C^n$ with

$$P_j(Z) = a_j + \sum_{k=1}^{n}\sum_{l=1}^{n_{jk}}(a_{kl} \cos (lZ_k) + b_{kl} \sin (lZ_k)),\ j = 1, \cdots, n\ .$$

We call $q_j = \max_k\{n_{jk}\}$ the degree of P_j for $j = 1, \cdots, n$, and P a triangular polynomial system with degree $(q_1, \cdots, q_n)$.

Suppose $P : C^n \to C^n$ is given as above. Let a_{ij} be the coefficient of $\cos(q_i Z_j)$ in P_i, b_{ij} be the coefficient of $\sin(q_i Z_j)$ in P_i, and $A = ((a_{ij})|(b_{ij}))$. Let a_i be the constant term of P_i, and $a = (a_1, \cdots, a_n)$. Define B to be the other coefficients of P. Obviously, (a, A, B) uniquely defines P. We write $P(\cdot)$ as $P(\cdot, a, A, B)$.

Notice that $\cos Z_j = (e^{iZ_j} + e^{-iZ_j})/2$ and $\sin Z_j = (e^{iZ_j} - e^{-iZ_j})/2i$. Then, the next theorem is direct from Theorem 8.

Theorem 9. *Let $P(\cdot, a, A, B)$ be a triangular polynomial system with degree $(q_1, \cdots, q_n)$. Then for all B, and for almost all $a \in C^n$ and $A \in C^{2n^2}$, $P(\cdot, a, A, B)$ has exactly $\Pi_{j=1}^n(2q_j)$ zeroes in D.*

§4. Relation Between Exponential Systems and Polynomial Systems

It may seem natural to solve an exponential system by transforming it into the corresponding polynomial system. This section shows why it is unreasonable.

Let $E(\cdot, a, A, B)$ be an exponential system with degree $(q_1, \cdots, q_n)$. Since $e^{iq_j Z_k}$ are nonzero for all $j, k = 1, \cdots, n$, multiplying $E_j(Z, a, A, b, B)$ by $e^{iq_j Z_1} \cdots e^{iq_j Z_n}$ for $j = 1, \cdots, n$, we obtain an exponential system containing only terms as

$$e^{im_1 Z_1} \cdots e^{im_n Z_n}\ ,$$

where all m_j are nonnegative. Denote the system by E^1. It is clear that E and E^1 have the same zeroes in D. Let $\bar{Z}_j = e^{iZ_j}$ for $j = 1, \cdots, n$. Then the system E^1 becomes a polynomial system. Denote it by PE^1. It is easy to know that the degree of PE^1 is $((n+1)q_1, \cdots, (n+1)q_n)$. Let $\bar{P}\bar{E}^1$ be the homogeneous system of PE^1. Since $\bar{P}\bar{E}^1$ has nontrivial zeroes, PE^1 is a deficient polynomial system. That is , the number of its isolated zeroes is less than its total degree $(n+1)^n \Pi_{j=1}^n q_j$.

Example. Let $E = (E_1, E_2): C^2 \to C^2$,

$$E_1(Z) = a_{11}e^{iq_1Z_1} + a_{12}e^{iq_1Z_2} + b_{11}e^{-iq_1Z_1} + b_{12}e^{-iq_1Z_2} + a_1 \, ,$$
$$E_2(Z) = a_{21}e^{iq_2Z_1} + a_{22}e^{iq_2Z_2} + b_{21}e^{-iq_2Z_1} + b_{22}e^{-iq_2Z_2} + a_2 \, .$$

Then

$$\begin{aligned} E_1^1(Z) &= a_{11}e^{i2q_1Z_1+iq_1Z_2} + a_{12}e^{iq_1Z_1+i2q_1Z_2} \\ &\quad + b_{11}e^{iq_1Z_2} + b_{12}e^{iq_1Z_1} + a_1e^{iq_1Z_1+iq_1Z_2} \, , \\ E_2^1(Z) &= a_{21}e^{i2q_2Z_1+iq_2Z_2} + a_{22}e^{iq_2Z_1+i2q_2Z_2} \\ &\quad + b_{21}e^{iq_2Z_2} + b_{22}e^{iq_2Z_1} + a_2e^{iq_2Z_1+iq_2Z_2} \, , \end{aligned}$$

and

$$PE_1^1(Z) = a_{11}Z_1^{2q_1}Z_2^{q_1} + a_{12}Z_1^{q_1}Z_2^{2q_1} + b_{11}Z_2^{q_1} + b_{12}Z_1^{q_1} + a_1Z_1^{q_1}Z_2^{q_1} \, ,$$
$$PE_2^1(Z) = a_{21}Z_1^{2q_2}Z_2^{q_2} + a_{22}Z_1^{q_2}Z_2^{2q_2} + b_{21}Z_2^{q_2} + b_{22}Z_1^{q_2} + a_2Z_1^{q_2}Z_2^{q_2} \, .$$

Since

$$\bar{P}\bar{E}_1^1(Z) = a_{11}Z_1^{2q_1}Z_2^{q_1} + a_{12}Z_1^{q_1}Z_2^{2q_1} \, ,$$
$$\bar{P}\bar{E}_2^1(Z) = a_{21}Z_1^{2q_2}Z_2^{q_2} + a_{22}Z_1^{q_2}Z_2^{2q_2} \, ,$$

for general $(a_{11}, a_{12}, a_{21}, a_{22}) \in C^4$, the zero set of $(\bar{P}\bar{E}_1^1, \bar{P}\bar{E}_2^1)$ is $\{(Z_1, 0), (0, Z_2) : Z_1, Z_2 \in C\}$. Hence (PE_1^1, PE_2^1) is a deficient polynomial system.

For the deficient polynomial system PE^1, if we use homotopy in [4] to locate all of its isolated zeroes, the majority of the homotopy curves will diverge to infinity. On the other hand, since E has at most $\Pi_{j=1}^n(2q_j)$ zeroes in D and the total degree of PE^1 is $(n+1)^n\Pi_{j=1}^n q_j$, it is unimaginable to follow $(n+1)^n\Pi_{j=1}^n q_j$ curves to find $\Pi_{j=1}^n q_j$ zeroes of PE^1. Hence, unless we have efficient methods to locate all isolated zeroes of the deficient system PE^1, it is unreasonable to transform exponential systems into corresponding polynomial systems to find all isolated zeroes of the exponential systems.

§5. Numerical Experiments

A program was written for zeroes of the exponential systems on the basis of the algorithm in [5]. The following are some examples calculated with homotopy (2.1).

Example 1. $E: C^2 \to C^2$ is defined by

$$E_1(Z) = 0.1e^{2Z_1} - e^{2Z_2} + 0.2e^{-2Z_1} - e^{-2Z_2} + e^{Z_2} + 2 + i \, ,$$
$$E_2(Z) = e^{Z_1} + 0.5e^{Z_2} + e^{-Z_1} + e^{-Z_2} + 2 + i \, .$$

The eight resulting zeroes of E are

$$(-0.2071+i\ 1.8105,\ -0.3726+i\ 2.5745)\ ,$$
$$(-1.5888+i\ 2.7418,\ -0.9827+i\ 5.9463)\ ,$$
$$(-1.3033+i\ 2.7651,\ 1.0035+i\ 0.2538)\ ,$$
$$(-0.5836+i\ 1.9454,\ 0.1767+i\ 3.7927)\ ,$$
$$(\ 1.2651+i\ 3.5091,\ 0.8664+i\ 0.2234)\ ,$$
$$(\ 0.5869+i\ 4.3628,\ 0.1382+i\ 3.7405)\ ,$$
$$(\ 0.2224+i\ 4.4742,\ -0.3556+i\ 2.5919)\ ,$$
$$(\ 1.4607+i\ 3.5199,\ -0.7264+i\ 6.0031)\ .$$

Example 2. Let $E:\ C^2 \to C^2$ be

$$E_1(Z) = \sin\,(2Z_1) + \sin\,(Z_2) + \cos\,(Z_2) + 1 + i\ ,$$
$$E_2(Z) = \cos\,(Z_2) + \sin\,(Z_1) + 1 + i\ .$$

The eight resulting zeroes of E are

$$(1.5192 - i\ 0.7354,\ 2.7119 + i\ 1.5708)\ ,$$
$$(1.0421 + i\ 0.9421,\ 3.7709 - i\ 1.6949)\ ,$$
$$(2.8864 + i\ 0.3374,\ 2.5369 + i\ 0.9993)\ ,$$
$$(3.6998 - i\ 0.2027,\ 4.4177 - i\ 1.0325)\ ,$$
$$(5.6272 - i\ 0.6207,\ 1.8145 + i\ 0.4727)\ ,$$
$$(4.4859 + i\ 0.1143E - 01,\ 4.6944 - i\ 0.8796)\ ,$$
$$(5.6707 + i\ 0.8982,\ 1.6556 + i\ 1.3718)\ ,$$
$$(6.4865 - i\ 0.6303,\ 3.5311 - i\ 0.8074)\ .$$

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