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NON-CLASSICAL ELLIPTIC PROJECTIONS AND L^2-ERROR ESTIMATES FOR GALERKIN METHODS FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract

In this paper we shall define a so-called "non-classical" elliptic projection associated with an integro-differential operator. The properties of this projection will be analyzed and used to obtain the optimal L^2 error estimates for the continuous and discrete time Galerkin procedures when applied to linear integro-differential equations of parabolic type.

§1. Introduction

Let Ω be an open bounded subset in $R^n (n \geq 1)$ with smooth boundary $\partial\Omega$ and consider the following integro-differential equation of parabolic type:

$$\rho(x)u_t(x,t) = \nabla \cdot [a(x)\nabla u(x,t)] + \int_0^t \nabla \cdot [b(x,t,\tau)\nabla u(x,\tau)]d\tau + f(x,t), \quad \text{in } Q_T, \tag{1.1}$$

$$u(x,0) = u_0(x), \ x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad \text{on} \quad S_T = \partial\Omega \times [0,T], \tag{1.3}$$

where $Q_T = \Omega \times (0,T], T > 0; \nabla$ is the gradient operator in $R^n; \rho(x), a(x), b(x,t,\tau)$ and $f(x,t)$ are known functions which are assumed to be as smooth as needed throughout this paper. In addition, we assume that there exist two positive constants c_*, c^* such that

$$0 < c_* \leq \rho(x), \quad a(x) \leq c^*, \quad x \in \Omega. \tag{1.4}$$

Recently, some attention has been given to numerical approximations to the solution of (1.1)–(1.3). Sloan and Thomée[15] considered the time discretization approximations, Cannon and Li and Lin[4] have formulated a Galerkin procedure for general linear equations. Optimal L^2 error estimates in the case when $a(x) = 1$ and $b(x,t,\tau) = b(t,\tau)$ appear in [11]. The problems of existence, uniqueness and stability of the solution can be found in [9, 12, 13, 17].

When $a(x) \neq$ const. and $b(x,t,\tau)$ is dependent upon x, the method developed in [11] fails to provide the desired L^2 error estimates. The main reason of this is that there are two second order operators on the right-hand side of (1.1), so the usual elliptic projection method discovered by Wheeler in [18] does not work in this case in general. This suggests that we need to treat the operator

$$\nabla \cdot [a(x)\nabla u] + \int_0^t \nabla \cdot [b(x,t,\tau)\nabla u(x,\tau)]d\tau \tag{1.5}$$

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as a single unit. In this paper we shall define a "non-classical" elliptic projection suitable for (1.5). In the special case when $b = 0$, our new projection reduces to the usual elliptic projection defined in [18].

Let $H^s(\Omega)$ denote Sobolev spaces on Ω and $\|\cdot\|_s$ the related norm, with $H^0(\Omega) = L^2(\Omega)$ with norm $\|\cdot\|$. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_1$.

Let $\{S_h\}_{0<h\leq 1}$ be the finite-dimensional subspaces in H_0^1 which satisfy the following approximation property:

$$\inf_{\chi\in S_h}\{\|v-\chi\| + h\|v-\chi\|_1\} \leq Ch^s\|v\|_s, \quad v\in H^s\cap H_0^1, \quad s\geq 1, \tag{1.6}$$

where C is a positive constant independent of h and $v \in H^s \cap H_0^1$.

If X is a normed space with norm $\|\cdot\|_X$ and $\phi : [0,T] \to X$, we define

$$\|\phi\|^2_{L^2(X)} = \int_0^T \|\phi(t)\|_X^2 dt, \quad \|\phi\|_{L^\infty(X)} = \operatorname{ess}\sup_{0\leq t\leq T}\|\phi(t)\|_X.$$

The continuous Galerkin approximation to the solution u of (1.1)–(1.3) is defined to be a map $U(t) : [0,T] \to S_h$ such that

$$(\rho U_t, \chi) + \Big(a\nabla U + \int_0^t b\nabla U(\tau)d\tau, \nabla\chi\Big) = (f,\chi), \quad t>0, \quad \chi\in S_h, \tag{1.7}$$

$$U(0,) - u_0 \text{ small}, \tag{1.8}$$

where

$$(\phi,\psi) = \int_\Omega \phi(x)\psi(x)dx$$

for scalar and vector functions, respectively. The choice of $U(0)$ will be described later. We know that (1.7)–(1.8) is actually a system of ordinary integro-differential equations and it can be easily checked that for any $U(0) \in S_h$ there exists a unique $U(t)$ for $t > 0$.

Let N be a positive integer, $\Delta t = T/N, t_m = m\Delta t$ and $t_{m+1/2} = (m+1/2)\Delta t$; then we define $f_m = f(t_m)$and $f_{m+1/2} = (1/2)(f_{m+1} + f_m)$. For $t(\tau), g(\tau)$ smooth, we know that

$$\int_{t_k}^{t_{k+1}} f(\tau)g(\tau)d\tau = \Delta t f(t_{t+1/2})g_{k+1/2} + \varepsilon_k(f,g).$$

Since it is easy to verify

$$\int_{t_k}^{t_{k+1}} fg d\tau = \Delta t(fg)_{k+1/2} + \frac{1}{2}\int_{t_k}^{t_{k+1}}(t_{k+1}-\tau)(t_k-\tau)\frac{d^2(fg)}{d\tau^2}d\tau,$$

$$(fg)_{k+1/2} = f_{k+1/2}g_{k+1/2} + \frac{1}{4}\Big(\int_{t_k}^{t_{k+1}}\frac{df}{d\tau}d\tau\Big)\Big(\int_{t_k}^{t_{k+1}}\frac{dg}{d\tau}\Big)d\tau,$$

$$f_{k+1/2} = f(t_{k+1/2}) + \frac{1}{2}\Big[\int_{t_{k+1/2}}^{t_{k+1}}(t_{k+1}-\tau)\frac{d^2f}{d\tau^2}d\tau + \int_{t_{k+1/2}}^{t_k}(t_k-\tau)\frac{d^2f}{d\tau^2}d\tau\Big],$$

we see that the error $\varepsilon_k(f,g)$ can be represented by

$$\varepsilon_k(f,g) = \frac{1}{2}\int_{t_k}^{t_{k+1}}(t_{k+1}-\tau)(t_k-\tau)\frac{d^2(fg)}{d\tau^2}d\tau + \frac{\Delta t}{4}\Big(\int_{t_k}^{t_{k+1}}\frac{df}{d\tau}d\tau\Big)\Big(\int_{t_k}^{t_{k+1}}\frac{dg}{d\tau}\Big)d\tau$$

$$+\frac{\Delta t}{2}\Big[\int_{t_{k+1/2}}^{t_{k+1}}(t_{k+1/2}-\tau)\frac{d^2f}{d\tau^2}d\tau + \int_{t_{k+1/2}}^{t_k}(t_k-\tau)\frac{d^2f}{d\tau^2}d\tau\Big]g_{k+1/2}.$$

Thus, a discrete time Crank-Nicolson Galerkin approximation to (1.1)–(1.3) is defined to be a family $\{U_m\}_{m=0}^N$ in S_h such that

$$(\rho\partial_t U_{m+1},\chi)+(a\nabla U_{m+1/2}+\Delta t\sum_{k=0}^{m} b_{mk}\nabla U_{k+1/2},\nabla\chi)=(f(t_{m+1/2}),\chi),\quad \chi\in S_h, \tag{1.9}$$

$$U_0-u_0 \text{ small}, \tag{1.10}$$

where

$$\partial_t U_{m+1}=(\Delta t)^{-1}(U_{m+1}-U_m),$$

$$b_{mk}=\frac{1}{2}b(x,t_{m+1},t_{k+1/2})+\frac{1}{2}b(x,t_m,t_{k+1/2}),\quad k=1,2,\cdots,m-1,$$

$$b_{mm}=\frac{1}{2}b(x,t_{m+1},t_{m+1/2}).$$

U_0 can be approximated by the L^2 projection of u_0 into S_h or any other similar approximation. Notice that (1.9) is $O((\Delta t)^2)$ order in time.

The main results of this paper are the following theorems:

Theorem 1. *Let u be a solution to (1.1)–(1.3) such that $u\in L^\infty(0,T;H^s), u_t\in L^2(0,T;H^s)$ and assume that U is the solution to (1.7)–(1.8) with $U(0)$ chosen properly. Then we have*

$$\|u-U\|_{L^\infty(L^2)}=O(h^s). \tag{1.11}$$

Theorem 2. *Let the solution u of (1.1)–(1.3) be such that $u\in L^\infty(0,T;H^s), u_t\in L^2(0,T;H^s), u_{tt}\in L^2(0,T;H^1)$ and $u_{ttt}\in L^2(0,T;L^2)$. Then there exists a positive constant σ such that if $\{U_m\}_{m=0}^N$ is the Crank-Nicolson Galerkin approximations, with U_0 chosen properly, it follows that for all $0<\Delta t<\sigma$,*

$$\max_{0\le m\le N}\|u_m-U_m\|=O(h^s+(\Delta t)^2). \tag{1.12}$$

In this paper we shall use the following version of Gronwall's lemma: If $f(t), g(t)$ are nonnegative real-valued functions which satisfy

$$f(t)\le Cg(t)+C\int_0^t f(\tau)d\tau,\quad 0\le t\le T,$$

then we have

$$f(t)\le Ce^{CT}\Big\{g(t)+\int_0^t g(\tau)d\tau\Big\}.$$

Here and in what follows we denote by C a generic constant which may be different upon each occurrence.

In Section 2 we shall define a "non-classical" elliptic projection and study its properties. The proofs of Theorem 1 and Theorem 2 will be given in Section 3 using the projection defined in Section 2.

§2. Non-Classical Elliptic Projection

For approximation to the solution of parabolic equations, it is thought that, in order to obtain optimal L^2 error estimates, we have to use an auxiliary elliptic projection introduced by Wheeler in [18]. Here we shall modify her idea and for u, the solution of (1.1)–(1.3), define a map $W(t):[0,T]\to S_h$ such that

$$(a\nabla(W-u)+\int_0^t b\nabla(W-u)(\tau)d\tau,\nabla\chi)=0,\quad x\in S_h. \tag{2.1}$$

We call this W a non-classical elliptic projection of u into S_h. It is easy to see that (2.1) is an integral equation of Volterra type. For example, if $S_h = \text{span}\{\psi_k\}_{k=1}^N$, where ψ_k are linearly independent, and if we assume that

$$W(x,t) = \sum_{k=1}^{N} C_k(t)\psi_k(x),$$

then (2.1) can be rewritten as

$$AC(t) + \int_0^t B(t,\tau)C(\tau)d\tau = F(t), \tag{2.2}$$

where A, B are matrices and F is a vector, and

$$C(t) = (C_1(t),\cdots,C_N(t))^T,\ F(t) = (F_1(t),\cdots,F_N(t))^T,$$

$$F_l(t) = \Big(a\nabla u + \int_0^t b\nabla u d\tau,\ \nabla\psi_l\Big),\quad l = 1,2,\cdots,N,$$

$$A = ((a\nabla\psi_k, \nabla\psi_l)), B(t,\tau) = ((b\nabla\psi_k,\nabla\psi_l)).$$

Since A is positive definite, it follows from the general theory of integral equations that there exists a unique solution $C(t)$ for (2.2). Consequently, we see that the W in (2.1) is well-defined.

We shall now prove some lemmas which will be used in the next section. For any $u \in H^k(0,T;H^s(\Omega))$, we define

$$\|u(t)\|_{s,k}^2 = \sum_{j=0}^{k}\Big\{\Big\|\frac{\partial^j}{\partial t^j}u(t)\Big\|_s^2 + \int_0^t \Big\|\frac{\partial^j}{\partial t^j}u(\tau)\Big\|_s^2 d\tau\Big\}$$

where

$$H^k(0,T;H^s) = \Big\{u \in L^2(0,T;H^s)\Big|\frac{\partial^j u}{\partial t^j} \in L^2(0,T;H^s),\ j = 0,1,\cdots,k\Big\}.$$

Remark. $\|\cdot\|_{s,k}$ is not a norm on $H^k(0,T;H^s(\Omega))$.

Lemma 1. *Let $\eta = W - u$. For $u \in L^2(0,T;H^s(\Omega))$, there exists a positive constant C independent of u and h such that*

$$\|\eta\| + h\|\eta\|_1 \le Ch^s\|u\|_{s,0}. \tag{2.3}$$

Proof. Since $W \in S_h$, we have from (2.1) that

$$\begin{aligned}\Big(a\nabla\eta + \int_0^t b\nabla\eta(\tau)d\tau, \nabla\eta\Big) &= \Big(a\nabla\eta + \int_0^t b\nabla\eta(\tau)d\tau, \nabla(\chi - u)\Big)\\ &\le c^*\|\nabla\eta\|\,\|\nabla(u-\chi)\| + C\Big(\int_0^t \|\nabla\eta(\tau)\|d\tau\Big)\|\nabla(u-\chi)\|\\ &\le \frac{c_*}{4}\|\nabla\eta\|^2 + C\int_0^t \|\nabla\eta(\tau)\|^2 d\tau + C\|\nabla(u-\chi)\|^2.\end{aligned}$$

Thus, it follows that

$$\begin{aligned}c_*\|\nabla\eta\|^2 - C\int_0^t \|\nabla\eta(\tau)\|^2 d\tau - \frac{c_*}{4}\|\nabla\eta\|^2\\ \le Ch^{2s-2}\|u\|_s^2 + \frac{c_*}{4}\|\nabla\eta\|^2 + C\int_0^t \|\nabla\eta(\tau)\|^2 d\tau.\end{aligned} \tag{2.4}$$

Gronwall's inequality implies that

$$\|\nabla\eta\|^2 \le Ch^{2s-2}\Big\{\|u\|_s^2 + \int_0^t \|u\|_s^2 d\tau\Big\} \le Ch^{2s-2}\|u\|_{s,0}^2. \tag{2.5}$$

For any function $v \in H_0^1$, its elliptic projection $\tilde{v}$ is defined by

$$(\nabla(v-\tilde{v}), \nabla\chi) = 0, \quad \chi \in S_h. \tag{2.6}$$

For $\phi \in L^2(\Omega)$, let ψ be the solution of

$$-\nabla\cdot a\nabla\psi = \phi, \ \text{ in } \ \Omega, \tag{2.7}$$

$$\psi = 0, \ \text{ on } \partial\Omega. \tag{2.8}$$

Thus, we have $\|\psi\|_2 \le C\|\phi\|$. Employing this construction we can show that

$$\begin{aligned}
\Big(\eta + \frac{1}{a}\int_0^t b\eta(\tau)d\tau, \phi\Big) &= \Big(\nabla\eta + \int_0^t \frac{b}{a}\nabla\eta(\tau)d\tau + \int_0^t (\nabla\frac{b}{a})\eta(\tau)d\tau, a\nabla\psi\Big)\\
&= \Big(a\nabla\eta + \int_0^t b\nabla\eta(\tau)d\tau, \nabla\psi\Big) + \Big(\int_0^t (\nabla\frac{b}{a})\eta(\tau)d\tau, a\nabla\psi\Big)\\
&= \Big(a\nabla\eta + \int_0^t b\nabla\eta(\tau)d\tau, \nabla(\psi-\tilde{\psi})\Big) + \Big(\int_0^t (\nabla\frac{b}{a})\eta(\tau)d\tau, a\nabla\psi\Big)\\
&\le C\Big(\|\nabla\eta\| + \int_0^t \|\nabla\eta(\tau)\|d\tau\Big)\|\nabla(\psi-\tilde{\psi})\| + C\Big(\int_0^t \|\eta(\tau)\|d\tau\Big)\|\nabla\psi\|\\
&\le Ch\Big(\|\nabla\eta\| + \int_0^t \|\nabla\eta(\tau)\|d\tau\Big)\|\psi\|_2 + C\int_0^t \|\eta(\tau)d\tau\|\psi\|_2\\
&\le C\Big\{h\Big(\|\nabla\eta\| + \int_0^t \|\nabla\eta(\tau)\|d\tau\Big) + \int_0^t \|\eta(\tau)\|d\tau\Big\}\|\phi\|,
\end{aligned} \tag{2.9}$$

so that we have

$$\|\eta\| \le C\int_0^t \|\eta(\tau)\|d\tau + Ch\Big(\|\nabla\eta\| + \int_0^t \|\nabla\eta(\tau)\|d\tau\Big). \tag{2.10}$$

From Gronwall's inequality together with (2.5) and (2.10) it follows that

$$\|\eta\| \le Ch^s\|u\|_{s,0}. \tag{2.11}$$

This completes the proof.

Lemma 2. *If $u \in H^1(0,T;H^s(\Omega))$, then there exists a positive constant C such that*

$$\|\eta_t\| + h\|\eta_t\|_1 \le Ch^s\|u\|_{s,1}. \tag{2.12}$$

Proof. We differentiate (2.1) to obtain

$$\Big(a\nabla\eta_t + b(t,t)\nabla\eta + \int_0^t b_t\nabla\eta(\tau)d\tau, \nabla\chi\Big) = 0. \tag{2.13}$$

Hence

$$\Big(a\nabla\eta_t + b(t,t)\nabla\eta + \int_0^t b_t\nabla\eta(\tau)d\tau, \nabla\eta_t\Big) = \Big(a\nabla\eta_t + b(t,t)\nabla\eta + \int_0^t b_t\nabla\eta(\tau)d\tau, \nabla(\chi - u_t)\Big). \tag{2.14}$$

If we take $\chi = \tilde{u}_t$, then we see that

$$\frac{c_*}{2}\|\nabla\eta_t\|^2 - C\Big(\|\nabla\eta\|^2 + \int_0^t \|\nabla\eta(\tau)\|d\tau\Big) \le \frac{c_*}{4}\|\nabla\eta_t\|^2$$
$$+C\Big(\|\nabla\eta\|^2 + \int_0^t \|\nabla\eta(\tau)\|^2 d\tau\Big) + C\|\nabla(u_t - \tilde{u}_t)\|^2. \tag{2.15}$$

So it follows from Lemma 1 that

$$\|\nabla\eta_t\|^2 \le Ch^{2s-2}\Big(\|u_t\|_s^2 + \|u\|_{s,0}^2 + C\int_0^t \|\nabla\eta(\tau)\|^2 d\tau\Big) \le Ch^{2s-2}\|u\|_{s,1}^2. \tag{2.16}$$

Similarly to the estimate of $\|\eta\|$, we consider

$$\Big(\eta_t + \frac{1}{a}\Big(b(t,t)\eta + \int_0^t b_t\eta(\tau)d\tau\Big), \phi\Big) = \Big(a\nabla\eta_t + b(t,t)\nabla\eta + \int_0^t b_t\nabla\eta(\tau)d\tau, \nabla\psi\Big)$$
$$+\Big(\Big(\nabla\frac{b(t,t)}{a}\Big)\eta + \int_0^t (\nabla\frac{b_t}{a})\eta(\tau)d\tau, a\nabla\psi\Big)$$
$$= \Big(a\nabla\eta_t + b(t,t)\nabla\eta + \int_0^t b_t\nabla\eta(\tau)d\tau, \nabla(\psi - \tilde{\psi})\Big) + \Big((\nabla\frac{b(t,t)}{a})\eta$$
$$+\int_0^t (\nabla\frac{b_t}{a})\eta(\tau)d\tau, a\nabla\psi\Big) \le Ch\Big(\|\nabla\eta_t\| + \|\nabla\eta\| + \int_0^t \|\nabla\eta(\tau)\|d\tau + \|\eta\|$$
$$+\int_0^t \|\eta(\tau)\|d\tau\Big)\|\psi\|_2 \le Ch^s\|u\|_{s,1}\|\phi\|, \tag{2.17}$$

from which and Lemma 1 we have

$$\|\eta_t\| \le C\Big\{h^s\|u\|_{s,1} + \|\eta\| + \int_0^t \|\eta(\tau)\|d\tau\Big\} \le Ch^s\|u\|_{s,1}. \tag{2.18}$$

Thus, Lemma 2 has been proved.

Lemma 3. *If $u \in H^2(0,T;H^s(\Omega))$, then we have*

$$\|\eta_{tt}\| + h\|\nabla\eta_{tt}\| \le Ch^s\|u\|_{s,2}. \tag{2.19}$$

Proof. We differentiate (2.13) to obtain

$$\Big(a\nabla\eta_{tt} + b(t,t)\nabla\eta_t + 2b_t(t,t)\nabla\eta + \int_0^t b_{tt}\nabla\eta(\tau)d\tau, \nabla\chi\Big) = 0. \tag{2.20}$$

The remainder of the proof is essentially the same as that of Lemma 2, so we omit it here.

It is easy to see from the above three lemmas that

Lemma 4. *If $u \in H^2(0,T;H^1(\Omega))$, then there exists a positive constant C, independent of h and u, such that*

$$\|W\|_1 \le C\|u\|_{1,0}, \quad \|W_t\|_1 \le C\|u\|_{1,1}, \quad \|W_{tt}\|_1 \le C\|u\|_{1,2}. \tag{2.21}$$

§3. The Optimal L^2 Error Estimates

We shall now prove Theorem 1 and Theorem 2 as stated in Section 1.

Proof of Theorem 1. Let $u - U = (u - W) + (W - U) = -\eta + \theta$, so that it suffices to estimate θ. We see from (1.1)-(1.3) and (2.1) that

$$(\rho W_t, \chi) + \Big(a\nabla W + \int_0^t b\nabla W(\tau)d\tau, \nabla\chi\Big) = (f,\chi) + (\rho(W_t - u_t),\chi), \quad \chi \in S_h. \tag{3.1}$$

Subtracting (1.7) from (3.1), we have

$$(\rho\theta_t,\chi)+\left(a\nabla\theta+\int_0^t b\nabla\theta(\tau)d\tau,\nabla\chi\right)=(\rho\eta_t,\chi),\quad \chi\in S_h. \tag{3.2}$$

Setting $\chi=\theta\in S_h$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\theta\|^2+\|a^{1/2}\nabla\theta\|^2\le C\int_0^t\|\nabla\theta(\tau)\|\,\|\nabla\theta(t)\|d\tau+C\|\eta_t\|\,\|\theta\|$$

$$\le\frac{c_*}{2}\|\nabla\theta\|^2+C(\|\eta_t\|^2+\int_0^t\|\nabla\theta(\tau)\|^2d\tau). \tag{3.3}$$

Applying Gronwall's lemma, we have

$$\|\theta\|^2\le C\{\|\theta(0)\|^2+\|\eta_t\|^2+\int_0^t\|\eta_t(\tau)\|^2d\tau\}. \tag{3.4}$$

If we approximate $U(0)$ and $W(0)$ such that

$$\|U(0)-u_0\|+\|W(0)-u_0\|\le Ch^s\|u_0\|_s, \tag{3.5}$$

for example, by taking them both to be the L^2 projection of u_0 into S_h, then Theorem 1 follows from Lemma 1 and (3.4)–(3.5).

Proof of Theorem 2. We see from (2.1) that

$$\left(a\nabla W_{m+1/2}+\Delta t\sum_{k=0}^m b_{mk}\nabla W_{k+1/2}+\varepsilon_m(W),\nabla\chi\right)$$

$$=\left(a\nabla u_{m+1/2}+\Delta t\sum_{k=0}^m b_{mk}\nabla u_{k+1/2}+\varepsilon_m(u),\nabla\chi\right),\quad \chi\in S_h, \tag{3.6}$$

where

$$\|\varepsilon_m(u)\|^2\le C(\Delta t)^3\int_{t_m}^{t_{m+1}}(\|u\|_1^2+\|u_t\|_1^2)d\tau+C(\Delta t)^4\|u\|^2_{H^2(0,T;H^1)},$$

$$\|\varepsilon_m(W)\|^2\le C(\Delta t)^3\int_{t_m}^{t_{m+1}}(\|W\|_1^2+\|W_t\|_1^2)d\tau+C(\Delta t)^4\|W\|^2_{H^2(0,T;H^1)}.$$

Thus,

$$(\rho\partial_t W_m,\chi)+\left(a\nabla W_{m+1/2}+\Delta t\sum_{k=0}^m b_{mk}\nabla W_{k+1/2},\nabla\chi\right)$$

$$=(f(t_{m+1/2})+\rho_m+\rho\partial_t\eta_m,\chi)+((\rho_m(u)-\rho_m(W)),\nabla\chi),\quad \chi\in S_h, \tag{3.7}$$

where

$$\rho_m=\partial_t u_m-u_t(t_{m+1/2})$$

and

$$\|\rho_m\|^2\le C(\Delta t)^3\int_{t_{m-1}}^{t_{m+1}}\|u_{ttt}\|^2d\tau.$$

We subtract (1.9) from (3.7) and obtain

$$(\rho\partial_t\theta_m,\chi)+\left(a\nabla\theta_{m+1/2}+\Delta t\sum_{k=0}^m b_{mk}\nabla\theta_{k+1/2},\nabla\chi\right)$$

$$=(\rho\partial_t\eta_m+\rho_m,\chi)+(\rho_m(u)-\rho_m(W),\nabla\chi),\chi\in S_h. \tag{3.8}$$

If we set $\chi = \theta_{m+1/2}$, multiply (3.8) by $2\Delta t$ and sum on m, we see that

$$\|\rho^{1/2}\theta_{m+1}\|^2 - \|\rho^{1/2}\theta_0\|^2 + 2\Delta t\sum_{l=0}^{m}\|a^{1/2}\nabla\theta_{l+1/2}\|^2$$

$$\le C(\Delta t)^2\sum_{l=0}^{m}\sum_{k=0}^{l}(b_{lk}\nabla\theta_{k+1/2},\nabla\theta_{l+1/2}) + C\Delta t\sum_{l=0}^{m}(\rho\partial_t\eta_l + \rho_l,\theta_{l+1/2})$$

$$+C\Delta t\sum_{l=0}^{m}(\rho_l(u)-\rho_l(W),\nabla\theta_{l+1/2}) \le C(\Delta t)^2\sum_{l=0}^{m}\sum_{k=0}^{l}\|\nabla\theta_{k+1/2}\|\,\|\theta_{l+1/2}\|$$

$$+C\Delta t\sum_{l=0}^{m}(\|\partial_t\eta_l\| + \|\rho_l\|)\|\theta_{l+1/2}\| + C\Delta t\sum_{l=0}^{m}(\|\rho_l(W)\| + \|\rho_l(u)\|)\|\nabla\theta_{l+1/2}\|$$

$$\le C(\Delta t)^2\sum_{l=0}^{m}\|\nabla\theta_{l+1/2}\|^2 + C(\Delta t)^2\sum_{l=0}^{m-1}\sum_{k=0}^{l}\|\nabla\theta_{k+1/2}\|^2$$

$$+C\Delta t\Big\{\sum_{l=0}^{m}\|\partial_t\eta_l\|^2 + \sum_{l=0}^{m}(\|\rho_l\|^2 + \|\rho_l(W)\|^2 + \|\rho_l(u)\|^2)\Big\}. \tag{3.9}$$

We know that

$$\Delta t\sum_{l=0}^{m}\|\partial_t\eta_l\|^2 \le C\int_0^T\|\eta_t(\tau)\|d\tau, \tag{3.10}$$

and it is easy to check from Lemma 4 that

$$\Delta t\sum_{l=0}^{m}\|\rho_l(u)\|^2 \le C(\Delta t)^4\|u\|^2_{H^1(0,T;L^2(\Omega))}, \tag{3.11}$$

$$\Delta t\sum_{l=0}^{m}\|\rho_l(W)\|^2 \le C(\Delta t)^4\|W\|^2_{H^1(0,T;L^2(\Omega))} \le C\|u\|^2_{H^1(0,T;L^2(\Omega))}. \tag{3.12}$$

Thus, we have

$$\|\theta_{m+1}\|^2 + (1-C\Delta t)\Delta t\sum_{l=0}^{m}\|\nabla\theta_{l+1/2}\|^2 \le C\{\|\theta_0\|^2 + h^{2s} + (\Delta t)^4\}$$

$$+C(\Delta t)\sum_{l=0}^{m-1}\Big(\Delta t\sum_{k=0}^{l}\|\nabla\theta_{k+1/2}\|^2\Big). \tag{3.13}$$

Select $\sigma > 0$ such that $1 - C\Delta t \ge (1/2)$ for all $0 < \Delta t \le \sigma$. Hence, it follows from the discrete version of Gronwall's lemma that

$$\max_{1\le m\le P}\|\theta_m\|^2 \le C\{\|\theta_0\|^2 + h^{2s} + (\Delta t)^4\}. \tag{3.14}$$

Theorem 2 is a consequence of this last result and Lemma 1 provides that (3.5) holds.

Remark. It is easy to see from the analysis in this paper that the non-classical elliptic projection method can be modified and used to obtain the optimal L^2 error estimates for nonlinear equations and other similar type of equations, e.g. equations which contain two or more higher order derivatives (the Sobolev equation is such an example).

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