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A CLASS OF SINGLE STEP METHODS WITH A LARGE INTERVAL OF ABSOLUTE STABILITY*1)2)

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Abstract

In this paper, a class of integration formulas is derived from the approximation so that the first derivative can be expressed within an interval $[nh,(n+1)h]$ as

$$\frac{dy}{dt}=-P(y-y_n)+f_n+Q_n(t).$$

The class of formulas is exact if the differential equation has the shown form, where P is a diagonal matrix, whose elements

$$-p_j=\frac{\partial}{\partial y_j}f_j(t_n,y_n),\quad j=1,\cdots,m$$

are constant in the interval $[nh,(n+1)h]$, and $Q_n(t)$ is a polynomial in t.

Each of the formulas derived in this paper includes only the first derivative f and

$$\frac{\partial}{\partial y_j}f_j(t_n,y_n).$$

It is identical with a certain Runge-Kutta method as P tends to zero and thus correct to the order of such Runge-Kutta method. In particular, when $Q_n(t)$ is a polynomial of degree two, one of our formulas is an extension of Treanor's method, and possesses better stability properties. Therefore the formulas derived in this paper can be regarded as a modified or an extended form of the classical Runge-Kutta methods. Preliminary numerical results indicate that our fourth order formula is superior to Treanor's in stability properties.

§1. Introduction

It is well known that the classical Runge-Kutta method, generally very satisfactory for non-stiff systems, fails badly in handling stiff systems. Thus, it is desirable to have a class of explicit formulas which can handle stiff systems (at least some special stiff systems) but which can provide proper speed and accuracy, of course, where a little special treatment is required.

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In many practical applications, one encounters systems of ordinary differential equations which can often be expressed approximately as

$$y' = -P(y - y_n) + f_n + \tilde{Q}_n(t, y),$$

where P is a diagonal matrix, $p_j = -\partial f_j/\partial y_j$ is a large quantity, $\tilde{Q}_n(t, y)$ is a function of t and y varies slowly and thus can be approximated by a polynomial $Q_n(t)$ in t. By virtue of this fact Treanor has proposed a modified Runge-Kutta method in [1], however, the interval of absolute stability is still small for handling stiff systems [2], [3]. Though the approach we will use is almost the same as Treanor's, one of the formulas we will provide, when $Q_n(t)$ is a polynomial of degree two, is really an extension of Treanor's method and possesses even better stability properties. Therefore, our formulas may also be regarded as a modified or an extended form of the classical Runge-Kutta methods.

Finally, eleven test problems arising mainly in chemistry from [4], [5] are chosen for numerical experiment. Preliminary numerical results indicate that our fourth order formula is superior to Treanor's in stability properties. However, just like Treanor's method, generally speaking our formulas are suitable only for the cases in which the main diagonal elements of the Jacobian matrix are large and the off-diagonal elements are comparatively small.

§2. Derivation of Integration Formulas

In this section a class of numerical integration formulas of the stiff initial value problem

$$\begin{cases} y' = f(t, y), \\ y|_{t=0} = y_0 \end{cases} \tag{1}$$

will be derived.

Assume that the equations of (1) at point (t_n, y_n) can be expressed approximately as

$$y' = -P(y - y_n) + f_n + Q_n(t), \tag{2}$$

where P is a diagonal matrix with elements

$$-p_j = \frac{\partial}{\partial y_j} f_j(t_n, y_n), \quad j = 1, 2, \cdots, m$$

and $Q_n(t)$ is a polynomial in t containing unknown parameters which are determined in the course of the integration.

To simplify notation, we restrict our discussion to the scalar equation, and set

$$F_0(h) = e^{-ph}, \quad F_l(h) = \frac{F_{l-1}(h) - \dfrac{1}{(l-1)!}}{(-ph)}, \quad l = 1, 2, \cdots. \tag{3}$$

Now the formulas for different $Q_n(t)$ are derived as follows. First of all we consider the simplest case, namely Eq. (1) can be expressed approximately in the interval $[nh, (n+1)h]$ as

$$y' = -p(y - y_n) + f_n + A(t - t_n). \tag{2.1}$$

Then the ordinary differential equation

$$y' = -p(y - y_n) + f_n,$$

in the interval $[nh, (n+\frac{1}{2})h]$ and Eq. (2.1) in the interval $[nh, (n+1)h]$ will be integrated, respectively. We obtain then solutions

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2}F_1\left(\frac{h}{2}\right)f_n, \quad y'_{n+\frac{1}{2}} = f\left(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}\right) = f_{n+\frac{1}{2}}$$

and

$$y_{n+1} = y_n + hF_1(h)f_n + Ah^2F_2(h).$$

By virtue of (2.1) and $y'_{n+\frac{1}{2}}$, Ah can be evaluated from

$$y'_{n+\frac{1}{2}} = -p\left(y_{n+\frac{1}{2}} - y_n\right) + f_n + \frac{h}{2}A.$$

Finally we obtain the integration formula

$$\text{(I)}\quad \begin{cases} y_{n+\frac{1}{2}} = y_n + \frac{h}{2}F_1\left(\frac{h}{2}\right)f_n, \quad f_{n+\frac{1}{2}} = f\left(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}\right), \\ y_{n+1} = y_n + hF_1(h)f_n + 2hF_2(h)\left\{\left(f_{n+\frac{1}{2}} - f_n\right) + p\left(y_{n+\frac{1}{2}} - y_n\right)\right\}. \end{cases}$$

Secondly, when Eq.(1) can be expressed approximately in the interval $[nh, (n+1)h]$ as

$$y' = -P(y - y_n) + f_n + A(t - t_n) + B(t - t_n)^2. \tag{2.2}$$

One can in the same way as above obtain the integration formula

$$\text{(II)}\quad \begin{cases} y_{n+\frac{1}{3}} = y_n + \frac{h}{3}F_1\left(\frac{h}{3}\right)f_n, \quad f_{n+\frac{1}{3}} = f\left(t_{n+\frac{1}{3}}, y_{n+\frac{1}{3}}\right), \\ y_{n+\frac{2}{3}} = y_n + \frac{2h}{3}F_1\left(\frac{2h}{3}\right)f_n + \frac{4}{3}hF_2\left(\frac{2h}{3}\right)\left\{\left(f_{n+\frac{1}{3}} - f_n\right) + p\left(y_{n+\frac{1}{3}} - y_n\right)\right\}, \\ f_{n+\frac{2}{3}} = f\left(t_{n+\frac{2}{3}}, y_{n+\frac{2}{3}}\right), \\ y_{n+1} = y_n + hF_1(h)f_n + 3hF_2(h)\left\{\left(f_{n+\frac{1}{3}} - f_n\right) + p\left(y_{n+\frac{1}{3}} - y_n\right)\right\} \\ \qquad + \frac{9}{2}hF_3(h)\left\{f_{n+\frac{2}{3}} - 2f_{n+\frac{1}{3}} + f_n + p\left(y_{n+\frac{2}{3}} - 2y_{n+\frac{1}{3}} + y_n\right)\right\}. \end{cases}$$

Similarly, to obtain a modified Treanor's method we take respectively

$$y^{(1)}_{n+\frac{1}{2}} = y_n + \frac{h}{2}F_1\left(\frac{h}{2}\right)f_n, \quad f^{(1)}_{n+\frac{1}{2}} = f\left(t_{n+\frac{1}{2}}, y^{(1)}_{n+\frac{1}{2}}\right)$$

and

$$y^{(2)}_{n+\frac{1}{2}} = F_0\left(\frac{h}{2}\right)y_n + \frac{h}{2}F_1\left(\frac{h}{2}\right)\left(f^{(1)}_{n+\frac{1}{2}} + py^{(1)}_{n+\frac{1}{2}}\right), \quad f^{(2)}_{n+\frac{1}{2}} = f\left(t_{n+\frac{1}{2}}, y^{(2)}_{n+\frac{1}{2}}\right)$$

instead of

$$y^{(1)}_{n+\frac{1}{2}} = y_n + \frac{h}{2}f_n, \quad f^{(1)}_{n+\frac{1}{2}} = f\left(t_{n+\frac{1}{2}}, y^{(1)}_{n+\frac{1}{2}}\right)$$

and

$$y^{(2)}_{n+\frac{1}{2}} = y_n + \frac{h}{2}f^{(1)}_{n+\frac{1}{2}}, \quad f^{(2)}_{n+\frac{1}{2}} = f\left(t_{n+\frac{1}{2}}, y^{(2)}_{n+\frac{1}{2}}\right).$$

From this the integration formula

$$
\text{(III)}\quad\begin{cases}
y^{(1)}_{n+\frac12} = y_n + \frac{h}{2}F_1\left(\frac{h}{2}\right)f_n, \quad f^{(1)}_{n+\frac12} = f\left(t_{n+\frac12}, y^{(1)}_{n+\frac12}\right),\\
y^{(2)}_{n+\frac12} = F_0\left(\frac{h}{2}\right)y_n + \frac{h}{2}F_1\left(\frac{h}{2}\right)\left(f^{(1)}_{n+\frac12} + py^{(1)}_{n+\frac12}\right),\\
f^{(2)}_{n+\frac12} = f\left(t_{n+\frac12}, y^{(2)}_{n+\frac12}\right),\\
\bar{y}_{n+1} = y_n + hF_1(h)f_n + 2hF_2(h)\left\{\left(f^{(2)}_{n+\frac12} - f_n\right) + p\left(y^{(2)}_{n+\frac12} - y_n\right)\right\},\\
\bar{f}_{n+1} = f(t_{n+1}, \bar{y}_{n+1}),\\
y_{n+1} = y_n + h\{F_1(h)f_n + (4F_3(h) - 3F_2(h))(f_n + py_n) + (2F_2(h)\\
\qquad -4F_3(h))\left[\left(f^{(1)}_{n+\frac12} + py^{(1)}_{n+\frac12}\right) + \left(f^{(2)}_{n+\frac12} + py^{(2)}_{n+\frac12}\right)\right]\\
\qquad +(4F_3(h) - F_2(h))(\bar{f}_{n+1} + p\bar{y}_{n+1})\},
\end{cases}
$$

can therefore be obtained, where $p_j = -\dfrac{\partial f_j}{\partial y_j}$ are taken instead of $\tilde{p}_j = -\dfrac{f^{(2)}_{n+\frac12} - f^{(1)}_{n+\frac12}}{y^{(2)}_{n+\frac12} - y^{(1)}_{n+\frac12}}$ in Treanor's method. Obviously, it is inconvenient for our formula (III) to take such a form of $\tilde{p}$, because it requires the solution of a set of nonlinear equations at each step.

§3. Basic Properties of the Integration Formulas

In this section, we will discuss some basic properties of the integraton formulas derived in Section 2, such as the order of accuracy, stability, the relation between the formulas derived in Section 2, and Runge-Kutta methods.

The formulas derived in Section 2 may be regarded as a particular case of the general explicit one-step methods. The definitions of their order and consistancy may be given analogously, and thus are omitted here. In order to save space, we consider only formula (I) in detail.

First of all, it is easy to see that formula (I) becomes the classical Runge-Kutta method of order two as p tends to zero:

$$
\text{(I)}'\quad\begin{cases}
k_1 = f(t_n, y_n),\\
k_2 = f\left(t_{n+\frac12}, y_n + h\frac{k_1}{2}\right),\\
y_{n+1} = y_n + hk_2.
\end{cases}
$$

The order of accuracy. Expanding formula (I) in a Taylor series at the point t_n gives its local truncation error (LTE) T_2,

$$T_2 = T_{RK(2)} + \left(\frac18 p^2 f_n + \frac{7}{24}pf'_n\right)h^3 = T_{RK(2)} + T_{E(2)}(p)h^3,$$

where $T_{RK(2)}$ is the LTE of the Runge-Kutta method (I)′. Therefore formulas (I) is of order two. Similarly, we can verify that formulas (II) and (III) are of order three and four respectively.

It should be noted that $T_{E(2)}(p)$ is not only a function of f and f' but also a polynomial of degree two in p. Numerical results indicate that for the fixed steplength h, if the three methods, namely Runge-Kutta's, Treanor's and ours, applied to the same initial problem of ODEs are all stable, then the numerical solutions obtained using these methods can be ordered, as far as accuracy is concerned, in general as follows: the Runge-Kutta method is the best, Treanor's is in between, and ours is the worst. This fact means that $T_{E(4)}$ plays a more important part in the LTE than $T_{RK(4)}$. However, in handling stiff systems, generally p is not equal to zero (if $p = 0$ the formula becomes one of Runge-Kutta's), and is often very large. Therefore a further investigation on how to construct the formulas in Section II so that $T_{E(q)}(p)$ is as small as possible is worthwhile.

The stability properties. We apply formula (I) to the scalar test equation $y' = \lambda y, \lambda < 0$. When $p = -\lambda$, we obtain easily

$$y_{n+1} = y_n e^{\lambda h},$$

hence formula (I) is stable. When $p \neq -\lambda$ we have

$$y_{n+1} = R(\lambda,\ ph) y_n,$$

where

$$R(\lambda, ph) = 1 + \frac{\lambda h}{ph}\left(1 - e^{-ph}\right) + 2\lambda h\left(1 + \frac{\lambda h}{ph}\right)\left(1 - e^{-ph/2}\right)\left(\frac{e^{-ph} - 1 + ph}{p^2h^2}\right).$$

Set

$$\theta_2(ph) = \min\left\{\left(1 + e^{-ph/2}\right)\Big/\frac{(e^{-ph} - 1 + ph)}{p^2h^2/2},\quad \left(1 + e^{-ph}\right)\Big/\frac{(1 - e^{-ph})}{ph}\right\}.$$

It can be proved that if $0 \le -\lambda h \le ph + \theta_2(ph)$, then $|R(\lambda, ph)| \le 1$ holds. It follows that formula (I) is stable as $\lambda h \in [-(ph + \theta_2(ph)), 0]$. It is easily seen that $\theta_2(0) = 2$ as ph tends to zero, and $|R(\lambda, 0)| \le 1$ as $\lambda h \in [-2, 0]$, namely, the interval of absolute stability of the usual Runge-Kutta method with order two. We believe that a result analogous to formula (I) for formulas (II) and (III) also holds. Hence formula (III) derived in Section II has better stability properties than Treanor's. Here we note that it is difficult to prove directly the stability properties of formulas (II) and (III) as we do formula (I). But the numerical results in the next section will show that the conclusion is trustworthy.

§4. Numerical Results

In this section, we present some preliminary numerical results which can be used to make a comparison between formula (III) and Treanor's method when they are used in component form with fixed steplength. The first aim of the numerical experiment is to confirm numerically that the conclusions stated in the previous section are true, namely formula (III) has better stability properties than Treanor's. Another aim is to demonstrate mumerically that, as a general rule, formula (III) is very efficient for stiff systems when its Jacobian matrix has large main diagonal elements and no large off-diagonal elements.

To this end, we are particularly concerned with eleven test systems arising mainly in chemistry and two numerical examples on parabolic equations. The test problems can be found in [4]-[6], and are specified the appendix in detail.

All numerical experiments were run in FORTRAN double precision on the IBM 3083 of the computing center of the Gesellschaft für Mathematik und Datenverabeitung in Bonn.

The numerical results are presented in Table 1 which includes the following parameters:

ALMH : the allowable maximum steplength

MRER : the maximal relative errors at t_f

H : steplength h.

Our numerical experiments for each test problem are always started with the steplength $h = 0.1$. When numerical results overflowed or diverged with $h = 0.1$, we used a steplength smaller than 0.1 and so on; hence ALMH=0.1 in Table 1.

As we can see from Table 1, the conclusions stated in Section III are demonstrated again.

Table 1

	Treanor's		Formula III			
Test Problem No.	ALMH	MRER	H	MRER	ALMH	MRER
1	0.01	2.4(-2)	0.01	6.3(-3)	0.05	4.3(-2)
2	.0001	6.1(-5)	.0001	1.7(-2)	0.1	diverge
3	0.002	2.8(-8)	0.002	7.6(-6)	.0025	1.3(-5)
4	0.01	2.2(-2)	0.01	4.5(-2)	0.1	7.8(-2)
5	0.01	6.7(-7)	0.01	4.8(-6)	0.1	6.1(-5)
6	0.01	1.4(-5)	0.01	4.2(-4)	0.1	1.4(-2)
7	0.001	diverge	0.01	7.7(-8)	0.1	4.0(-6)
8	0.1	4.0(-6)			0.1	2.8(-5)
9	0.1	4.9(-5)			0.1	9.7(-5)
10	0.0001	overflow	0.0001	3.4(-6)	0.1	6.5(-3)
11	0.0001	overflow	0.0001	4.5(-6)	0.1	9.3(-2)

Numerical example on parabolic equations. Here, numerical results in the two dimensional case for parabolic equations which are aimed at showing the effectiveness of formula III for some special stiff systems will be shown. Let Ω^2 be a square domain of $I\!R^2$ given by

$$\Omega = \{(x_1, x_2) \in I\!R^2, 0 < x_1 < 1, \quad 0 < x_2 < 1\}.$$

Now we give the following problems[6].

Problem 1 (Dirichlet type).

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - 10^{-t} && \text{in } \Omega, \quad t > 0, \\ u(x_1, x_2, t) &= 0 && \text{on } \Gamma, \quad t > 0, \\ u(x_1, x_2, 0) &= u_0(x_1, x_2) && \text{in } \Omega, \end{aligned}$$

where

$$u_0(x_1, x_2) = \begin{cases} x_2, & \text{if } x_2 \le x_1 \le 1 - x_2, \quad x_2 \ge 0, \\ 1 - x_1, & \text{if } 1 - x_1 \le x_2 \le x_1, \quad x_1 \le 1, \\ 1 - x_2, & \text{if } 1 - x_2 \le x_1 \le x_2, \quad x_2 \le 1, \\ x_1, & \text{if } x_1 \le x_2 \le 1 - x_1, \quad x_1 \ge 0. \end{cases}$$

Problem 2 (Neumann type).

$$\frac{\partial u}{\partial t} = \Delta u + (x_2 - x_1) * 10^{-t} \quad \text{in } \Omega, \quad t > 0,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad t > 0,$$

$$u(x_1, x_2) = 0 \quad \text{in } \Omega,$$

Let Ω be subdivided uniformly with $\Delta x_1 = \Delta x_2 = 1/4$, and employ $\Delta t = 0.01$, 0.1, 0.2, 0.5, and 1.0 respectively (according to the stability conditions in [6] for piecewise linear finite element solutions by using explicit schemes, $\Delta t < 1/96$). The numerical results are presented in Tables 2 and 3.

Table 2. Numerical Solution for Problem 1 at Point (0.5, 0.5)

t \ Δt	0.01	0.1	0.2	0.5	1.0
0.5	-.02555	-.02653		-.01253	
1.0	-.00809	-.00846	-.00931	-.01168	-.00760
1.5	-.00256	-.00268		-.00404	
2.0	-.00081	-.00085	-.00093	-.00129	-.00209
2.5	-.00026	-.00027		-.00041	
3.0	-.000081	-.000085	-.000093	-.000129	-.000240

Table 3. Numerical Solution for Problem 2 at Point (0.5, 0.75)

t \ Δt	0.01	0.1	0.5	1.0
0.5	.00576	.00583	.00778	
1.0	.00183	.00185	.00204	.00343
1.5	.00058	.00058	.00067	
2.0	.00018	.00019	.00021	.00006
2.5	.00006	.00006	.00007	
3.0	.000018	.000019	.000021	.000029

As can be seen from the three tables, our numerical results demonstrate that formula III has better stability properties than Treanor's and is very efficient for some special stiff systems. Also, it can be seen from Table 1 that as far as accuracy is concerned, Treanor's method is better than formula III. This indicates that a properly combined version of formula III, Treanor's method and the Runge-Kutta method with order four can be favourable for numerical integration of some special stiff systems. Therefore it is suggested that one should take $|ph| < 2.78$ in the Runge-Kutta method with order four, $2.78 \le |ph| < 8.0$ in Treanor's, and $|ph| \geq 8.0$ in formula III respectively. Of course, it is convenient for Treanor's method to take $p_j = -\dfrac{\partial f_j}{\partial y_j}$ instead of

$$p_j = -\frac{f^{(2)}_{n+\frac{1}{2}} - f^{(1)}_{n+\frac{1}{2}}}{y^{(2)}_{n+\frac{1}{2}} - y^{(1)}_{n+\frac{1}{2}}}$$

in the entire algorithm as stated above.

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Appendix: Specification of Test Problems

Problem 1.
$$y_1' = 77.27(y_2 - y_1 y_2 + y_1 - 8.375 * 10^{-6} y_1^2), \quad y_1(0) = 4,$$
$$y_2' = -(y_2 + y_1 y_2 - y_3)/77.27, \quad y_2(0) = 1.1,$$
$$y_3' = .161(y_1 - y_3), \quad y_3(0) = 4,$$
$$t_1 = 300.$$

Problem 2.
$$y_1' = y_3 - 100 y_1 y_2, \quad y_1(0) = 1,$$
$$y_2' = y_3 + 2y_4 - 100 y_1 y_2 - 2 * 10^4 y_2^2, \quad y_2(0) = 1,$$
$$y_3' = -y_3 + 100 y_1 y_2, \quad y_3(0) = 0,$$
$$y_4' = -y_4 + 10^4 y_2^2, \quad y_3(0) = 0,$$
$$t_1 = 20.$$

Problem 3.
$$y_1' = -0.04 y_1 + 0.01 y_2 y_3, \quad y_1(0) = 1,$$
$$y_2' = 400 y_1 - 100 y_2 y_3 - 3000 y_2^2, \quad y_2(0) = 0,$$
$$y_3' = 30 y_2^2, \quad y_3(0) = 0,$$
$$t_1 = 40.$$

Problem 4.
$$y_1' = -7.89 * 10^{-10} y_1 - 1.1 * 10^7 y_1 y_3, \quad y_1(0) = 1.76 * 10^{-3}$$
$$y_2' = 7.89 * 10^{-10} y_1 - 1.13 * 10^9 y_2 y_3, \quad y_2(0) = 0$$
$$y_3' = 7.89 * 10^{-10} y_1 + 1.13 * 10^3 y_4 + y_1' + y_2', \quad y_3(0) = 0$$
$$y_4' = 1.1 * 10^7 y_1 y_3 - 1.13 * 10^3 y_4, \quad y_4(0) = 0,$$
$$t_f = 1000.$$

Problem 5.
$$y_1' = -0.013 y_1 - 1000 y_2 y_3, \quad y_1(0) = 1,$$
$$y_2' = -2500 y_2 y_3, \quad y_2(0) = 1,$$
$$y_3' = -0.013 y_1 - 1000 y_1 y_3 - 2500 y_2 y_3, \quad y_3(0) = 0,$$
$$t_1 = 50.$$

Problem 6.
$$y_1' = .01 - [1 + (y_1 + 1000)(1 + y_1)](.01 + y_1 + y_2), \quad y_1(0) = 0,$$
$$y_2' = .01 - (1 + y_2^2)(.01 + y_1 + y_2), \quad y_2(0) = 0,$$
$$t_1 = 100.$$

Problem 7.
$$y_1' = 1.3(y_3 - y_1) + 10400ky_2, \quad y_1(0) = 761,$$
$$y_2' = 1880[y_4 - y_2(1+k)], \quad y_2(0) = 0,$$
$$y_3' = 1752 - 269y_3 + 267y_1, \quad y_3(0) = 600,$$
$$y_4' = .1 + 320y_2 - 321y_4, \quad y_4(0) = 0.1,$$
$$t_1 = 1000, \textbf{where } k = \exp(20.7 - 1500/y_1).$$

Problem 8.
$$y_1' = -y_1 - y_1y_2 + 294y_2, \quad y_1(0) = 1,$$
$$y_2' = y_1(1 - y_2)/98 - 3y_2, \quad y_2(0) = 0,$$
$$t_1 = 240.$$

Problem 9.
$$y_1' = .2(y_2 - y_1), \quad y_1(0) = 0,$$
$$y_2' = 10y_1 - (60 - .125y_3)y_2 + .125y_3, \quad y_2(0) = 0,$$
$$y_3' = 1, \quad y_3(0) = 0,$$
$$t_1 = 400.$$

Problem 10.
$$y_1' = 10^{11}(-3y_1y_2 + .0012y_4 - 9y_1y_3), \quad y_1(0) = 3.365 * 10^{-7},$$
$$y_2' = -3 * 10^{11}y_1y_2 + 2 * 10^7 y_4, \quad y_2(0) = 8.261 * 10^{-3},$$
$$y_3' = 10^{11}(-9y_1y_3 + .001y_4), \quad y_3(0) = 1.642 * 10^{-3},$$
$$y_4' = 10^{11}(3y_1y_2 - .0012y_4 + 9y_1y_3), \quad y_4(0) = 9.38 * 10^{-6},$$
$$t_1 = 100.$$

Problem 11.
$$y_1' = -y_1 + 10^8 y_3(1 - y_1), \quad y_1(0) = 1,$$
$$y_2' = -10y_2 + 3 * 10^7 y_3(1 - y_2), \quad y_2(0) = 0,$$
$$y_3' = -y_1' - y_2', \quad y_3(0) = 0,$$
$$t_1 = 1.$$

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