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RECURRENCE RELATIONS FOR THE COEFFICIENTS IN ULTRASPHERICAL SERIES SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract

A method is presented for obtaining recurrence relations for the coefficients in ultraspherical series of linear differential equations. This method applies Doha's method (1985) to generate polynomial approximations in terms of ultraspherical polynomials of $y(zx), -1 \leq x \leq 1, z \in \underset{\sim}{C}, |z| \leq 1$, where y is a solution of a linear differential equation. In particular, rational approximations of $y(z)$ result if x is set equal to unity. Two numerical examples are given to illustrate the application of the method to first and second order differential equations. In general, the rational approximations obtained by this method are better than the corresponding polynomial approximations, and compare favourably with Padé approximants.

§1. Introduction

The truncated Chebyshev series has been widely used in numerical analysis as a good numerical approximation to $y \in C[-1, 1]$ using the supremum norm

$$\|y\|_\infty = \sup_{x\in[-1,1]} |y(x)|.$$

Lanczos[1] and Handscomb[2] have compared the performance of truncated Chebyshev series with truncated ultraspherical expansions. Light[3-4] has investigated conditions under which approximation to continuous functions on $[-1, 1]$ by series of Chebyshev polynomials is superior to approximation by other ultraspherical orthogonal expansions. In particular, he has derived conditions on the Chebyshev coefficients which guarantee that the Chebyshev expansion of the corresponding functions converges more rapidly than expansions in Legendre polynomials or Chebyshev polynomials of the second kind.

As candidates for the efficient representation of mathematical functions by easily computed expressions, rational functions are often to be preferred to polynomials. Indeed it has been found empirically that, in general, rational approximations can achieve a smaller maximum error for the same amount of computation than polynomial approximations; see for instance, Ralston and Rabinowitz[5].

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The well-known effective means of producing rational approximations to a function $y(z)$ in a complex variable is to develop elements of Padé table from its Taylor series. This table is a two-dimensional array whose (m,n) element is defined as that rational function of degree m in the numerator and n in the denominator whose own Taylor series expansion, $\sum_{r=0}^{\infty} c_r z^r$ say, agrees with that of $y(z)$ up to and including the term in z^{m+n}. Discussion of Padé table and its convergence is considered in Bender and Orszag [6].

In Doha [7], a method for obtaining simultaneously polynomial and rational approximations from Chebyshev and Legendre series for functions defined by linear differential equations with its associated boundary conditions has been described.

Our principal aim in the present paper is twofold:

(i) to give an extension of Doha's method, but the function $y(x)$ and its derivatives are expanded in ultraspherical polynomials $C_n^{(\alpha)}$. This extension, however, could be useful in applications.

(ii) to compare computationally the performance of the rational approximation obtained from Chebyshev series of the first kind and those obtained from the ultraspherical series for varying α.

The ultraspherical expansions are defined by

$$y(x) = \sum_{n=0}^{\infty} a_n C_n^{(\alpha)}(x)$$

where the coefficients a_n are given by

$$a_n = \int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} y(x) C_n^{(\alpha)}(x)dx \Big/ \int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} \{C_n^{(\alpha)}(x)\}^2 dx \tag{1}$$

and $C_n^{\alpha)}(x)$ satisfy the orthogonality relation

$$\int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x)dx = 0, \quad m \neq n, \quad \alpha > -\frac{1}{2}.$$

For our present purposes it is convenient to standardize the ultraspherical polynomials so that

$$C_n^{(\alpha)}(1) = \Gamma(n+2\alpha)/\Gamma(2\alpha)n!$$

In this form the polynomials may be generated using the recurrence formula

$$(n+1)C_{n+1}^{(\alpha)}(x) = 2(n+\alpha)xC_n^{(\alpha)}(x) - (n+2\alpha-1)C_{n-1}^{(\alpha)}(x), \quad n = 1,2,3,\cdots$$

starting from $C_0^{(\alpha)}(x) = 1$ and $C_1^{(\alpha)}(x) = 2\alpha x$, or obtained from Rodrigue's formula

$$C_n^{(\alpha)}(x) = \frac{(-1)^n \Gamma(n+2\alpha)\Gamma(n+1/2)}{2^n n! \Gamma(2\alpha)(n+\alpha+1/2)} (1-x^2)^{\frac{1}{2}-\alpha} D^n[(1-x^2)^{n+\alpha-\frac{1}{2}}]$$

where $D \equiv \frac{d}{dx}$. Certain values of α correspond to more familiar sets of orthogonal polynomials, the Legendre polynomials given by $C_n^{(1/2)}(x) = P_n(x)$, the Chebyshev polynomials of the second kind by $C_n^{(1)}(x) = U_n(x)$ and the Chebyshev polynomials of the first kind by

$$T_n(x) = \frac{n}{2} \lim_{\alpha \to 0} \frac{1}{\alpha} C_n^{(\alpha)}(x).$$

It is to be noted here that the usual powers of x are given by

$$x^n = \frac{n!}{2^n} \lim_{\alpha\to\infty} \frac{C_n^{(\alpha)}(x)}{\alpha^n}$$

which enables one to obtain the Taylor series expansion to $y(x)$.

Further details of the properties of these polynomials may be found in Abramowitz and Stegun [8]. From the orthogonality of the ultraspherical polynomials, the coefficients a_n of (1) takes the form

$$\epsilon_n = \frac{n!(n+\alpha)\Gamma(\alpha)\Gamma(2\alpha)}{\sqrt{\pi}\,\Gamma(n+2\alpha)\Gamma(a+\frac{1}{2})} \int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x) y(x) dx. \tag{2}$$

In general, it is not possible to evaluate the integral occurring in (2) explicitly, and to find a_n, recourse has to be made to a suitable quadrature technique.

The present method enables one to find these coefficients directly, provided that the function $y(x)$ should satisfy a linear differential equation with appropriate boundary conditions. The solution of linear differential equations in series of Chebyshev polynomials $T_n(x)$ has been given by Lanczos[9], Clenshaw[10], Fox[11], Morris and Horner[12], Olaofe[13] and Horner[14]. An extension of Clenshaw's method had been described by Elliott[15], and an extension of Clenshaw's method into the complex domain has been given in Doha [7]. The proposed method may also be considered as an extension of Elliott's method into the complex domain, which consequently yields rational approximation and as a special case the polynomial one.

The method is described Sections 2 and 3, and is illustrated by numerical examples in Section 4. Numerical results and comparisons and some concluding remarks are given in Sections 5 and 6 respectively.

§2. Ultraspherical Series Solution for Linear Differential Equations

Let $y(x)$ satisfy the linear differential equation of order m,

$$\sum_{i=0}^{m} p_i(x) D^i y(x) = q(x), \quad p_m(x) \neq 0 \tag{3}$$

where $q(x)$ and $p_i(x)$, $i = 0, 1, 2, \cdots, m$, are functions in x, and, in addition, let $y(x)$ be a function defined in $[-1, 1]$ which has the uniformly convergent expansion

$$y(x) = \sum_{n=0}^{\infty} a_n C_n^{(\alpha)}(x) \tag{4}$$

where the coefficients a_n are to be determined. Now assume that the kth derivative can be expanded in a uniformly convergent series

$$y^{(k)}(x) = \sum_{n=0}^{\infty} a_n^{(k)} C_n^{(\alpha)}(x), \quad k = 0, 1, 2, \cdots, m. \tag{5}$$

The method of determining the coefficients a_n depends basically upon the following two recurrence relations for the ultraspherical polynomials $C_n^{(\alpha)}(x)$, namely

$$2(n+\alpha)C_n^{(\alpha)}(x) = D(C_{n+1}^{(\alpha)}(x) - C_{n-1}^{(\alpha)}(x)) \tag{6}$$

and

$$2(n+\alpha)xC_n^{(\alpha)}(x) = (n+1)C_{n+1}^{(\alpha)}(x) + (n+2\alpha-1)C_{n-1}^{(\alpha)}(x) \tag{7}$$

both of which are valid for $n \geq 1$. In view of the formulas (4) and (6) we find that

$$2a_n^{(k)} = \frac{a_{n-1}^{(k+1)}}{n+\alpha-1} - \frac{a_{n+1}^{(k+1)}}{n+\alpha+1}, \quad n \geq 1. \tag{8}$$

We define a related set of coefficients $b_n^{(k)}$ by writing

$$a_n^{(k)} = (n+\alpha)b_n^{(k)}, \quad n \geq 0, \quad k = 0, 1, 2, \cdots, m, \tag{9}$$

and accordingly, equation (8) takes the form

$$2(n+\alpha)b_n^{(k)} = b_{n-1}^{(k+1)} - b_{n+1}^{(k+1)}, \quad n \geq 1. \tag{10}$$

Again, let $C_n(y)$ denote the coefficient of $C_n^{(\alpha)}(x)$ in the expansion of y. Then, from the recurrence relation (7) we see that

$$C_n(xy) = \frac{n+\alpha}{2}[\beta(n)b_{n-1} + \gamma(n)b_{n+1}], \quad n \geq 1$$

where

$$\beta(n) = \begin{cases} n/(n+\alpha), & \alpha \neq 0 \\ 1, & \alpha = 0 \end{cases}; \quad \gamma(n) = 2 - \beta(n).$$

In what follows, a generalization of the previous relation is needed. Define for an arbitrary function μ of the variable n

$$\mu^+(n) = \mu(n+1), \quad \mu^-(n) = \mu(n-1).$$

By induction, one finds

$$C_n(x^k y) = \frac{n+\alpha}{2^n} \sum_{j=0}^{k} \lambda_{kj}(n) b_{n-k+2j}, \quad n, k \geq 0 \tag{11}$$

where

$$\lambda_{kj}(n) = \binom{k}{j}, \quad \alpha = 0; \quad \lambda_{00}(n) = 1,$$

$$\lambda_{kj}(n) = \begin{cases} \beta(n)\lambda_{k-1,0}^-(n), & j = 0, \\ \beta(n)\lambda_{k-1,j}^-(n) + \gamma(n)\lambda_{k-1,j-1}^+(n), & 1 \leq j \leq k-1, \quad k \geq 1, \\ \gamma(n)\lambda_{k-1,k-1}^+(n), & \alpha \neq 0. \end{cases}$$

In equation (11) we may replace $y(x)$ by $y^{(k)}(x)$ provided b_n is changed by $b_n^{(k)}$. From (11), the quantities $C_n(xy), C_n(x^2y), \cdots, C_n(x^m y)$ can easily be found, and so in Equation (3) $C_n(p_i(x)D^i y)$ can de written down if $p_i(x)$ is a polynomial in x. In cases where $p_i(x)$, $i =$

$0,1,2,\cdots,m$, are not polynomials in x, it is sometimes best to replace them by suitable polynomial approximations.

We might have a use for coefficients with negative order, $n<0$. If $2\alpha=m, m$ a nonnegative integer, then we will take

$$b_{-n}^{(k)}=\begin{cases}0, & 1\le n\le m-1,\\ b_{n-m}^{(k)}, & n\ge m;\end{cases}$$

while if 2α is not an integer, we take

$$b_{-n}^{(k)}=0,\quad n\ge 1.$$

In general, boundary conditions on the solution $y(x)$ are given at $x=0$ or $x=\pm1$. For these points we have

$$C_n^{(\alpha)}(1)=(-1)^n C_n^{(\alpha)}(-1)=\Gamma(n+2\alpha)/\Gamma(2\alpha)n!$$

$$C_{2n+1}^{(\alpha)}(0)=0,\ C_{2n}^{(\alpha)}(0)=(-1)^n\Gamma(n+\alpha)/n!\Gamma(\alpha),\quad \alpha\ne 0.$$

If the series (4) and (5) are substituted into the differential equation (3) and the result is combined with (10) and (11), we obtain relations for the coefficients $b_n^{(k)}$ for $k=0,1,2,\cdots,m$, for all n. These relations and those obtained from the boundary conditions are equivalent to an infinite system of linear equations in the unknowns $b_n^{(k)}$. The numerical solution of these equations can be performed by any of the algorithms described in Wimp [16], or the two well-known methods described in detail by Chenshaw[10]. These are the method of recurrence and the iterative method.

The starting point of the method of recurrence is to assume that $b_n^{(k)}=0$ for $k=0,1,2,\cdots,m$ and $n>N$, where N is some arbitrary positive integer not known a priori, and to assign arbitrary values to $b_N^{(k)}$. The values of $b_n^{(k)}$ for $n=N-1,N-2,\cdots,0$ may then be obtained from the recurrence relations. Finally, a multiplying factor is determined so that the initial or boundary conditions are fulfilled. It may happen that the recurrence method yields b_0 without making use of any equations near $n=0$. In order that the remaining equations be satisfied, the recurrence process is repeated as often as required with different values of $b_N^{(k)}$. An appropriate linear combination of the trial solutions gives the required solution. The precision of results obtained by this method may be increased by taking a larger value of N. If, however, the selected N is larger than that required to achieve the desired precision. little effort need be wasted.

The iterative method starts with some initial guess for the b_n which satisfies the boundary conditions. From these values (10) can be used to compute $b_n^{(k)}$, $1\le k\le m$. These values can be used to compute a new b_n from the recurrence relations, again satisfying the boundary conditions. This procedure is continued until the desired precision is reached. Such schemes often do not converge, or converge slowly. Since the recurrence method is often quite rapidly convergent, the iterative method is perhaps most useful in correcting small errors due to rounding which arise in the application of the recurrence method.

§3. Extension of the Method in the Complex Domain

To extend the method into the complex domain we consider instead the function $y(zx)=R^{(\alpha)}(z,x)$, where x is the independent variable, $-1\le x\le 1$, and z is regarded as a parameter which may take any real or complex values. The function $R^{(\alpha)}(z,x)$ satisfies the differential equation

$$\sum_{i=0}^{m} p_i(zx)\frac{d^i R^{(\alpha)}(z,x)}{z^i dx^i} = q(zx). \tag{12}$$

Assuming that

$$R^{(\alpha)}(z,x) = \frac{1}{\gamma(z)}\sum_{n=0}^{\infty} a_n(z)C_n^{(\alpha)}(x) \tag{13}$$

and using Clenshaw's method described in the previous section, the coefficients $a_n(z)$ can be found so that the preceding differential equation (12) is satisfied. If, for example, the function $y(x)$ satisfies the initial condition $y(0) = 1$, then we get

$$\gamma(z) = \sum_{n=0}^{\infty} a_n(z)C_n^{(\alpha)}(0).$$

With this $\gamma(z)$ and with $x = 1$, we find that

$$R^{(\alpha)}(z,1) = y(z) = \sum_{n=0}^{\infty} a_n(z)C_n^{(\alpha)}(1)\Big/\sum_{n=0}^{\infty} a_n(z)C_n^{(\alpha)}(0). \tag{14}$$

Actually we do not work with infinite sums, but we start with a finite sum, say $R_N^{(\alpha)}(z,x)$, for $y(zx)$ in the form

$$y(zx) \simeq R_N^{(\alpha)}(z,x) = \frac{1}{\gamma(z)}\sum_{n=0}^{N} a_n(z)C_n^{(\alpha)}(x) \tag{15}$$

where the multiplying factor $1/\gamma(z)$ results from the satisfaction of an initial condition. Finally, putting $x = 1$ in (15) yields a finite rational function $R_N^{(\alpha)}(z,1) = Y_N(z)$, say, which approximates $y(z)$ for any real or complex values of z, $|z| \le 1$.

It is worth mentioning that, if we put $z = 1$ in (13) and (15), then we get the series expansion and the usual polynomial approximation for the function $y(x)$ respectively.

Since there is no easy way to estimate analytically the error in a rational approximation derived by the method explained above, probably the best way to investigate the accuracy of an approximation is to tabulate and inspect the absolute error function of the approximation for selected arguments. It is appropriate at this point to define the quantity

$$E_N^{(\alpha)} = \sup_{|z|\le 1} |R_N^{(\alpha)}(z,1) - y(z)|$$

to denote the maximum absolute error associated with the rational approximation to $y(z)$ obtained from $C_n^{(\alpha)}$.

§4. Numerical Examples

Example 1. Consider the solution of equation

$$xD^2y + Dy + 16xy = 0; \quad y(0) = 1,\ y'(0) = 0 \tag{16}$$

in the range $0 \le x \le 1$. This corresponds to the solution of Bessel's equation for $J_0(4x)$.

Inspection of (16) and the boundary conditions shows that the solution is an even function of x. Let $x \to xz$, $R^{(\alpha)}(z,x) = Y(zx)$, and $-1 \le x \le 1$. Then equation (16) takes the form

$$x\frac{d^2R^{(\alpha)}(z,x)}{dx^2} + \frac{dR^{(\alpha)}(z,x)}{dx} + 16z^2xR^{(\alpha)}(z,x) = 0, \quad R^{(\alpha)}(z,0) = 1, \quad \frac{dR^{(\alpha)}(z,0)}{dx} = 0. \quad (17)$$

Also, let $R^{(\alpha)}(z,x)$ be given by (13). Comparing the coefficients of $C_n^{(\alpha)}(x)$ in the expansion of the terms of equation (17), and making use of (11), we can easily show that

$$nb_{n-1}^{(2)} + (n+2\alpha)b_{n+1}^{(2)} + 2(n+\alpha)b_n^{(1)} + 16z^2[nb_{n-1} + (n+2\alpha)b_{n+1}] = 0. \quad (18)$$

Repeated use of (10) enables one to put (18) in the form

$$b_{n-2} = \frac{1}{(n-1)(n+\alpha+1)}[(n+2\alpha+1)(n+\alpha-1)b_{n+2} - 2\alpha(n+\alpha)b_n]$$
$$-\frac{(n+\alpha-1)}{8z^2(n-1)}[(n+2\alpha)b_{n+1}^{(1)} + nb_{n-1}^{(1)}]. \quad (19)$$

This and the equation

$$b_{n-1}^{(1)} = b_{n+1}^{(1)} + 2(n+\alpha)b_n \quad (20)$$

can be used alternatively to get a rational approximate solution for any value of $\alpha, \alpha > -1/2$.

The complete solution for the case $\alpha = 0$ has been given in detail in Doha [7].

In the following we consider the two important cases $\alpha = \frac{1}{2}$ and $\alpha = 1$, which correspond to the expansion in Legendre polynomials $P_n(x)$ and Chebyshev polynomials of the second kind $U_n(x)$.

Taking $b_{10} = 10$ (i.e. $N = 10$), with $b_{12} = b_{14} = \cdots b_{11}^{(1)} = b_{13}^{(1)} = \cdots = 0$, and rounding the coefficients of the powers of z in the other coefficients as they are calculated, we obtain the trial solutions shown in Table 1. We find that when these trial solutions have been computed, both equations (19) and (20) for $\alpha = \frac{1}{2}$ and $\alpha = 1$ have been satisfied to a certain accuracy. Thus, only the condition $R_{10}^{(\alpha)}(z,0) = 1$ remains to be satisfied. This condition gives for $\alpha = 1/2$

$$\gamma(z) = \sum_{n=0}^{5} a_{2n}(z)C_{2n}^{(1/2)}(0)$$

where the coefficients a's are given in Table 1. With this $\gamma(z)$ and with $x = 1$, we get the sought-for rational approximation in the form

$$R_{10}^{(1/2)}(z,1) = \frac{1 - 3.61738z^2 + 2.55143z^4 - 0.56159z^6 + 0.04035z^8 - 0.00069z^{10}}{1 + 0.38262z^2 + 0.08192z^4 + 0.01338z^6 + 0.00196z^8 + 0.00031z^{10}} \quad (21)$$

and the corresponding polynomial approximation as

$$R_{10}^{(1/2)}(1,x) = 0.25620P_0(x) - 1.07057P_2(x) + 0.48785P_4(x) - 0.07638P_6(x)$$
$$+0.00599P_8(x) - 0.00027P_{10}(x). \quad (22)$$

For the case $\alpha = 1$, we get the rational and polynomial approximations as

$$R_{10}^{(1)}(z,1) = \frac{1 - 3.63652z^2 + 2.61931z^4 - 0.60562z^6 + 0.04754z^8 - 0.00105z^{10}}{1 + 0.36348z^2 + 0.07322z^4 + 0.01113z^6 + 0.00161z^8 + 0.00024z^{10}} \quad (23)$$

and

$$R_{10}^{(1)}(1,x) = 0.38274U_0(x) - 0.45707U_2(x) + 0.14110U_4(x) - 0.01784U_6(x) \\ +0.00120U_8(x) - 0.00005U_{10}(x). \tag{24}$$

For the sake of comparison, it is worth while to write down the rational and polynomial approximations obtained from Chebyshev polynomials' $T_n(x)(\alpha = 0)$; these are given respectively by (see Doha [7]):

$$R_{10}^{(0)}(z,1) = \frac{1 - 3.60000z^2 + 2.49000z^4 - 0.52222z^6 + 0.03354z^8 - 0.00042z^{10}}{1 + 0.40000z^2 + 0.9000z^4 + 0.01556z^6 + 0.00243z^8 + 0.00042z^{10}} \tag{25}$$

and

$$R_{10}^{(0)}(1,x) = 0.05014T_0(x) - 0.66526T_2(x) + 0.24898T_4(x) - 0.033240T_6(x) \\ +0.00230T_8(x) - 0.00009T_{10}(x). \tag{26}$$

Evaluations of $J_0(4x)$ based on rational approximation formulae (21), (23) and (25) are given in Table 2 compared with the exact values of this function. The exact values were evaluated by taking the sum of the first twenty-five terms of the power series expansion of that function. This table shows that all three of them are essentially of the same accuracy, but $R_{10}^{(1/2)}(z,1)$ is better near the ends of the interval $[-1,1]$, while $R_{10}^{(0)}(z,1)$ is the best near the middle of that interval.

Example 2. It is well-known that rational approximations to $e^{-x}, x \in [-1,1]$, arise quite naturally in the numerical solution of heat conduction problems and in the study of numerical methods for ordinary differential equations; see Cody, Meinardus, and Varga [17]. This function satisfies the differential equations

$$(D+1)y = 0, \quad y(0) = 1. \tag{27}$$

If $R^{(\alpha)}(z,x) = y(zx), x \in [-1,1]$, then (27) takes the form

$$\frac{dR^{(\alpha)}(z,x)}{dx} + zR^{(\alpha)}(z,x) = 0, \quad R^{(\alpha)}(z,0) = 1.$$

Let $R^{(\alpha)}(z,x)$ be given by (13). Comparing the coefficients of $C_n^{(\alpha)}(x)$ in the expansion of the terms of the preceding differential equation, we find

$$a_n^{(1)} = -za_n.$$

This equation and equation (20) can be used to compute b_n and hence a_n. Since the equations are homogeneous, the values of a_n have to be substituted in equation (15) to satisfy the initial condition $R_N^{(\alpha)}(z,0) = 1$, which in turn gives

$$\gamma(z) = \sum_{n=0}^{N} a_n(z) C_n^{(\alpha)}(0).$$

In Table 3, we give the rational approximations for e^{-x} in $[-1,1]$, for $\alpha = 0, \frac{1}{2}$ and 1; $N = 2(2)10$. Table 4 also contains the explicit values of $E_N^{(\alpha)}$ corresponding to each approximation. In Table 5, we give the corresponding polynomial approximations to $e^{-x}, x \in [-1.1]$, for $\alpha = 0, 1/2$ and 1; $n = 2(2)10$.

§5. Numerical Results and Comparisons

From the results of Examples 1 and 2, it is not difficult to show that the polynomial approximation obtained for $\alpha = 0$ is better than those for $\alpha = 1/2$ and 1; meanwhile the approximation for $\alpha = 1/2$ is also better than that of $\alpha = 1$. This certifies that the terms of the Chebyshev expansion of the first kind are asymptotically smaller in maximum absolute value than the corresponding terms of any other ultraspherical expansion.

Results of Table 4 show that $E_N^{(0)} < E_N^{(1/2)} < E_N^{(1)}, 2 \leq N \leq 10$. This also certifies that the rational approximation obtained from Chebyshev polynomials of first kind ($\alpha = 0$) is the best of any other rational approximations obtained from ultraspherical polynomials corresponding to $\alpha > 0$.

Some comparison with the well-known technique of Padé would be appropriate. Now the (m, n)-th degree Padé approximant to e^{-z} is given by Braess[18]:

$$e^z \simeq R_{mn}(z) = P_m(z)/Q_n(z)$$

where

$$P_m(z) = \int_0^\infty (t+z)^m t^n e^{-t} dt,$$

$$Q_n(z) = \int_0^\infty (t-z)^n t^m e^{-t} dt.$$

After performing these integrals we get

$$e^z \simeq R_{mn}(z) = \sum_{k=0}^{m} \binom{m}{k} (n+k)! z^{m-k} \Big/ \sum_{k=0}^{n} \binom{n}{k} (m+k)! (-z)^{n-k}. \tag{28}$$

Now the rational approximations given in Table 3 compare favorably with the Padé approximants obtained from (28) expressible in a similar form; for example the rational functions $R_{10}^{(1)}$ and $R_{10,10}(z)$ were compared, and it is found that $R_{10}^{(1)}$ is better than $R_{10,10}(z)$. Evaluations of $e^{-z}, z = 0(0.1)1$, based on rational approximation functions $R_{10}^{(0)}(z), R_{10}^{(1/2)}(z), R_{10}^{(1)}(z)$ and $R_{10,10}(z)$, are given in Table 6 compared with the exact values of this function.

§6. Conclusion

In this paper we have described a method which enables us to find simultaneously the rational and—as a special case—polynomial approximations for an arbitrary function $f(x)$ expanded in an infinite series of ultraspherical polynomials $C_n^{(\alpha)}(x)$. The coefficients of expansion may be obtained to any degree of accuracy. The function $f(x)$ is assumed to satisfy some linear differential equation with associated initial or boundary conditions. The differential equation can then be solved directly to give the unknown coefficients of expansion.

The rational approximation to be obtained by this method can be considered as an extension of Elliott's method[15] and Doha's method[7] into the complex domain. It is of fundamental importance to note that the polynomial approximation for $f(x)$ is obtained directly from its rational one, $R^{(\alpha)}(z, x)$, for any $\alpha > -\frac{1}{2}$ simply by putting the parameter z equal to unity.

Table 1*. Computation of trial solution for Example 1 in terms of a_n using Legendre and Chebyshev polynomials respectively.

n	Trial $z^{10-n}a_n(z)$	1	z^2	z^4	z^6	z^8	z^{10}	$a_n(1)$
0	$z^{10}a_0$	−266313	+253187	−99003.5	+11642.5	−496	−8	+0.256198
		−1586340	+1009740	−332721	+30524	−1320	−54	+0.382736
2	z^8a_2	+710167.5	−336990	+50717.5	−1925	+37.5		−1.070566
		+1586340	−613155	+80286	−2427	+72		−0.457070
4	z^6a_4	−243486	+54405	−3213	−13.5			+0.487854
		−396585	+76175	−4035	−35			+0.141098
6	z^4a_6	+32792.5	−2756	+71.5				−0.076379
		+44065	−5122	+91				−0.017843
8	z^2a_8	−2354.5	−8.5					+0.005994
		−2754	−18					+0.001205
10	a_{10}	+105						−0.000266
		+110						−0.000048

Note. $z^4a_6 = 32792.5 - 2756z^2 + 71.5z^4$.

Table 2. Evaluation of $J_0(4x)$ based on rational approximations $R_{10}^{(0)}(z,1)$, $R_{10}^{(1/2)}(z,1)$, $R_{10}^{(1)}(z,1)$ compared with the exact values.

x	$R_{10}^{(1)}(z,1)$	$R_{10}^{(1/2)}(z,1)$	$R_{10}^{(1)}(z,1)$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.960398227	0.960398256	0.960398227	0.960398227
0.2	0.846287353	0.846287463	0.846287351	0.846287353
0.3	0.671132744	0.671132967	0.671132702	0.671132744
0.4	0.455402169	0.455402511	0.455401767	0.455402168
0.5	0.223890793	0.223891236	0.223888549	0.223890779
0.6	0.002507756	0.002508254	0.002498785	0.002507683
0.7	−0.185035873	−0.185035372	−0.185064389	−0.185036033
0.8	−0.320188549	−0.320187900	−0.320265318	−0.320188170
0.9	−0.391773852	−0.391771880	−0.391956217	−0.391768984
1.0	−0.397173242	−0.397165636	−0.397566534	−0.397149810

Table 3. The rational approximations for e^{-x} in $[-1,1]$, for $\alpha = 0, \frac{1}{2}$, and 1; $N = 2(2)10$,

$$e^{-x} = \sum_{i=0}^{N} p_i x^i \Big/ \sum_{i=0}^{N} q_i x^i, \quad -1 \le x \le 1.$$

	$\alpha = 0$		$\alpha = 1/2$		$\alpha = 1$	
i	p_i	q_i	p_i	q_i	p_i	q_i
			$N = 2$			
0	8	8	10	10	12	12
1	-8	0.0	-10	0.0	-12	0.0
2	3	-1	4	-1	5	-1

* In each row, the first line gives the coefficients corresponding to using Legendre polynomials, while the second line gives those corresponding to using Chebyshev polynomials of the second kind.

$N = 4$

0	192	192	126	126	1920	1920
1	−192	0.0	−126	0.0	−1920	0.0
2	84	−12	56	−7	864	−96
3	−20	0.0	−14	0.0	−224	0.0
4	2.5	0.5	2	0.25	35	3

$N = 6$

0	23040	23040	67567.5	67567.5	80640	80640
1	−23040	0.0	−67567.5	0.0	−80640	0.0
2	10560	−960	31185	−2598.75	37440	−2880
3	−2880	0.0	−8662.5	0.0	−10560	0.0
4	504	24	1575	59.0625	1980	60
5	−56	0.0	−189	0.0	−252	0.0
6	3.5	−0.5	14	−1.09375	21	−1

$N = 8$

0	5160960	5160960	17229712.5	17229712.5	92897280	92897280
1	−5160960	0.0	−17229712.5	0.0	−92897280	0.0
2	2419200	−161280	8108100	−506756.25	43868160	−2580480
3	−698880	0.0	−2364862.5	0.0	−12902400	0.0
4	137280	2880	472972.5	8445.9375	2620800	40320
5	−19008	0.0	−67567.5	0.0	−384384	0.0
6	1848	−40	6930	−108.28125	41184	−480
7	−120	0.0	−495	0.0	−3168	0.0
8	4.5	0.5	22.5	1.230473	165	5

$N = 10$

0	1857945600	1857945600	6874655288	6874655288	40874803200	40874803200
1	−1857945600	0.0	−6874655288	0.0	−40874803200	0.0
2	882524160	−46448640	3273645376	−163682269	19508428800	−928972800
3	−263208960	0.0	−982093612.5	0.0	−5883494400	0.0
4	54835200	645120	206756550	2153714.063	1250242560	11612160
5	−8386560	0.0	−32162130	0.0	−197406720	0.0
6	960960	−6720	3783780	−21114.84375	23761920	−107520
7	−82368	0.0	−337837.5	0.0	−2196480	0.0
8	5148	60	22522.5	173.37474	154440	840
9	−220	0.0	−1072.5	0.0	−8008	0.0
10	5.5	−0.5	33	−1.353514	286	−6

Table 4. Values of $E_N^{(\alpha)}$ for $\alpha = 0, \frac{1}{2}$, and 1; $N = 2(2)10$.

N	$E_N^{(0)}$	$E_N^{(1/2)}$	$E_N^{(1)}$
2	6.1×10^{-2}	7.0×10^{-2}	8.7×10^{-2}
4	8.3×10^{-4}	1.3×10^{-3}	1.6×10^{-3}
6	3.5×10^{-6}	7.8×10^{-6}	1.3×10^{-5}
8	1.4×10^{-8}	3.0×10^{-8}	5.7×10^{-8}
10	2.6×10^{-11}	3.0×10^{-10}	4.0×10^{-10}

Table 5. The polynomial approximations for e^{-x} in $[-1,1]$, for $\alpha = 0, \frac{1}{2}$, and $1; N = 2(2)10$,

$$e^{-x} = \sum_{i=0}^{N} a_i C_i^{(\alpha)}(x).$$

	$\alpha = 0$	$\alpha = \frac{1}{2}$	$\alpha = 1$
i	a_i	a_i	a_i
		$N = 2$	
0	1.285714286	1.185185185	1.136363636
1	-0.444444444	-1.111111111	-0.545454545
2	0.111111111	0.370370370	0.136363636
		$N = 4$	
0	1.265927978	1.175122292	1.130268199
1	-1.130193906	-1.103563941	-0.542966612
2	0.271468144	0.357791754	0.133004926
3	-0.044321329	-0.070440252	-0.021893815
4	0.005540166	0.010062893	0.002736727
		$N = 6$	
0	1.266066460	1.175122292	1.130318431
1	-1.130318728	-1.103638649	-0.542990786
2	0.271495464	0.357814456	0.133010576
3	-0.044336870	-0.070455654	-0.021896966
4	0.005474246	0.009965134	0.002714633
5	-0.000542900	-0.001099548	-0.000269857
6	0.000045242	0.000099959	0.000022488

	$\alpha = 0$	$\alpha = \frac{1}{2}$	$\alpha = 1$
i	a_i	a_i	a_i
		$N = 8$	
0	1.266065876	1.175201193	1.130318207
1	-1.130318207	-1.103638323	-0.542990678
2	0.271495339	0.357814350	0.133010550
3	-0.044336849	-0.070455634	-0.021896962
4	0.005474240	0.009965128	0.002714632
5	-0.000542926	-0.001099586	-0.000269864
6	0.000044977	0.000099454	0.000022389
7	-0.000003198	-0.000007620	-0.000001594
8	0.000000199	0.000000508	0.000000010
		$N = 10$	
0	1.266065878	1.175201194	1.303182080
1	-1.130318208	-1.103638324	-0.542990679
2	0.271495340	0.357814351	0.133010550
3	-0.044336850	-0.070455634	-0.021896962
4	0.005474240	0.009965128	0.002714632
5	-0.000542926	-0.001099586	-0.000269864
6	0.000044977	0.000099454	0.000022389
7	-0.000003198	-0.000007620	-0.000001594
8	0.000000199	0.000000506	0.000000099
9	-0.000000001	-0.000000030	-0.000000006
10	0.000000000	0.000000002	0.000000000

Table 6. Evaluation of e^{-x} based on rational approximations $R_{10}^{(\alpha)}(x,1)(\alpha = 0, \frac{1}{2}, 1)$, Padé approximant $R_{10,10}(x)$ Compared with exact values.

x	$R^{(0)}(x,1)$	$R^{(1/2)}(x,1)$	$R^{(1)}(x,1)$	$R_{10,10}(x)$	Exact
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9048374180	0.9048374180	0.9048374182	0.9048374086	0.9048374180
0.2	0.8187307532	0.8187307531	0.8187307531	0.8187307480	0.8187307531
0.3	0.7408182207	0.7408182207	0.7408182206	0.7408182182	0.7408182207
0.4	0.6703200460	0.6703200461	0.6703200461	0.6703200499	0.6703200460
0.5	0.6065306597	0.6065306597	0.6065306598	0.6065306526	0.6065306597
0.6	0.5488116361	0.5488116361	0.5488116360	0.5488116337	0.5488116361
0.7	0.4965853038	0.4965853039	0.4965853038	0.4965852976	0.4965853038
0.8	0.4493289641	0.4493289643	0.4493289641	0.4493289624	0.4493289641
0.9	0.4065696597	0.4065696600	0.4065696598	0.4065696559	0.4065696597
1.0	0.3678794412	0.3678794415	0.3678794414	0.3678794373	0.3678794412

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