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A DIRECT METHOD FOR PITCHFORK BIFURCATION POINTS*

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Abstract

To overcome the difficulty caused by the singularity at the pitchfork bifurcation points, we introduce the homotopy parameter so that the problem of computing the pitchfork bifurcation points can be transferred to that of computing the fold points of degree 3 with respect to the homotopy parameter. An extended system for pitchfork bifurcation points is given. The regularity of the extended system is proved. Finally, the numerical examples show the effectiveness of our method.

§1. Introduction

The pitchfork bifurcation is one of the basic bifurcation phenomena in nonlinear problems. We consider the following nonlinear problem with one parameter

$$f(\lambda, x) = 0, \tag{1.1}$$

where λ is a real parameter, $x \in X$, a Banach space, f is a C^3 Fredholm operator with index 0 from $R \times X$ to X. To determine the bifurcation points from the trivial solution we only need computing the eigenvalues of the corresponding linearized operator in the nonlinear problems. It brings us difficulties to compute the pitchfork bifurcation points from the non-trivial solutions, which are unknown in advance. What Concerns us in this paper is to compute such kind of pitchfork bifurcation points, which are called the secondary pitchfork bifurcation points, in the nonlinear problem (1.1).

Several extended systems for computing the secondary bifurcation points have been proposed in [1]–[3]. We introduce a homotopy parameter so that the problem of computing the pitchfork bifurcation points can be transferred to the problem of computing the fold points of degree 3 with respect to the homotopy parameter, which can be solved by using our algorithm in [4]. In Section 3 an extended system for pitchfork bifurcation points is given and its regularity is proved. Finally, in Section 4 we give numerical examples to show the effectiveness of our method.

§2. Pitchfork Bifurcation Point

Let $f_\lambda^0, f_x^0, f_{\lambda\lambda}^0, f_{\lambda x}^0, f_{xx}^0, f_{xxx}^0, \cdots$ denote the partial Fréchet-derivatives of f at $a_0 = (\lambda_0, x_0)$. Denote the dual pairing of $x \in X$ and $\psi \in X^*$ by $\langle\psi, x\rangle$, where X^* is the conjugate space of X.

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Definition 2.1. *A point* $a_0 = (\lambda_0, x_0)$ *is bifurcation point of* (1.1) *with respect to* λ *if*

$$f(a_0) = 0, \tag{2.1}$$

$$\text{Dim Ker } f_x(a_0) \geq 1, \tag{2.2}$$

$$f_\lambda^0 \in \text{ Range } f_x^0, \tag{2.3}$$

where Ker f_x^0 *is the unll space of Jacobian* f_x^0.

We always assume that

$$\text{Dim Ker } f_x^0 = \text{Codim Range} f_x^0 = 1. \tag{2.4}$$

In this case there exist nontrivial $\phi_0 \in X$ and $\psi_0 \in X^*$ such that

$$N = \text{Ker} f_x^0 = \{r\phi_0 | r \in R\}, \tag{2.5}$$

$$M = \text{Range} f_x^0 = \{x \in X | < \psi_0, x >= 0\}. \tag{2.6}$$

We can decompose $X = N \otimes V_0$ where V_0 is a complement of N in X. (2.3) in Definition 2.1 implies $< \psi_0, f_\lambda^0 >= 0$ and there is a unique $v_0 \in V_0$ such that

$$f_x^0 v_0 + f_\lambda^0 = 0. \tag{2.7}$$

Definition 2.2. *A bifurcation point* a_0 *of* (1.1) *with respect to* λ *is a pitchfork bifurcation point of* (1.1) *with respect to* λ *if*

$$< \psi_0, f_{xx}^0 \phi_0 \phi_0 >= 0, \tag{2.8}$$

$$< \psi_0, f_{\lambda x}^0 \phi_0 + f_{xx}^0 \phi_0 v_0 > \neq 0, \tag{2.9}$$

$$< \psi_0, 3 f_{xx}^0 \phi_0 u_0 + f_{xxx}^0 \phi_0 \phi_0 \phi_0 > \neq 0, \tag{2.10}$$

where ϕ_0, v_0 *are given in* (2.5), (2.7), *and* $u_0 \in V_0$ *is uniquely determined by*

$$f_x^0 u_0 + f_{xx}^0 \phi_0 \phi_0 = 0. \tag{2.11}$$

§3. Regular extended system

We introduce a homotopy path of (1.1):

$$F(\lambda, \mu, x) = f(\lambda, x) - \mu f(\lambda^*, x^*) = 0, \tag{3.1}$$

where μ is a homotopy parameter. In probability 1 we can choose such λ^*, x^*, that $f(\lambda^*, x^*)$ $\bar{\in}$ Range f_x^0 i.e. $< \psi_0, f(\lambda^*, x^*) > \neq 0$ (see [5]).

We propose an extended system for the pitchfork bifurcation points of (1.1) with respect to λ as follows:

$$\begin{cases} f(\lambda, x) - \mu f(\lambda^*, x^*) = 0, & f_x \phi = 0, \\ l\phi - 1 = 0, \quad f_{xx}\phi\phi + f_x u = 0, & lu = 0, \end{cases} \tag{3.2}$$

where $l \in X^*$. We take such l that $lx = 0$ means $x \in V_0$ and $l\phi_0 = 1$. Usually, lw can be taken as the r-th component of w for the actual computation. Next, we are going to discuss the regularity of (3.2) at the pitchfork bifurcation points. The Jacobian of (3.2) at $(x_0, \phi_0, u_0, \lambda_0, 0)$ is

$$J = \begin{pmatrix} f_x^0 & 0 & 0 & f_\lambda^0 & f(\lambda^*, x^*) \\ f_{xx}^0\phi_0 & f_x^0 & 0 & f_{\lambda x}^0\phi_0 & 0 \\ [g]_x & 2f_{xx}^0\phi_0 & f_x^0 & [g]_\lambda & 0 \\ 0 & l & 0 & 0 & 0 \\ 0 & 0 & l & 0 & 0 \end{pmatrix} \tag{3.3}$$

where $g = g(\phi_0, u_0; f) = f_x^0 u_0 + f_{xx}^0\phi_0\phi_0$. We suppress the superscript $\circ$ for simplicity of the notations. Consider

$$JW = T, \tag{3.4}$$

where $W = (w_0, w_1, w_2, c_0, c_1)^T, T = (t_0, t_1, t_2, b_0, b_1)^T$, $w_i, t_i \in X, c_j, b_j \in R$. Expanding (3.4) we have

$$f_x w_0 + c_0 f_\lambda + c_1 f(\lambda^*, x^*) = t_0, \tag{3.5}$$
$$f_{xx}\phi_0 w_0 + f_x w_1 + c_0 f_{\lambda x}\phi_0 = t_1, \tag{3.6}$$
$$l w_1 = b_0, \tag{3.7}$$
$$[g]_x w_0 + 2f_{xx}\phi_0 w_1 + f_x w_2 + c_0[g]_\lambda = t_2, \tag{3.8}$$
$$l w_2 = b_1. \tag{3.9}$$

From (3.5) we obtain

$$w_0 = c\phi_0 + c_0 v_0 + c_1 z_1^* + s_0, \tag{3.10}$$

where $v_0, z_1^*, s_0 \in V_0$ are uniquely determined by

$$f_x v_0 = -Q[f_\lambda] = -f_\lambda, \tag{3.11}$$
$$f_x z_1^* = -Q[f(\lambda^*, x^*)], \tag{3.12}$$
$$f_x s_0 = Q[t_0], \tag{3.13}$$

where Q is the project operator on M. Applying ψ_0 on (3.5) leads to

$$c_1 < \psi_0, f(\lambda^*, x^*) >=< \psi_0, t_0 > . \tag{3.14}$$

Because $< \psi_0, f(\lambda^*, x^*) > \neq 0$, we can get $c_1 = c_1^* =< \psi_0, t_0 > / < \psi_0, f(\lambda^*, x^*) >$. Substituting (3.10) into (3.6) we have

$$f_{xx}\phi_0(c\phi_0 + c_0 v_0 + c_1^* z_1^* + s_0) + f_x w_1 + c_0 f_{\lambda x}\phi_0 = t_1. \tag{3.15}$$

Projecting (3.15) on M and noticing (3.7) yield

$$w_1 = b_0\phi_0 + c u_0 + c_0\bar{z}_2 + c_1^* z_2^* + s_1, \tag{3.16}$$

where $u_0, \bar{z}_2, z_2^*, s_1 \in V_0$ are uniquely given by (2.11) and

$$f_x\bar{z}_2 = -Q[f_{xx}\phi_0 v_0 + f_{\lambda x}\phi_0], \tag{3.17}$$
$$f_x z_2^* = -Q[f_{xx}\phi_0 z_1^*], \tag{3.18}$$
$$f_x s_1 = Q[t_1 - f_{xx}\phi_0 s_0], \tag{3.19}$$

and c_0 should satisfy

$$c_0 < \psi_0, \bar{F}_1 > + c_1^* < \psi_0, F_1^* > = < \psi_0, t_1 - f_{xx}\phi_0 s_0 >, \tag{3.20}$$

where $\bar{F}_1 = f_{xx}\phi_0 v_0 + f_{\lambda x}\phi_0, F_1^* = f_{xx}\phi_0 z_1^*$. From (3.20) we get $c_0 = c_0^* = (< \psi_o, t_1 - f_{xx}\phi_0 s_0 > - c_1^* < \psi_0, F_1^* >)/ < \psi_0, \bar{F}_1 >$ due to (2.9). Similarly, we can obtain

$$w_2 = b_1\phi_0 + cz_0 + c_0^*\bar{z}_3 + c_1^* z_3^* + s_2, \tag{3.21}$$

where $z_0, \bar{z}_3, z_3^*, s_2 \in V_0$ are uniquely determined by

$$f_x z_0 + Q[3 f_{xx}\phi_0 u_0 + f_{xxx}\phi_0\phi_0\phi_0] = 0, \tag{3.22}$$

$$f_x \bar{z}_3 = -Q[\bar{F}_2], \tag{3.23}$$

$$f_x z_3^* = -Q[F_2^*], \tag{3.24}$$

$$f_x s_2 = Q[t_2 - [g]_x s_0 - 2 f_{xx}\phi_0(b_0\phi_0 + s_1)], \tag{3.25}$$

$\bar{F}_2 = f_{xxx}\phi_0\phi_0 v_0 + f_{xx}(u_0 v_0 + 2\phi_0\bar{z}_2) + f_{\lambda xx}\phi_0\phi_0 + f_{\lambda x}u_0, F_2^* = f_{xxx}\phi_0\phi_0 z_1^* + f_{xx}(u_0 z_1^* + 2\phi_0 z_2^*)$. We can also get

$$c = c^* = \frac{< \psi_0, t_2 - [g]_x s_0 - 2 f_{xx}\phi_0(b_0\phi_0 + s_1) - c_0^*\bar{F}_2 - c_1^* F_2^* >}{< \psi_0, 3 f_{xx}\phi_0 u_0 + f_{xxx}\phi_0\phi_0\phi_0 >}.$$

So far solution $W = (w_0, w_1, w_2, c_0, c_1)^T$ of (3.4) is completely determined for any T. Obviously, if $T = 0$, the solution of (3.4) $W = 0$ from the above procedure. According to the open mapping theorem we have finished the proof of the following theorem:

Theorem 3.1. *The extended system* (3.2) *of* (1.1) *is regular at the pitchfork bifurcation points of* (1.1) *with respect to* λ *as* $\mu = 0$.

Remark. To locate the pitchfork bifurcation points, we can solve the extended system (3.2) by using Newton's method because of the regularity of (3.2). In fact, $(x_0, \phi_0, u_0, \lambda_0, 0)$ is the fold point of degree 3 with respect to the homotopy parameter μ (see [4] for the definition of the fold point and its degree). So we can use our algorithm in [4] to solve (3.2) effectively.

§4. Numerical Examples

Example 1. We consider a model of a two-cell exothermic reaction (see [6]):

$$y_1 + \varepsilon(y_1 - y_2) - \lambda \exp[y_1/(1 + \gamma y_1)] = 0, \tag{4.1a}$$

$$y_2 + \varepsilon(y_2 - y_1) - \lambda \exp[y_2/(1 + \gamma y_2)] = 0, \tag{4.1b}$$

where y_i is the temperature of cell i $(i = 1, 2)$, ε is the coupling coefficient between two cells, and $\lambda \exp[y_i/(1 + \gamma y_i)]$ is the Arrhenius reaction rate term. Fix $\gamma = 0.23$ and ε, and (4.1) is a nonlinear system with one parameter λ. The following is the results of the computation. There exist two families of the pitchfork bifurcation points of (4.1) with respect to λ as ε varies.

ε	y_i $(i = 1, 2)$	λ	ε	y_i $(i = 1, 2)$	λ
0.01	2.61876	0.51086	0.01	7.21852	0.47864
0.02	2.85148	0.50955	0.02	6.62940	0.47987
0.03	3.16475	0.50687	0.03	5.97318	0.48240
0.04	3.70370	0.50124	0.04	5.10397	0.48782

Example 2. The diffusion term is added in Example 1.

$$\begin{cases} -d^2y_1/dx^2 + \varepsilon(y_1 - y_2) - \lambda \exp[y_1/(1+\gamma y_1)] = 0, \\ -d^2y_2/dx^2 + \varepsilon(y_2 - y_1) - \lambda \exp[y_2/(1+\gamma y_2)] = 0, \end{cases} \quad 0 < x < 1, \tag{4.2}$$

$$y(0) = y(1) = y(0) = y(1).$$

Fix $\gamma = 0.2$ and ε, and (4.2) is a nonlinear problem with one parameter λ. We use the central difference on the mesh points $x = jh$ ($j = 1, \cdots, n-1$), where $nh = 1$, to discretize (4.2). The following is the results, which show that there exist two families of the pitchfork bifurcation points with respect to λ on the non-trivial solutions of (4.2) as ε varies in different discretization sizes.

	$n = 10$		$n = 20$	
ε	λ	$y_i(\frac{1}{2})(i = 1, 2)$	λ	$y_i(\frac{1}{2})(i = 1, 2)$
0.01	3.48135	15.35818	3.50292	15.34223
0.1	3.48312	14.68741	3.50468	14.67613
0.5	3.53070	11.83959	3.55193	11.84642
1.0	3.75134	8.09363	3.77017	8.12716
1.1	3.87388	6.97313	3.88912	7.03054

	$n = 10$		$n = 20$	
ε	λ	$y_i(\frac{1}{2})(i = 1, 2)$	λ	$y_i(\frac{1}{2})(i = 1, 2)$
0.01	4.61907	2.35430	4.64071	2.36242
0.1	4.61687	2.45699	4.63853	2.46480
0.5	4.55832	3.02216	4.58043	3.02765
1.0	4.30065	4.37623	4.32555	4.36841
1.1	4.16739	5.06942	4.19599	5.03980

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