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ON THE RELATION BETWEEN AN INVERSE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS AND AN INITIALBOUNDARY VALUE PROBLEM FOR A HYPERBOLIC SYSTEM *1)

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Abstract

This paper deals with a coefficient inverse problem of a system of ODEs whose coefficient matrix is the so-called generalized negative definite matrix. To solve the problem, an initial-boundary value problem of a hyperbolic system of PDEs is constructed. The existence and uniqueness of its solution and its asymptotic convergence with respect to one of the variables to the original inverse problem are proved. As a result, the solution of the inverse problem is reduced to the solution of the direct problem. A few numerical examples were solved to show the effectiveness of the method.

§1. Introduction

In this paper, the following coefficient inverse problem of a system of ODEs is discussed:

$$\frac{dx(t)}{dt} = Ax(t) + F(t) \tag{1.1}$$

where $x(t)$ and $F(t)$ are given, and A is a generalized negative definite $n \times n$ unknown matrix. Here by generalized negative definite matrix we mean the real parts of eigenvalues of which are all negative and there exists a constant $C > 0$ such that the inequality

$$(Ax, x) \leq -C(x, x) \tag{1.2}$$

is valid for all $x \in R^n$.

Such kind of problems is often met in practice. For example, coefficient inverse problems of parabolic differential equations can be reduced to it, if the spatial variables are discretized or integral transformations with respect to the spatial variables are applied. In this case the matrix A is a strict negative definite matrix. Another important example is the identification problem of compartmental models, which have been widely used in a variety of areas, such as medicine, biology, economics, and so on. In this problem the matrix A is the compartmental matrix with the properties: 1) Its off-diagonal entries are nonnegative; 2) Its diagonal entries are negative; 3) It is diagonally dominant with respect to the columns.

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It is well known that the real parts of eigenvalues of the matrices which possess the Properties 2) and 3) are negative. Under some additional conditions the compartmental matrices satisfy inequality (1.2) so that they are generalized negative definite.

In 1983, H.W. Alt et al.[1] developed an elegant method for solving coefficient inverse problems of differential equations. They considered the elliptic equation

$$-\nabla\cdot(a\nabla u)=f, \tag{1.3}$$

where the solution $u(x,y)$ is given, and the problem is to find a positive definite matrix $a(x,y)$ to fit (1.3). They have proved that the solution of the inverse problem mentioned above is the limit of the solution of a certain direct problem of a system of partial differential equations when one of its parameters tends to infinity (in the rest part of this paper we will call this direct problem asymptotically convergent to the original inverse problem). In this way the solution of the inverse problem is reduced to the solution of the direct problem, for which there are well known methods, thus making the things much easier.

In this paper, a similar approach is used for solving (1.1). Differently from [1], a system of evolutionary equations is considered here. Instead of discretizing the time variable in advance (as the authors of [1] suggested), we consider an auxiliary direct problem of a hyperbolic system of differential equations which asymptotically converges to the inverse problem (1.1) itself, rather than to its time discrete form. Moreover, in [1], all results are obtained only for the finite dimensional approximation of the original problem. The striking difference of this paper is that we have proved the asymptotic convergence for the infinite dimensional case.

§2. An Auxiliary Problem

We write the problem which was proposed at the beginning of this paper as follows.

Problem (P): Suppose that $F^*(t)$ and $x^*(t)$ are given. Find a generalized negative definite matrix A^* such that

$$\frac{dx^*(t)}{dt}=A^*x^*(t)+F^*(t). \tag{2.1}$$

Here the existence of the solution A^* is assumed.

To solve this inverse problem, we introduce an auxiliary direct problem

$$\left.\begin{aligned}&\frac{\partial x(t,\tau)}{\partial\tau}+\frac{\partial x(t,\tau)}{\partial t}-A(\tau)x(t,\tau)-F^*(t)=0 &(2.2)\\ &\frac{dA(\tau)}{d\tau}+\frac{1}{T}\int_0^T[x(t,\tau)-x^*(t)]\otimes x(t,\tau)dt=0 &(2.3)\end{aligned}\right\},\quad (t,\tau)\in(0,T)\times(0,\infty),$$

$$x(t,0)=x_0(t),\quad t\in[0,T], \tag{2.4}$$

$$x(0,\tau)=x^*(0),\tau\in[0,\infty), \tag{2.5}$$

$$A(0)=A_0, \tag{2.6}$$

where the operation $\otimes$ is defined for two R^n-vectors V_1,V_2 by

$$V_1\otimes V_2:=(v_{1i}v_{2j}),\quad i,j=1,2,\cdots,n\in R^{n\times n},$$

$x_0(t)$ is an arbitrary vector function, and A_0 arbitrary matrix.

It will be proved that the matrix function $A(\tau; x_0, A_0)$ in the solution $(x(t,\tau;x_0,A_0), A(\tau;x_0,A_0))$ of (2.2)–(2.6) approaches the solution A^* of the problem (P) as $\tau \to \infty$. (In the rest of this paper, we shall omit x_0 and A_0 in $x(t,\tau;x_0,A_0)$ and $A(\tau;x_0,A_0)$.) Before doing this, we will show the existence and uniqueness of the solution of (2.2)–(2.6).

Certain restrictions should be imposed on the known functions $F^*(t)$, $x^*(t)$ and $x_0(t)$:

1) $x^*(t) \in L_{\infty'}[0,T]$, i.e., there exists a constant M such that

$$\sup_{0\le t\le T} \|x^*(t)\| \le M. \tag{2.7}$$

2) $F^*(t)$ and $x_0(t)$ are bounded variation functions on $[0,T]$, i.e., for any subdivision $0 = t_0 < t_1 < \cdots < t_N = T$, there exist constants C_T and C_X such that

$$\sum_{i=0}^{N-1} \|F^*(t_{i+1}) - F^*(t_i)\| \le C_F, \qquad \sum_{i=0}^{N-1} \|x_0(t_{i+1}) - x_0(t_i)\| \le C_X. \tag{2.8}$$

As a consequence of 2), we have

3) $F^*(t), x_0(t) \in L_\infty[0,T]$, i.e.,

$$\sup_{0\le t\le T} \|F^*(t)\| \le M, \qquad \sup_{o\le t\le T} \|x_0(t)\| \le M. \tag{2.9}$$

(Without loss of generality, the constant M here is taken to be the same as that in (2.7).)

Theorem 1 (Uniqueness). *If the bounded solution of* (2.2)-(2.6) *exists, then it is unique.*

Proof. Suppose that both $(x(t,\tau), A(\tau))$ and $(y(t,\tau), B(\tau))$ are the bounded solutions of (2.2)–(2.6). Denoting $z = x - y$ and $C = A - B$, we can get the following equations by subtracting the corresponding equations (2.2)–(2.3):

$$\frac{\partial z}{\partial t} + \frac{\partial z}{\partial \tau} - Az - Cy = 0, \tag{2.10}$$

$$\frac{dC}{d\tau} + \frac{1}{T}\int_0^T \{z \otimes y + (x - x^*) \otimes z\} dz = 0. \tag{2.11}$$

Multiplying (2.10) and (2.11) by $z(t,\tau)$ and matrix C respectively and integrating them, and then adding up these two equations, one obtains

$$\begin{aligned}
&\frac{1}{2T}\Big\{\int_0^T z^2(t,\tau)d\tau + \int_0^T z^2(t,\tau)dt\Big\} + \frac{1}{2}C^2(\tau)\\
&= \frac{1}{T}\int_0^T\int_0^T \{z^T(t,\tau)A(\tau)z(t,\tau) + [x(t,\tau) - x^*(t)]^T C(\tau) z(t,\tau) dt d\tau\Big\}.\\
&\le K\Big\{\int_0^\tau\int_0^T z^2 dt d\tau + \int_0^\tau C^2 d\tau\Big\}, \qquad \text{for a certain} \quad K > 0.
\end{aligned}$$

Set $\phi(\tau) = \int_0^\tau\int_0^T z^2 dt d\tau + \int_0^\tau C^2 d\tau$. It follows that $\frac{d\phi(\tau)}{d\tau} \le L\phi(\tau)$, where L is a constant, $L > 0$. Using Gronwalls inequality, we get $0 \le \phi(\tau) \le \phi(0)e^{L^\tau}$. Since $\phi(0) = 0$, we obtain $\phi(\tau) = 0$, and this proves the uniqueness.

Now we are going to prove the existence of the bounded solution of (2.2)–(2.6). To do this, we will consider a finite-difference approximation to (2.2)–(2.6), and then prove the convergence of its solution to the (weak) solution of (2.2)–(2.6).

Lemma 1. *Let the real parts of eigenvalues of A^* be all negative. Then the solution $x_i^*, i = 1, \cdots, n$, of the following finite-difference equations*

$$\left.\begin{aligned} &\frac{x_{i+1}^* - x_i^*}{\Delta t} = A^* x_{i+1}^* + F_{i+1}^*, \quad i = 1, \cdots, N-1, \quad \Delta t = \frac{T}{N} \\ &x_0^* = x^*(0) \end{aligned}\right\} \tag{2.12}$$

is convergent to the solution $x^*(t)$ of the system of equations (2.1) as $\Delta t \to 0$. Here $F_i^* = F^*(i\Delta t)$. Consequently, $x_i^*, \quad i = 1, 2, \cdots, n$, are bounded.

Proof. (2.12) is a consistent approximation to the (2.1). Since the eigenvalues of A^* are all negative, the scheme (2.12) is stable. From Lax's equivalence theorem, the consistency and stability imply the convergence. It follows that x_i^* are bounded. This completes the proof.

Consider the following finite difference approximation to (2.2)–(2.6):

$$\left.\begin{aligned} &x_{i+1}^{n+1} = x_i^n + (A^{n+1} x_{i+1}^{n+1} + F_{i+1}^*)h, \\ &A^{n+1} = A^n - \frac{1}{T}\sum_{i=0}^{N-1} \left(x_{i+1}^{n+1} - x_{i+1}^*\right) \otimes x_{i+1}^{n+1} h^2, \\ &x_i^0 = x_0(ih), \quad A^0 = A_0, \quad x_0^n = x^*(0), \\ &\qquad i = 0, 1, \cdots, N-1; \quad n = 0, 1, \cdots, \end{aligned}\right\} \tag{2.13}$$

where $h = T/N, F_{i+1}^* = F^*((i+1)h)$, and X_{i+1}^* is the solution of (2.12). In (2.13) the same mesh length in t and τ directions is taken, i.e., $\Delta t = \Delta \tau = h$.

Let I be the set of integers not less than zero, and I_N the set of integers $0, 1, \cdots, N-1$.

Lemma 2. *For every $i \in I_N$ and $n \in I$, A^n and $\sum_{i=1}^{N}(x_i^n - x_i^*)^2 h$ are uniformly bounded, i.e.,*

$$\|A^n\| \le M, \quad \sum_{i=1}^{N} x_i^n - x_i^*)^2 h \le M. \tag{2.14}$$

Here M is independent of h. (Without loss of generality, here we take the same M as that in (2.7) and (2.9).)

Proof. Discretize (2.1) corresponding to (2.13)

$$x_{i+1}^* = x_i^* + (A^* x_{i+1}^* + F_{i+1}^*)h, \quad i = 0, 1, \cdots, N-1. \tag{2.15}$$

Subtracting (2.15) from the first equation of (2.13), multiplying the equality by $x_{i+1}^{n+1} - x_{i+1}^*$, and employing the inequality

$$(x_{i+1}^{k+1} - x_{i+1}^*)(x_i^k - x_i^*) \le \frac{1}{2}(x_{i+1}^{k+1} - x_{i+1}^*)^2 + \frac{1}{2}(x_i^k - x_i^*)^2$$

we have $\frac{1}{2}(x_{i+1}^{k+1} - x_{i+1}^*)^2 \le \frac{1}{2}(x_i^k - x_i^*)^2 + (x_{i+1}^{k+1} - x_{i+1}^*)^T (A^{k+1} - A^*) x_{i+1}^{k+1} h + (x_{i+1}^{k+1} - x_{i+1}^*)^T A^* (x_{i+1}^{k+1} - x_{i+1}^*)h$. Taking $k = 0, 1, \cdots, n$ and $i = 0, 1, \cdots, N-1$, adding all these inequalities together, and then dividing the obtained inequality by T, we get

$$\frac{1}{2T}\sum_{i=0}^{N-1}(x_{i+1}^{n+1}-x_{i+1}^*)^2+\frac{1}{2T}\sum_{k=1}^{n}(x_N^k-x_N^*)^2\le\frac{1}{2T}\sum_{i=0}^{N-1}(x_i^0-x_i^*)^2$$
$$+\frac{1}{T}\sum_{i=0}^{N-1}\sum_{k=0}^{n}(x_{i+1}^{k+1}-x_{i+1}^*)^T(A^{k+1}-A^*)x_{i+1}^{k+1}h \tag{2.16}$$
$$+\frac{1}{T}\sum_{i=0}^{N-1}\sum_{k=0}^{n}(x_{i+1}^{k+1}-x_{i+1}^*)A^*(x_{i+1}^{k+1}-x_{i+1}^*)h.$$

Similarly, from the second equation of (2.13), multiplying it by $A^{k+1}-A^*$ and taking $k=0,1,\cdots,n$, we can obtain

$$\frac{1}{2}(A^{n+1}-A^*)^2\le\frac{1}{2}(A^0-A^*)^2-\frac{1}{T}\sum_{i=0}^{N-1}\sum_{k=0}^{n}(A^{k+1}-A^*)(x_{i+1}^{k+1}-x_{i+1}^*)\otimes x_{i+1}^{k+1}h^2. \tag{2.17}$$

Adding (2.16)$\times h$ and (2.17), we have

$$\frac{1}{2}(A^{n+1}-A^*)^2+\frac{1}{2T}\sum_{i=0}^{N-1}(x_{i+1}^{n+1}-x_{i+1}^*)^2h+\frac{1}{2T}\sum_{k=1}^{n}(x_N^k-x_N^*)^2h\le\frac{1}{2}(A^0-A^*)^2$$
$$+\frac{1}{2T}\sum_{i=0}^{N-1}(x_i^0-x_i^*)^2h+\frac{1}{T}\sum_{i=0}^{N-1}\sum_{k=0}^{n}(x_{i+1}^{k+1}-x_{i+1}^*)^TA^*(x_{i+1}^{k+1}-x_{i+1}^*)h^2.$$

Since the eigenvalues of A^* are negative,

$$(A^{n+1}-A^*)^2+\sum_{i=0}^{N-1}(x_{i+1}^{n+1}-x_{i+1}^*)^2h+\sum_{k=1}^{n}(x_N^k-x_N^*)^2h\le C'\Big\{(A^0-A^*)^2+\sum_{i=0}^{N-1}(x_i^0-x_i^*)^2h\Big\}.$$

From the assumptions, the right side is bounded. It follows that

$$(A^{n+1}-A^*)^2+\sum_{i=0}^{N-1}(x_{i+1}^{n+1}-x_{i+1}^*)^2h\le C.$$

But A^* is bounded. It follows that A^{n+1} is bounded. And $\sum_{i=0}^{N-1}(x_{i+1}^{n+1}-x_{i+1}^*)^2h$ is bounded, which proves the lemma.

Lemma 3. *For any $n\ge 1$, there exists a constant C independent of h such that the solution x_i^n of (2.13) has the property*

$$\sum_{i=0}^{N-1}\|x_{i+1}^n-x_i^n\|\le C \tag{2.18}$$

if $h<1/M$, where M is the bound given in (2.14).

Proof. Denote $x_i^n=x_{i+1}^n-x_i^n$. From the first equation of (2.13),

$$x_{i+1}^n=x_i^{n-1}+(A^nx_{i+1}^n+F_{i+1}^*)h,\quad x_i^n=x_{i-1}^{n-1}+(A^nx_i^n+F_i^*)h.$$

Thus

$$(I-A^nh)X_i^n=X_{i-1}^{n-1}+(F_{i+1}^*-F_i^*)h.$$

If $h < 1/M$, the coefficient matrix $I - A^n h$ is nonsingular. Then

$$X_i^n = (I - A^n h)^{-1} X_{i-1}^{n-1} + (I - A^n h)^{-1}(F_{i+1}^* - F_i^*)h,$$
$$\|X_i^n\| \le \frac{1}{1-Mh}\|X_{i-1}^{n-1}\| + \frac{h}{1-Mh}\|F_{i+1}^* - F_i^*\|, \quad i = 1, \cdots, N-1.$$

By using the induction, for the case of $n \le N, n > N$, respectively, it is easy to get

$$\begin{aligned}\sum_{i=1}^{N-1}\|X_i^n\| \le{}& \sum_{i=1}^{n-1}\frac{\|X_0^{n-i}\|}{(1-Mh)^i} + \sum_{i=n}^{N-1}\frac{\|X_{i-n}^0\|}{(1-Mh)^n}\\ &+\sum_{i=1}^{n-1}\sum_{j=1}^{i}\frac{h}{(1-Mh)^j}\|F_{i-j+2}^* - F_{i-j+1}^*\|\\ &+\sum_{i=n}^{N-1}\sum_{j=1}^{n}\frac{h}{(1-Mh)^j}\|F_{i-j+2}^* - F_{i-j+1}^*\|,\end{aligned} \tag{2.19}$$

$$\sum_{i=1}^{N-1}\|X_i^n\| \le \sum_{i=1}^{n-1}\frac{\|X_0^{n-i}\|}{(1-Mh)^i} + \sum_{i=1}^{N-1}\sum_{j=1}^{i}\frac{h}{(1-Mh)^j}\|F_{i-j+2}^* - F_{i-j+1}^*\|. \tag{2.20}$$

Letting $i = 0$ in the first equation of (2.13), we have

$$\begin{aligned}&(I - A^k h)x_1^k = x_0^{k-1} + F_1^* h = x^*(0) + F_1^* h, \quad \|x_1^k\| \le \frac{M+1}{1-Mh},\\ &X_0^k = x_1^k - x_0^k = (A^k x_1^k + F_1^*)h, \|X_0^k\| \le \frac{M(M+2)h}{1-Mh}, \quad k = 0, 1, \cdots.\end{aligned} \tag{2.21}$$

From (2.19), (2.20), (2.21) and (2.8), we obtain that for $n \le N$,

$$\begin{aligned}\sum_{i=0}^{N-1}\|X_i^n\| &\le \frac{M(M+2)h}{1-Mh} + \frac{M(M+2)Nh}{(1-Mh)^N} + \frac{C_X}{(1-Mh)^N} + \frac{NC_Fh}{(1-Mh)^N}\\ &\le \frac{1}{(1-Mh)^N}\{M(M+2)h + M(M+2)T + C_X + X_FT\}\\ &\le e^{MT}\{M + 2 + M(M+2)T + C_X + X_FT\} = C\end{aligned}$$

and for $n > N$,

$$\begin{aligned}\sum_{i=0}^{N-1}\|X_i^n\| &\le \frac{M(M+2)h}{i-Mh} + \frac{M(M+2)Nh}{(I-Mh)^N} + \frac{NC_Fh}{(i-Mh)^N}\\ &\le e^{MT}\{M + 2 + M(M+2)T + X_FT\} = C'\end{aligned}$$

which completes the proof.

Corollary. For every $i \in I_N$ and $n \in I$, there exists a constant Mindependent of h such that $\|x_i^n\| \le M$.

Proof. From Lemma 3,

$$\|x_i^n - x^*(0)\| = \|x_i^n - x_0^n\| \le \sum_{j=1}^{i}\|x_j^n - x_{j-1}^n\| \le \sum_{j=0}^{N-1}\|x_{j+1}^n - x_j^n\| \le C.$$

Therefore, $\|x_i^n\| \le \|x^*(0)\| + C = M$. Since C is independent of h, so is M.

We are going to prove the convergence of the difference approximation. To do this, we construct families $\{x_h(t,\tau)\}$ and $\{A_h(\tau)\}$ defined on $0 \le t \le T, \tau \ge 0$ from $\{x_i^n\}$ and $\{A^n\}$:

$$x_h(t,\tau) = x_i^n, \qquad \text{if } ih \le t < (i+1)h,\ nh \le \tau < (n+1)h,$$

$$A_h(\tau) = A^n, \qquad \text{if } nh \le \tau < (n+1)h.$$

We now show that these two families of functions are convergent.

Lemma 4. *There exists a sequence $\{h_k\}$ such that $\{A_{h_k}\}$ converges to a certain matrix function $A(\tau)$, and $\{x_{h_k}\}$ converges almost everywhere to a certain vector function $x(t,\tau)$, that is,*

$$\|A_{h_k}(\tau) - A(\tau)\| \to 0, \quad h_k \to 0, \tag{2.22}$$

$$\left\| \int_0^T [x_{h_k}(t,\tau) - x(t,\tau)]dt \right\| \to 0, \quad h_k \to 0, \tag{2.23}$$

where $\|\cdot\|$ in (2.22) and (2.23) are matrix and vector norm, respectively. The limit function $A(\tau)$ is bounded, and $x(t,\tau)$ is bounded almost everywhere.

Furthermore, for any fixed real number R, the convergence is uniform with respect to τ, if $0 \le \tau \le R$.

Proof. The approach to the following proof is similar to that of Lemma 16.8 in [3].

From Lemma 2, the set of $\{A_h(\tau)\}$ is uniformly bounded on any line $\tau =$const. >0. So we can find a subsequence $\{A'_h\}$ which converges on this line. Let $\{\tau_m\}$ be a countable and dense subset of $(0,\infty)$. By a diagonal process, a subsequence $\{A_{h_k}\}$ can be selected from $\{A'_h\}$ so that $\{A_{h_k}\}$ converges on every line $\tau = \tau_m, m = 1,2,\cdots$, as $h_k \to 0$.

For the sake of brevity, set $A_k = A_{h_k}$. we show that $\{A_k\}$ converges at each $0 < \tau < \infty$. To do this, we first show that for every $\tau\{A_k\}$ is a Cauchy sequence, i.e.

$$I_{kl}(\tau) = \|A_k(\tau) - A_l(\tau)\| \to 0, \quad \text{as} \quad k,l \to \infty. \tag{2.22$_1$}$$

For any τ, we find a subsequence $\{\tau_{m_s}\} \subset \{\tau_m\}$ such that $\tau_{m_s} \to \tau$ as $s \to \infty$. Write $\tau_s = \tau_{m_s}$. Then

$$I_{kl}(\tau) \le \|A_k(\tau) - A_k(\tau_s)\| + \|A_k(\tau_s) - A_l(\tau_s)\| + \|A_l(\tau_s) - A_l(\tau)\| = I_1 + I_2 + I_3.$$

Since $\{A_k\}$ is convergent at $\tau = \tau_s$, $I_2 \to 0$ as $k,l \to \infty$. For I_1 we have

$$I_1 = \|A_k([\frac{\tau}{h_k}]h_k) - A_l([\frac{\tau_s}{h_k}]h_k)\| = \|A^{[\tau/h_k]} - A^{[\tau_s/h_k]}\| = \sum_{n=[\tau_s/h_k]+1}^{[\tau/h_k]} \|A^n - A^{n-1}\|.$$

From the second equation of (2.13) and the equivalence of the norms of matrices,

$$\begin{aligned} I_1 &= \frac{1}{T} \sum_{n=[\tau_s/h_k]+1}^{[\tau/h_k]} \| \sum_{i=0}^{N-1} (x_{i+1}^n - x_{i+1}^*) \otimes x_{i+1}^n h_k^2 \| \\ &\le \frac{1}{T} \sum_n \| \sum_{i=0}^{N-1} (x_{i+1}^n - x_{i+1}^*) \otimes (x_{i+1}^n - x_{i+1}^*) h_k^2 \| \\ &+ \frac{1}{T} \sum_n \| \sum_{i=0}^{N-1} (x_{i+1}^n - x_{i+1}^*) \otimes x_{i+1}^* h_k^2 \| \\ &\le C h_k \sum_n \left\{ \sum_{i=0}^{N-1} (x_{i+1}^n - x_{i+1}^*)^2 h_k \right\}^{1/2}. \end{aligned}$$

By using Lemma 2, $I_1 \le C'|\tau-\tau_s| \to 0$ as $s \to \infty$. similarly, $I_3 \to 0$ as $s \to \infty$. Therefore, if $\varepsilon > 0$ is given, choose τ_s so that $C'|\tau - \tau_s| < \varepsilon/4$. For this fixed s, choose k and l so large that $I_2 < \varepsilon/4$. Then we have $I_{kl}(\tau) < \varepsilon$, which proves $(2.22)_1$.

It follows that the sequence $A_k(\tau)$ has a limit $A(\tau)$ for each fixed τ. The convergence is uniform in τ if we consider a bounded interval $0 \le \tau \le R$, where R is any fixed real number. In fact, in this case, for given ε we can choose a finite set $S \subset \{\tau_m\}$ with the property that if $0 \le \tau \le R$, there is a τ_s in S such that $C'|\tau - \tau_s| < \varepsilon/4$. Then we select k and l so large that $I_2 < \varepsilon/2$ for all $\tau_m \in S$. For these k and l we have $I_{kl}(\tau) < \varepsilon$ uniformly in τ.

Next we prove (2.23). The procedure of the proof is similar to that mentioned above for (2.22). From Lemma 3 and its corollary, the set of $\{x_h(t,\tau)\}$, considered as functions of t, are uniformly bounded and have uniformly bounded total variation on $[0,T]$ on any line τ =const.> 0. By Helly's theorem, we can find a subsequence $\{x'_h\}$ which converges at each point on $[0,T]$ of this line. As stated above, we can select $\{\tau_m\}$ and a subsequence $\{x_k\}$ which converges on every line $\tau = \tau_m, m = 1, 2, \cdots$, as $h_k \to 0$.

Instead of $(2.22)_1$, we consider

$$I'_{kl}(\tau) = \| \int_0^T [x_k(t,\tau) - x_l(t,\tau)]dt \|. \tag{$(2.23)_1$}$$

Correspondingly,

$$\begin{aligned}
I'_1 &= \left\| \int_0^T \left\{ x_k\left(t, \left[\frac{\tau}{h_k}\right]h_k\right) - x_k\left(t, \left[\frac{\tau_s}{h_k}\right]h_k\right)\right\} dt \right\| \\
&= \left\| \sum_{i=0}^{N-1} \int_{ih_k}^{(i+1)h_k} \left\{ x_k\left(t, \left[\frac{\tau}{h_k}\right]h_k\right) - x_k\left(t, \left[\frac{\tau_s}{h_k}\right]h_k\right)\right\} dt \right\| \\
&= \left\| \sum_{i=0}^{N-1} \left\{ x_i^{[\tau/h_k]} - x_i^{[\tau_s/h_k]} \right\} h_k \right\| = \left\| \sum_{i=0}^{N-1} \sum_{n=[\tau_s/h_k]+1}^{[\tau/h_k]} (x_i^n - x_i^{n-1}) h_k \right\| \\
&\le \sum_n \left\| \sum_{i=0}^{N-1} x_i^n - \sum_{i=0}^{N-1} x_i^{n-1} \right\| h_k = \sum_n \left\| \sum_{i=0}^{N-1} (A^n x_i^n + F_i^*) h_k + x_0^{n-1} - x_N^{n-1} \right\| h_k \\
&\le \sum_n (M^2T + MT + 2M) h_k \le C|\tau - \tau_s| \to 0 \quad \text{as} \quad s \to \infty.
\end{aligned}$$

As in the proof of the first part of this lemma, finally, we obtain the existence of a limit function $x(t,\tau)$ of $\{x_{h_k}(t,\tau)\}$ in the sence that

$$\left\| \int_0^T [x_{h_k}(t,\tau) - x(t,\tau)]dt \right\| \to 0, \quad \text{as} \quad h_k \to 0$$

and the convergence is uniform on any finite interval.

The boundedness of $A(\tau)$ and $x(t,\tau)$ is a consequence of the boundedness of A^n and x_i^n, which completes the proof.

Lemma 5. *The functions $x(t,\tau)$ and $A(\tau)$ in Lemma 4 are a (weak) solution of problem* (2.2)-(2.6).

Proof. From the first equation of (2.13),

$$\frac{x_{i+1}^{n+1} - x_{i+1}^n}{h} + \frac{x_{i+1}^n - x_i^n}{h} - (A^{n+1} x_{i+1}^{n+1} + F_{i+1}^*) = 0. \tag{2.24}$$

Take any $\phi(t,\tau) \in C_0^1$. Denote $\phi_i^n = \phi(ih, nh)$. Multiplying (2.24) by ϕ_i^n and summing, changing the subscripts and superscripts, we obtain

$$\sum_{n=1}^{\infty}\sum_{i=0}^{N-1}\frac{\phi_i^{n-1}-\phi_i^n}{h}x_{i+1}^n h^2 - \sum_{i=0}^{N-1}\phi_i^0 x_{i+1}^0 h + \sum_{n=0}^{\infty}\sum_{i=1}^{N}\frac{\phi_{i-1}^n-\phi_i^n}{h}x_i^n h^2$$

$$-\sum_{n=0}^{\infty}\phi_0^n x_0^n h + \sum_{n=0}^{\infty}\phi_N^n x_N^n h - \sum_{n=0}^{\infty}\sum_{i=0}^{N-1}\phi_i^n(A^{n+1}x_{i+1}^{n+1} + F_{i+1}^*)h^2 = 0.$$

Since $\phi \in C_0^1, \phi_i^0 = 0(i = 0, 1, \cdots, N-1), \phi_0^n = \phi_N^n = 0(n = 0, 1, \cdots)$. From Lemma 4, x_i^n and A^n converge boundedly and almost everywhere to $x(t,\tau)$ and $A(\tau)$, respectively. By the limit process, we get

$$\int_0^{\infty}\int_0^T \Big\{x(t,\tau)\frac{\partial\phi}{\partial\tau} + x(t,\tau)\frac{\partial\phi}{\partial t} + [A(\tau)x(t,\tau) + F^*(t)]\phi\Big\}dtd\tau = 0. \tag{2.25}$$

From the second equation of (2.13), $\dfrac{A^{n+1}-A^n}{h} + \dfrac{1}{T}\sum_{i=0}^{N-1}(x_{i+1}^{n+1} - x_{i+1}^*) \otimes x_{i+1}^{n+1}h = 0.$

Take any $\psi(\tau) \in C_0^1$. Denote $\psi(nh) = \psi^n$. We have

$$\sum_{n=0}^{\infty}\psi^n\Big[\frac{A^{n+1}-A^n}{h} + \frac{1}{T}\sum_{i=0}^{N-1}(x_{i+1}^{n+1} - x_{i+1}^*) \otimes x_{i+1}^{n+1}h\Big]h = 0,$$

$$\sum_{n=1}^{\infty}\frac{\psi^{n-1}-\psi^n}{h}A^n h + \frac{1}{T}\sum_{n=0}^{\infty}\sum_{i=0}^{N-1}\psi^n(x_{i+1}^{n+1} - x_{i+1}^*) \otimes x_{i+1}^{n+1}h^2 = 0.$$

By the limit process,

$$\int_0^{\infty}\Big\{-\frac{\partial\psi}{\partial\tau}A(\tau) + \frac{\psi}{T}\int_0^T[x(t,\tau) - x^*(t)] \otimes x(t,\tau)dt\Big\}d\tau = 0. \tag{2.26}$$

(2.25) and (2.26) show that $x(t,\tau)$ and $A(\tau)$ satisfy (2.2) and (2.3) (in the weak sense). Take the limit process also in the initial and boundary conditions of (2.13), which shows that $x(t,\tau)$ and $A(\tau)$ satisfy (2.4)-(2.6). So $x(t,\tau)$ and $A(\tau)$ are a generalized solution of (2.2)-(2.6). This completes the proof.

The following theorems are obvious consequences of Lemma 5.

Theorem 2 (Existence). *The solution $(x(t,\tau), A(\tau))$ of the initial-boundary problem* (2.2)-(2.6) *exists, where $A(\tau)$ is bounded, and $x(t,\tau)$ bounded almost everywhere.*

Theorem 3 (Convergence). *The solution of the finite-difference equations* (2.13) *converges to the solution of problem* (2.2)-(2.6) *as $h \to 0$.*

Proof. Let x_i^n and $A^n(i = 0, 1, \cdots, N; n = 0, 1, \cdots)$ be a solution of (2.13). we know from the proof of Lemma 6 that all convergent subsequences of $\{x_i^n\}$ and $\{A^n\}$ tend to the solution of (2.2)-(2.6). But from Theorem 1, the solution of (2.2)-(2.6) is unique. So the whole sequences $\{x_i^n\}$ and $\{A^n\}$ are convergent, and the limit functions $(x(t,\tau), A(\tau))$ are the unique solution of (2.2)-(2.6), which completes the proof.

§3. Asymptotic Convergence Theorem

Lemma 6 (A priori estimate). *The solution* $(x(t,\tau), A(\tau))$ *of the problem* (2.2)–(2.6) *satisfies the following inequality:*

$$\begin{aligned}\sup_{\tau>0} A^2(\tau) \quad &+ \sup_{\tau>0}\int_0^T x^2(t,\tau)dt + \sup_{0\le t\le T}\int_0^\infty [x(t,\tau)-x^*(t)]^2 d\tau \\ &+\int_0^\infty\int_0^T [x(t,\tau)-x^*(t)]^2 dt d\tau \le C,\end{aligned} \tag{3.1}$$

where the constant $C>0$ *is only related to the initial data.*

Proof. Define

$$y(t,\tau)=x(t,\tau)-x^*(t), \quad B(\tau)=A(\tau)-A^*,$$

where $(x(t,\tau), A(\tau))$ solves the problem (2.2)–(2.6).

Subtracting (2.1) from (2.2) and multiplying the obtained equality by y, and then integrating over $[0,t]\times[0,\tau]$, where $0<t\le T, \tau>0$, we get

$$\begin{aligned}&\frac{1}{2T}\int_0^\tau y^2(t,\tau)d\tau + \frac{1}{2T}\int_0^t y^2(t,\tau)dt - \frac{1}{T}\int_0^\tau\int_0^t y^T(t,\tau)B(\tau)x(t,\tau)dtd\tau \\ &-\frac{1}{T}\int_0^\tau\int_0^t y^T(t,\tau)A^*y(t,\tau)dtd\tau = \frac{1}{2T}\int_0^t y^2(t,0)dt.\end{aligned} \tag{3.2}$$

By a similar method, we can also get

$$\frac{1}{2}B^2(\tau)+\frac{1}{T}\int_0^\tau\int_0^t y^T(t,\tau)B(\tau)x(t,\tau)dtd\tau = \frac{1}{2}B^2(0). \tag{3.3}$$

Adding (3.2) to (3.3), we have

$$\frac{1}{2}B^2(\tau)+\frac{1}{2T}\int_0^\tau y^2(t,\tau)d\tau+\frac{1}{2T}\int_0^t y^2(t,\tau)dt - \frac{1}{2T}\int_0^\tau\int_0^t y^T A^* y dtd\tau = C'. \tag{3.4}$$

Since A^* satisfies inequality (1.2), we can draw from (3.4) a conclusion

$$B^2(\tau)+\int_0^t y^2(t,\tau)dt+\int_0^\tau y^2(t,\tau)d\tau+\int_0^\tau\int_0^t y^2(t,\tau)dtd\tau \le C''. \tag{3.5}$$

It is clear that (3.5) gives (3.1). Here the constant $C>0$ only depends on x^*, A^*, x^0, A^0 and T. This completes the proof.

Theorem 4 (The asymptotic convergence theorem). *Let* $(x(t,\tau), A(\tau))$ *be the solution of the problem* (2.2)–(2.6). *Then the matrix function* $A(\tau)$ *has a limit as* $\tau\to\infty$:

$$\lim_{\tau\to\infty} A(\tau)=A_\infty$$

which satisfies equation (2.1). *That is, it is the solution of the problem* (P).

Proof. Since the matrix function $A(\tau)$ is uniformly bounded, there exists a sequence $\{\tau_m\}$ with $\tau_m\to\infty$ such that

$$\lim_{m\to\infty} A(\tau_m)=A_\infty.$$

We are going to prove that A_∞ satisfies equation (2.1).

For an arbitrary vector function $\xi(t) \in C_0^\infty(0,T)$ (which means that its every component is in $C_0^\infty(0,T)$), consider the following identity

$$\int_0^T \left\{ \frac{\partial x(t,\tau)}{\partial \tau} + \frac{\partial x(t,\tau)}{\partial t} - A(\tau)x(t,\tau) - F^*(t) \right\}^T \xi(t)dt = 0. \tag{3.6}$$

Integrating (3.6) from $\tau_m + s - 1$ to $\tau_m + s$ with $0 < s < 1$, we obtain

$$\begin{aligned} \int_{\tau_m+s-1}^{\tau_m+s} \int_0^T \frac{\partial x^T(t,\tau)}{\partial t}\xi(t)dtd\tau + \int_0^T \{x(t,\tau_m+s) - x(t,\tau_m+s-1)\}^T \xi(t)dt \\ - \int_{\tau_m+s-1}^{\tau_m+s} \int_0^T \xi^T(t)A(\tau)x(t,\tau)dtd\tau = \int_0^T F^{*T}(t)\xi(t)dt. \end{aligned} \tag{3.7}$$

Now let us estimate the three integrals on the left-hand side of (3.7) separately.

$$\text{The first integral } = -\int_{\tau_m+s-1}^{\tau_m+s} \int_0^T [x(t,\tau) - x^*(t)]^T \frac{d\xi}{dt}dtd\tau + \int_0^T \frac{dx^{*T}}{dt}\xi(t)dt,$$

$$\begin{aligned} \left| \int_{\tau_m+s-1}^{\tau_m+s} \int_0^T [x(t.\tau) - x^*(t)]^T \frac{d\xi}{dt}dtd\tau \right| &\le C(\xi)\left\{ \int_{\tau_m+s-1}^{\tau_m+s} \int_0^T [x(t,\tau) - x^*(t)]^2 dtd\tau \right\}^{1/2} \\ &\le C(\xi)\left\{ \int_{\tau_m-1}^{\tau_m+1} \int_0^T [x(t,\tau) - x^*(t)]^2 dtd\tau \right\}^{1/2}, \end{aligned}$$

which tends uniformly in s to zero as m goes to infinity by (3.1). Here $C(\xi) > 0$ is a constant related to $\xi(t)$. Therefore

$$\int_{\tau_m+s-1}^{\tau_m+s} \int_0^T \frac{\partial x^T(t,\tau)}{\partial t}\xi(t)dtd\tau \to \int_0^T \frac{dx^{*T}(t)}{dt}\xi(t)dt$$

uniformly in s when $m \to \infty$.

The third term of (3.7) can be written as

$$\begin{aligned} \int_0^T \xi^T(t)A(\tau_m)x^*(t)dt + \int_0^T \int_{\tau_m+s-1}^{\tau_m+s} \xi^T(t)A(\tau_m)[x(t,\tau) - x^*(t)]d\tau dt \\ + \int_0^T \xi^T(t)\left\{ \int_{\tau_m+s-1}^{\tau_m+s} [A(\tau) - A(\tau_m)]x(t,\tau)d\tau \right\}dt = I_1 + I_2 + I_3. \end{aligned}$$

By using Schwartz's inequality and (3.1), one can obtain the following result easily:

$$I_2 = \int_0^T \int_{\tau_m+s-1}^{\tau_m+s} \xi^T(t)A(\tau_m)[x(t,\tau) - x^*(t)]d\tau dt \to 0 \quad \text{uniformly in } s \text{ as } m \to \infty.$$

Consider

$$\begin{aligned} \|I_3'\| &= \left\| \int_{\tau_m+s-1}^{\tau_m+s} [A(\tau) - A(\tau_m)]x(t,\tau)d\tau \right\| = \left\| \int_{\tau_m+s-1}^{\tau_m+s} \left[\int_{\tau_m}^{\tau} \frac{dA(\eta)}{d\eta}d\eta \right] x(t,\tau)d\tau \right\| \\ &\le \frac{1}{T}\int_0^T \int_{\tau_m+s-1}^{\tau_m+s} \int_{\tau_m}^{\tau} \|x(t,\eta) - x^*(t)\| \; \|x(t,\eta)\| \; \|x(t,\tau)\| d\eta d\tau dt. \end{aligned}$$

By using Schwartz's inequality, and because of (3.1) and the boundedness of $x(t,\tau)$, we have

$$\|I_3'\| \le C\left\{ \int_0^T \int_{\tau_m-1}^{\tau_m+1} [x(t,\tau) - x^*(t)]^2 d\tau dt \right\}^{1/2}$$

which also approaches zero uniformly in s as $m \to \infty$. Hence, $I_3 \to 0$ as $m \to \infty$.

Now integrating (3.7) from $s=0$ to $s=1$ and letting $m \to \infty$, one gets

$$\int_0^T \frac{dx^{*T}(t)}{dt}\xi(t)dt - \int_0^T \xi^T(t)A_\infty x^*(t)dt + \lim_{m\to\infty}\int_0^1\int_0^T \{x(t,\tau_m+s) \\ -x(t,\tau_m+s-1)\}^T\xi(t)dtds = \int_0^T F^{*T}(t)\xi(t)dt. \tag{3.8}$$

It can be proved that $\lim_{m\to\infty}\int_0^1\left\{\int_0^T [x(t,\tau_m+s)-x(t,\tau_m+s-1)]^T\xi(t)dt\right\}ds = 0$. In fact, the above expression is

$$\int_0^T\int_{\tau_m}^{\tau_m+1}[x(t,\tau)-x^*(t)]^T\xi(t)d\tau dt - \int_0^T\int_{\tau_m-1}^{\tau_m}[x(t,\tau)-x^*(t)]^T\xi(t)d\tau dt \to 0, \text{ as } m\to\infty.$$

So we can conclude the following result from (3.8):

$$\int_0^T\left\{\frac{dx^*(t)}{dt} - A_\infty x^*(t) - F^*(t)\right\}^T\xi(t)dt = 0, \quad \text{for} \quad \forall\xi(t)\in C_0^\infty(0,T).$$

Thus $\frac{dx^*(t)}{dt} = A_\infty x^*(t) + F^*(t)$.

Next, we prove that A_∞ is independent of the choice of the sequence $\{\tau_m\}$. The way to do this is the same as in [1].

From (3.4),

$$\frac{d}{d\tau}\left\{\frac{1}{2}B^2(\tau)+\frac{1}{2}\int_0^T y^2(t,\tau)dt\right\} = -\frac{1}{2T}y^2(T,\tau)+\frac{1}{2T}\int_0^T y^TA^*ydt \le 0.$$

But the expression in the bracket is ≥ 0. So its limit L exists, when $\tau\to\infty$. Hence, for any $\{\tau_m\}$ we have $\lim_{m\to\infty}\int_{\tau_m}^{\tau_m+1}\left\{B^2(\tau)+\int_0^T y^2(t,\tau)dt\right\}d\tau = L$.

From (3.1), $\lim_{m\to\infty}\int_{\tau_m}^{\tau_m+1}\int_0^T y^2(t,\tau)dtd\tau = 0$.

It yields $\lim_{m\to\infty}\int_{\tau_m}^{\tau_m+1}B^2(\tau)d\tau = \lim_{m\to\infty}\int_{\tau_m}^{\tau_m+1}[A(\tau)-A^*]^2d\tau = L$. Therefore, $\|A_\infty - A^*\|^2 = L$. If there exist two limits A_∞^1 and A_∞^2, both satisfying (2.1). Thus $(A_\infty^1 - A_\infty^2)x^* = 0$. It follows that $A^* + \varepsilon(A_\infty^1 - A_\infty^2)$ also satisfies (2.1), and it is a generalized negative definite

matrix too if ε is small enough. Hence

$$\|A_\infty^1 - A^* - \varepsilon(A_\infty^1 - A_\infty^2)\|^2 = \|A_\infty^2 - A^* - \varepsilon(A_\infty^1 - A_\infty^2)\|^2.$$

From (3.9) we can get $(A_\infty^1 - A^*)(A_\infty^1 - A_\infty^2) = (A_\infty^2 - A^*)(A_\infty^1 - A_\infty^2)$. Consequently, $\|A_\infty^1 - A_\infty^2\| = 0$, i.e., $A_\infty^1 = A_\infty^2$. This completes the proof.

§4. Numerical Examples

All our computational work was conducted on a micro-computer IBM PC/AT.

Example 1. Two-compartment model. The input $F(t)$ and the "measurements" $x^*(t)$ are

$$F(t) = \{-e^{-2t}, e^{-t}\}^T, \quad x^*(t) = \{te^{-t}, e^{-t} + e^{-2t}\}^T,$$

respectively. The initial values $x^0(t)$ and A^0 for the auxiliary problem (2.2)–(2.6) were selected as follows:

$$x_1^0(t) = \begin{cases} \frac{2}{t}, & 0 \le t \le \frac{T}{2}, \\ 2 - \frac{t}{2}, & \frac{T}{2} \le t \le T, \end{cases}$$

$$x_2^0(t) = 2 - \frac{t}{2}, \quad 0 \le t \le T, \quad T = 8.$$

$$a_{11}^0 = -0.9, \quad a_{12}^0 = 1.5, \quad a_{21}^0 = 0, \quad a_{22}^0 = -2.5.$$

In computation, it was taken that $\Delta t = \Delta\tau = 0.4$. The results are listed in Table 1.

Table 1

τ	a_{11}	a_{12}	a_{21}	a_{22}	Time for comput.
200	-1.0928	1.0763	0.0000	-2.0993	34"
400	-1.0192	1.0478	0.0000	-2.0990	1'07"
600	-0.9996	1.0402	0.0000	-2.0990	1'42"
Exact sol.	-1.0000	1.0000	0.0000	-2.0000	

Example 2. The same example with $a_{11}^0 = a_{12}^0 = a_{21}^0 = a_{22}^0 = 0, \quad T = 6$. The results are

Table 2

τ	a_{11}	a_{12}	a_{21}	a_{22}	Time for comput.
200	-0.9463	1.0195	0.0000	-2.0990	27"
400	-0.9888	1.0360	0.0000	-2.0990	53"
600	-0.9944	1.0382	0.0000	-2.0990	1'18"
Exact sol.	-1.0000	1.0000	0.0000	-2.0000	

Example 3. In Example 1, x^0 was given as $x^0(t) = x^*(t)$. We obtained

Table 3

τ	a_{11}	a_{12}	a_{21}	a_{22}	Time for comput.
200	-1.0626	1.0646	0.0000	-2.0992	34"
400	-1.0108	1.0446	0.0000	-2.0990	1'07"
520	-1.0008	1.0407	0.0000	-2.0990	1'30"
Exact sol.	-1.0000	1.0000	0.0000	-2.0000	

Example 4. In Example 1, the data are disturbed by five per cent, i.e.,

$$x^*(t) = x^*(t) + 0.05.$$

The results are

Table 4

τ	a_{11}	a_{12}	a_{21}	a_{22}
400	-1.4821	1.3068	0.0000	-1.8311
1200	-1.5817	1.3517	0.0000	-1.8311
2000	-1.5961	1.3581	0.0000	-1.8311
2800	-1.5984	1.3592	0.0000	-1.8311

They show that $A(\tau)$ tends to a matrix A_∞ which is different from the matrix

$$\begin{pmatrix} -1.0 & 1.0 \\ 0.0 & -2.0 \end{pmatrix}.$$

Conclusions. The numerical results exhibited above show that: It has a fair accuracy; its computational speed is extremely satisfactory; no information about the solution is needed in advance for applying this method. That is, the initial values $x^0(t)$ and A^0 can be given completely arbitrarily. This is a conspicuous advantage compared with other existing methods.

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