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THE GLOBAL CONVERGENCE OF THE GMED ITERATIVE ALGORITHM*

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Abstract

In the late 1970's, Wiggins proposed a minimum entropy deconvolution (MED) which has become one of the most important deconvolution methods. He gave a varimax norm V_2^4 and a MED iterative procedure. Fortunately, for the last ten years in the practical using, the MED algorithm has never failed to reach a maximizer of the varimax norm. But so far, no theoratical proof has been given to show the convergence of the MED procedure. In this paper, we prove the global convergence of a generalized MED iterative procedure with respect to a generalized varimax norm $V_q^p(q = 2, p > 2)$.

§1 Introduction

The minimum entropy deconvolution (MED) was proposed by Wiggins in the late 1970's and is based on the assumption of spiky and sparse appearance of the reflectivity series, instead of the minimum phase source signals and the white reflectivity series. Because of its new idea, simple iterative algorithm and weak hypotheses over the components, etc, it has attracted attention and become one of the most important deconvolution methods [2,3]. The detailed studies about the mathematical theories and multi-maximizer property of the MED have been given in [4,5,6,7]. Many other varimax norms and generalized MED methods suitable for various practical applications have been proposed in [8,9,10].

The generalized minimum entropy deconvolution (GMED) norm is

$$V_q^p = \frac{\sum_{i=1}^{n} |y_i|^p}{\left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{p}{q}}} = \max, \quad p, q > 0; \; p > q, \tag{1.1}$$

where $\{y_i\}_{i=1}^n$ is the output of the filtering. when $p = 4, q = 2$, the GMED norm is just the wiggins' MED. A MED iterative algorithm for solving the preceding problem was given and experiments evidenced that it worked well. Many tests were made in [7] with different initial points, and none of them failed to reach a maximizer of V_2^4 . But no one could be sure that there were no exceptions. Hence, a theoretical verification of the algorithm's convergence is necessary.

In this paper, we generalize the iterative procedure of V_2^4 by that of $V_q^p(q = 2, \; p > q)$, and a proof of its global convergence is given.

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§2 The Iterative Procedure (GMED) of the Varimax Norm V_2^p

In this paper, the underlying field of a vector space is a set $\Re$ of the real numbers. $\Re^{m\times n}$ denotes the set of $m\times n$ matrixes; $\Re^m$, the set of the m-dimensional vectors. If not otherwise specified, a capital letter stands for a vector, and the corresponding common letter with a subscript represents a component of the vector.

Set $A \in \Re^{m\times n}$. Ker A is the null of A, and Im A is the image of A. Namely,

$$\mathrm{Ker}A \equiv \{U \in \Re^n \mid AU = 0\},$$

$$\mathrm{Im}A \equiv \{V \in \Re^m \mid V = AU, \text{ for any } U \in \Re^n\}.$$

Suppose $Y \in \Re^n, \alpha > 0$. Then, $Y^\alpha \in \Re^n$, $|Y| \in \Re^n$. That is,

$$Y^\alpha = (y_1 \mid y_1^{\alpha-1} \mid, y_2 \mid y_2^{\alpha-1} \mid, ..., y_n \mid y_n^{\alpha-1} \mid)^\tau,$$

$$|Y| = (|y_1|, |y_2|, ..., |y_n|)^\tau.$$

The minimum entropy deconvolution filter is a linear filter, where the observed signals are

$$W = (x_1, x_2, ..., x_m)^\tau.$$

Without loss of generality, let $x_1 \neq 0$. If the filter $F \in \Re^l$ ($F \neq 0$), then its output $Y \in \Re^n$, $n = m + l - 1$, and by the convolutional property of the filter, we have

$$Y = XF. \tag{2.1}$$

That is, $Y \in \mathrm{Im}X$, where

$$X = \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & x_2 & \ddots & x_1 \\ x_m & \vdots & \ddots & x_2 \\ & x_m & & \vdots \\ & & & x_m \end{pmatrix}_{n\times l}. \tag{2.2}$$

Obviously, rank $X = l$, Ker $X = \{0\}$, and $X^\tau X > 0$ [11].

By convention, we bave

$$\|Y\|_p = \Big(\sum_{i=1}^n |y_i|^p\Big)^{\frac{1}{p}}, \quad p \geq 1 \quad \text{(the Hölder norm)};$$

$$\|Y\|_\infty = \max_i |y_i| \quad \text{(the infinity norm)}.$$

Definition 2.1. *The generalized minimum entropy deconvolution of varimax norm* V_q^p *is the following maximizing problom:*

$$\mathrm{GMED}: \quad \max_{Y=XF} V_q^p = \max_{Y=XF} \Big(\frac{\|Y\|_p}{\|Y\|_q}\Big)^p, \quad p > q. \tag{2.3}$$

For convenience, we will use the following representations

$$V_2^p(Y) = \Big(\frac{\|Y\|_p}{\|Y\|_2}\Big)^p, \ Y \in \Re^n; \quad V_2^p(F) = \Big(\frac{\|XF\|_p}{\|XF\|_2}\Big)^p, \ F \in \Re^l.$$

They have the following two characteristics.

Characteristic 1. Set $\alpha \neq 0$, then

$$V_2^p(\alpha Y) = V_2^p(Y), \quad V_2^p(\alpha F) = V_2^p(F). \tag{2.4}$$

Characteristic 2. Let $S_n(\varepsilon), S_l(\varepsilon)$ denote the superspheres in $\Re^n, \Re^l$ with centers at the origins and radius ϵ respectively. Let $\Omega_n = \Re^n - S_n(\epsilon), \Omega_l = \Re^l - S_l(\epsilon)$. For $\mathrm{Ker}X = \{0\}$, when $F \neq 0$, we have $Y \neq 0$. So, $V_2^p(Y)$ is twice differentiable in Ω_n, and $V_2^p(F)$ is twice differentiable in Ω_l.

Let

$$K = X(X^\tau X)^{-1}X^\tau. \tag{2.5}$$

Obviously,

$$KK = K, \quad K^\tau = K, \tag{2.6}$$

$$\mathrm{Im}K = \mathrm{Im}X. \tag{2.7}$$

Hence, K is an orthogonal projection onto $\mathrm{Im}X$. So[11],

$$\| K \|_2 = 1; \ \frac{1}{\sqrt{n}} \leq \| K \|_\infty \leq \sqrt{n}.$$

Lemma 2.1. *For any $Y \in ImX$, we have*

$$KY = Y, \tag{2.8}$$

$$Y^\tau KY^{p-1} = \| Y^p \|_1, \tag{2.9}$$

$$(Y^{p-1})^\tau KY^{p-1} = \| KY^{p-1} \|_2^2 . \tag{2.10}$$

Proof. Since K is an orthogonal projection onto $\mathrm{Im}X$, $Y \in \mathrm{Im}X$, then

$$KY = Y.$$

Also,

$$Y^\tau KY^{p-1} = (KY)^\tau Y^{p-1} = Y^\tau Y^{p-1} = \| Y^p \|_1 .$$

From (2.6), we have

$$(Y^{p-1})^\tau KY^{p-1} = (KY^{p-1})^\tau (KY^{p-1}) = \| KY^{p-1} \|_2^2 .$$

(2.8)–(2.10) are proved.

Define a map J :

$$J(Y) = \| KY^{p-1} \|_\infty^{-1} KY^{p-1}. \tag{2.11}$$

Let $D(J)$, $R(J)$ denote the domain and the range of the operator J respectively. Assume

$$D(J) = \{Y \in \mathrm{Im}X \mid \| Y \|_\infty = 1\}. \tag{2.12}$$

From (2.7), obviously $R(J) \in D(J)$.

Lemma 2.2. *For any $Y \in D(J)$, there is*

$$\frac{1}{\sqrt{n}} \leq \| K \|_\infty^{-1} \leq \| KY^{p-1} \|_\infty^{-1} < n. \tag{2.13}$$

Proof. From (2.9) and the Cauchy inequality,

$$\| KY^{p-1} \|_2 \| Y \|_2 \geq (KY^{p-1})^\tau Y = \| Y^p \|_1 .$$

So,

$$\| KY^{p-1} \|_2^{-1} \leq \frac{\| Y \|_2}{\| Y^p \|_1} < \sqrt{n}.$$

Then,

$$\| KY^{p-1} \|_\infty^{-1} \leq \sqrt{n} \| KY^{p-1} \|_2^{-1} < n.$$

On the other hand,

$$\| KY^{p-1} \|_\infty \leq \| K \|_\infty \| Y^{p-1} \|_\infty = \| K \|_\infty \leq \sqrt{n}.$$

So, (2.13) holds.

The definitions and corresponding theorems of the boundary and the continuity for a nonlinear operator used hereafter can be found in [13].

Theorem 2.1. *$J(Y)$ is a bounded operator in $D(J)$. That is, there exists an $L > 0$, such that for any $Y1, Y2 \in D(J)$, we have*

$$\| J(Y1) - J(Y2) \|_\infty \leq L \| Y1 - Y2 \|_\infty . \tag{2.14}$$

Proof.

$$\begin{aligned}
\|J(Y1) - J(Y2)\|_\infty &= \| \| KY_1^{p-1} \|_\infty^{-1} KY_1^{p-1} - \| KY_2^{p-1} \|_\infty^{-1} KY_2^{p-1} \|_\infty \\
&= \| (\| KY_1^{p-1} \|_\infty^{-1} - \| KY_2^{p-1} \|_\infty^{-1}) KY_1^{p-1} + (KY_1^{p-1} - KY_2^{p-1}) \| KY_2^{p-1} \|_\infty^{-1} \|_\infty \\
&\leq (| \| KY_1^{p-1} \|_\infty^{-1} - \| KY_2^{p-1} \|_\infty^{-1} |) \| KY_1^{p-1} \|_\infty + \| KY_1^{p-1} - KY_2^{p-1} \|_\infty \| KY_2^{p-1} \|_\infty^{-1} \\
&= (| \| KY_1^{p-1} \|_\infty - \| KY_2^{p-1} \|_\infty |) \| KY_2^{p-1} \|_\infty^{-1} + \| KY_1^{p-1} - KY_2^{p-1} \|_\infty \| KY_2^{p-1} \|_\infty^{-1} \\
&\leq 2 \| KY_2^{p-1} \|_\infty^{-1} \| KY_1^{p-1} - KY_2^{p-1} \|_\infty \leq 2n(p-1) \| K \|_\infty \| Z_\theta \|_\infty \| Y_1 - Y2 \|_\infty ,
\end{aligned}$$

where

$$Z_\theta = \begin{pmatrix} | y_{\theta 1}^{p-2} | & & & 0 \\ & | y_{\theta 2}^{p-2} | & & \\ & & \ddots & \\ 0 & & & | y_{\theta n}^{p-2} | \end{pmatrix}.$$

By the mean-value theorem

$$y_{\theta i} = y_{1i} + \theta(y_{2i} - y_{1i}), \quad 0 < \theta < 1,$$

where y_{1i}, y_{2i} are the components of Y_1, Y_2 respectively. For $\max\{| y_{1i} |\} = \max\{| y_{2i} |\} = 1$, $\max\{| y_{\theta i} |\} \leq 1$. So, $\| Z_\theta \|_\infty \leq 1$, and we obtain

$$\|J(Y_1) - J(Y_2)\|_\infty \leq 2n(p-1)\sqrt{n} \|Y_1 - Y_2\|_\infty . \tag{2.14'}$$

Let $L = 2n\sqrt{n}(p-1)$. Then (2.14') is just (2-14). So, (2-14) is proved.

Theorem 2.2. *If $Y_* = XF_*$, and $Y_* \in D(J)$ satisfies*

$$Y_* = J(Y_*), \tag{2.15}$$

then F_* is a stationary point of $V_2^p(F)$[12]*. That is,*

$$\nabla V_2^p(F_*) = 0, \tag{2.16}$$

where

$$\nabla V_2^p(F_*) \equiv \left(\frac{\partial V_2^p}{\partial f_1}, \frac{\partial V_2^p}{\partial f_2}, \ldots, \frac{\partial V_2^p}{\partial f_l}\right)$$

Proof. For $Y \in D(J)$, $F \in \Omega_l$ and $V_2^p(F)$ is twice differentiable with respect to F. So we have

$$\begin{aligned}\nabla V_2^p(F) &= \frac{p}{\| Y^2 \|_1^{\frac{p}{2}+1}} X^\tau (\| Y^2 \|_1 Y^{p-1} - \| Y^p \|_1 Y) \\ &= \frac{p}{\| Y^2 \|_1^{\frac{p}{2}+1}} (X^\tau X) \left(\| Y^2 \|_1 (X^\tau X)^{-1} X^\tau Y^{p-1} - \| Y^p \|_1 F\right).\end{aligned}$$

Let $\nabla V_2^p(F) = 0$. We obtain

$$F = \frac{\| Y^2 \|_1}{\| Y^p \|_1} (X^\tau X)^{-1} X^\tau Y^{p-1}. \tag{2.17}$$

Since $\mathrm{Ker}X = \{0\}$, there is an equavalent equation of (2.17)

$$Y = \frac{\| Y^2 \|_1}{\| Y^p \|_1} K Y^{p-1}, \tag{2.18}$$

where K is determined by (2.5). For

$$Y_* = J(Y_*) = \| K Y_*^{p-1} \|_\infty^{-1} K Y_*^{p-1} \tag{2.19}$$

from (2.9) and (2.19) we have

$$\begin{aligned}\frac{\| Y_*^2 \|_1}{\| Y_*^p \|_1} K Y_*^{p-1} &= \frac{\| Y_*^2 \|_1}{Y_*^\tau K Y_*^{p-1}} \| K Y_*^{p-1} \|_\infty Y_* \\ &= \frac{\| Y_*^2 \|_1}{Y_*^\tau \| K Y_*^{p-1} \|_\infty Y_*} \| K Y_*^{p-1} \|_\infty Y_* = Y_*.\end{aligned}$$

So, $Y_* = XF_*$ satisfies equation (2.18) and hence (2.17). Consequently, F_* is a stationary point of V_2^p.

Since $\mathrm{Ker}X = \{0\}$, X is a bijection mapping from $\Re^l$ onto $\mathrm{Im}X$. For convenince, we also call $Y_* = XF_*$ a stationary point of V_2^p.

From (2.15), we construct the following iterative procedure (GMED):

$$\text{GMED}: \quad \begin{cases} Y_{(0)} \in D(J), \\ Y_{(k+1)} = J(Y_{(k)}). \end{cases}$$

If $Y_{(0)}$ is not a solution of (2.15), by (GMED), we can obtain a sequence $\{Y_{(k)}\}_{k=0}^\infty$ any of whose converging subsequences converges to a solution of (2.15). That is, the (GMED) is of global convergence. We will prove it in the ensuing section.

§3 The Proof of the Convergence

Let

$$g(Y) = \| Y^2 \|_1 Y^{p-1} - \| Y^p \|_1 Y. \tag{3.1}$$

Then,

$$\nabla V_2^p(Y) \equiv \left(\frac{\partial V_2^p}{\partial y_1}, \frac{\partial V_2^p}{\partial y_2}, \ldots, \frac{\partial V_2^p}{\partial y_n}\right)^\tau = \frac{p}{\| Y^2 \|_1^{\frac{p}{2}+1}} g(Y). \tag{3.2}$$

Lemma 3.1. *For any $Y \in \Re^n$ and $Y \neq 0$, there is*

$$Y^\tau g(Y) = 0. \tag{3.3}$$

Proof. Substituting (3.1) into (3.3), we have

$$Y^\tau g(Y) = \| Y^2 \|_1 Y^\tau Y^{p-1} - \| Y^p \|_1 Y^\tau Y = \| Y^2 \|_1 \| Y^p \|_1 - \| Y^p \|_1 \| Y^2 \|_1 = 0.$$

Lemma 3.2. *Assume $U, V \in \Re^n$, $u_i, v_i \geq 0$, $i = 1, 2, \ldots, n$. For any $t \in (0, 1]$, there exist*

$$\| U^2 \|_1 \| V^2 \|_1 \geq (U^t)^\tau (V^{2-t})(V^t)^\tau (U^{2-t}). \tag{3.4}$$

The equality holds iff

$$U = \alpha V, \quad \alpha \neq 0. \tag{3.5}$$

Proof. Assume $A, B \in \Re^n$, $a_i, b_i \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$. From the Hölder inequality, we have

$$\| A \|_p \| B \|_q \geq A^\tau B. \tag{3.6}$$

The equality holds iff $A^p = \beta B^q$, $\beta \neq 0$.

Let $t = \frac{2}{p} \in (0, 1]$ in (3.6), and let $A^p = U^2, B^q = V^2$. Then we have

$$\| U^2 \|_1^{\frac{t}{2}} \| V^2 \|_1^{1-\frac{t}{2}} \geq (U^t)^\tau (V^{2-t}). \tag{3.7}$$

Similarly, let $A^p = V^2, B^q = U^2$ in (3.6). Then we have

$$\| U^2 \|_1^{1-\frac{t}{2}} \| V^2 \|_1^{\frac{t}{2}} \geq (U^{2-t})^\tau V^t. \tag{3.8}$$

So, (3.4) is obtained by the product of (3.7) and (3.8).

Since the condition (3.5) is necessary and sufficient for the equality to hold in either (3.7) or (3.8), it is necessary and sufficient for the equality to hold in (3.4).

Lemma 3.3. *For any $Y \in \mathrm{Im}X$ and $\alpha > 0$, we have*

$$-Y^\tau g(\alpha K Y^{p-1} + Y) \geq 0. \tag{3.9}$$

The equality holds iff

$$Y = \beta K Y^{p-1}, \quad \beta \neq 0. \tag{3.10}$$

Proof. Note $W = \alpha K Y^{p-1} + Y$. Then

$$\begin{aligned} -Y^\tau g(W) &= -Y^\tau(\| W^2 \|_1 W^{p-1} - \| W^p \|_1 W) \\ &= \| W^p \|_1 (\alpha Y^\tau K Y^{p-1} + Y^\tau Y) - \| W^2 \|_1 Y^\tau W^{p-1} \\ &= \| W^p \|_1 (\alpha \| Y^p \|_1 + \| Y^2 \|_1) - \| W^2 \|_1 Y^\tau W^{p-1} \\ &= \| W^p \|_1 \| Y^p \|_1 (\alpha + \frac{\| Y^2 \|_1}{\| Y^p \|_1}) - \| W^2 \|_1 Y^\tau W^{p-1}. \end{aligned} \tag{3.11}$$

In Lemma 3.2, let $U = | W |^{\frac{p}{2}}, V = | Y |^{\frac{p}{2}}$, and $t = \frac{2}{p}$. We have

$$\begin{aligned} \| W^p \|_1 \| Y^p \|_1 &\geq (| Y |)^\tau (| W |)^{p-1} (| Y |^{p-1})^\tau | W | \geq (| Y |)^\tau (| W |)^{p-1} (Y^{p-1})^\tau W \\ &= (| Y |)^\tau (| W |)^{p-1} [\alpha \| K Y^{p-1} \|_2^2 + \| Y^p \|_1]. \end{aligned} \tag{3.12}$$

Substitute (3.12) into (3.11). Then

$$\begin{aligned} -Y^\tau g(W) &\geq (| Y |)^\tau | W |^{p-1} (\alpha \| K Y^{p-1} \|_2^2 + \| Y^p \|_1)(\alpha + \frac{\| Y^2 \|_1}{\| Y^p \|_1}) \\ &\quad - (| Y |)^\tau (| W |)^{p-1} (\alpha^2 \| K Y^{p-1} \|_2^2 + 2\alpha \| Y^p \|_1 + \| Y^2 \|_1) \\ &= \alpha (| Y |)^\tau (| W |)^{p-1} \Big(\frac{\| Y \|_2^2 \| K Y^{p-1} \|_2^2}{\| Y^p \|_1} - \| Y^p \|_1 \Big). \end{aligned} \tag{3.13}$$

From Lemma 2.1 and the Cauchy inequality, we have

$$\| Y \|_2^2 \| K Y^{p-1} \|_2^2 \geq \| Y^p \|_1^2 . \tag{3.14}$$

Substituting it into (3.13) and noticing $\alpha > 0$, we obtain (3.9).

In deducing inequality (3.9), the condition (3.10) is sufficient for all the equalities to hold; it is also necessary for the equalities in some inequalities such as (3.14) to hold. So it is necessary and sufficient for the equality to hold in (3.9).

From the previous results, we can conclude the following theorem.

Theorem 3.1. *For any $Y \in \mathrm{Im} X$, $Y \neq 0$, and $\alpha > 0$, we have*

$$V_2^p(\alpha K Y^{p-1}) \geq V_2^p(Y). \tag{3.15}$$

The equality holds iff (3.10) is satisfied.

Proof. From Ker $Y = \{0\}$ and Lemma 2.1, if $Y \neq 0$, then $K Y^{p-1} \neq 0$. From (2.9), $Y^\tau K Y^{p-1} > 0$. Then the line segment joining Y to $K Y^{p-1}$ is placed in Ω_n. Thus by the mean-value theorem, we have

$$V_2^p(\alpha K Y^{p-1}) = V_2^p(K Y^{p-1}) = V_2^p(Y) + (K Y^{p-1} - Y)^\tau \nabla V_2^p(Y_\theta),$$

where $Y_\theta = \theta Y + (1-\theta) K Y^{p-1}, 0 < \theta < 1$. From Lemma 3.1

$$\begin{aligned} (K Y^{p-1} - Y)^\tau \nabla V_2^p(Y_\theta) &= \Big[\frac{1}{1-\theta} Y_\theta - \frac{1}{1-\theta} Y \Big]^\tau \Big[\frac{p}{\| Y_\theta^2 \|_1^{\frac{p}{2}+1}} g(Y_\theta) \Big] \\ &= -\frac{1}{1-\theta} \frac{p}{\| Y_\theta^2 \|_1^{\frac{p}{2}+1}} \theta^{p+1} Y^\tau g\Big(Y + \frac{1-\theta}{\theta} K Y^{p-1} \Big). \end{aligned}$$

By Lemma 3.3, we have proved the theorem.

Corollary 3.1. Suppose $\{Y_{(k+1)}\}_{k=0}^{\infty}$ is a sequence produced by (GMED) procedure with initial point $Y_{(0)}$. Then

$$V_2^p(Y_{(k+1)}) \geq V_2^p(Y_{(k)}). \tag{3.16}$$

The equality holds iff $Y_{(k+1)} = Y_{(k)}$.

Proof. As $\{Y_{(k)}\} \subset D(J) \subset \mathrm{Im}X$, from Theorem 3.1 we obtain (3.16). Since

$$\| Y_{(k)} \|_{\infty} = 1, \quad k = 0, 1, 2, \ldots$$

condition (3.10) implies $Y_{(k+1)} = Y_{(k)}$.

Theorem 3.2 (Global convergence of GMED iteration procedure). *Suppose* $\{Y_{(k)}\}_{k=0}^{\infty}$ *is a sequence produced by* GMED *with initial point* $Y_{(0)}$. *Then any of its converging subsequences* $\{Y_{(k_n)}\}_{n=0}^{\infty}$ *converges to a stationary point of* $V_2^p(F)$.

Proof. Since $\| Y_{(k)} \|_{\infty} = 1$, the converging subsequence $\{Y_{(k_n)}\}$ of $\{Y_{(k)}\}$ exists. Set

$$\{Y_{(k_n)}\} \to Y_*, \quad n \to \infty.$$

As $D(J)$ is a complete set, so

$$Y_* \in D(J).$$

Let

$$Y_{(k_n+1)} = J(Y_{(k_n)}), \tag{3.17}$$

and suppose $\{Y_{(k_n+1)}\}$ has a converging subsequence $\{Y_{(j_n+1)}\}$ which has a limit $Y_*' \in D(J)$ and satisfies

$$Y_{(j_n+1)} = J(Y_{(j_n)}). \tag{3.18}$$

Obviously, $\{Y_{(j_n)}\}$ is a subsequence of $\{Y_{(k_n)}\}$. As the operator J is bounded, and hence continuous,

$$Y_{(j_n+1)} \to Y_*', \quad j_n + 1 \to \infty; \quad J(Y_{(j_n)}) \to J(Y_*), \quad j_n \to \infty.$$

From (3.18),

$$Y_*' = J(Y_*).$$

As $\{Y_{(j_n+1)}\}$ and $\{Y_{(j_n)}\}$ are subsequences of $\{Y_{(k)}\}$, from Corollary 3.1 we have

$$V_2^p(Y_*') = V_2^p(Y_*).$$

Hence

$$Y_* = Y_*',$$

and then

$$Y_* = J(Y_*).$$

By Theorem 2.2, Y_* is a stationary point of $V_2^p(F)$.

According to the theory of mathematical programming[12], the conclusions of Theorem 3.2. imply the global convergence of GMED iteration procedure.

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