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SUPERCONVERGENCE OF FEM FOR SINGULAR SOLUTION*

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Superconvergence of the finite element method (FEM) has been discussed extensively for the problem having smooth solution (See Krizek and Neittaanmaki [8]). A typical result in this direction is the following (see Lin and Xie [4] for details). Consider the model problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \boldsymbol{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$ and f is a smooth function. In order to keep the mesh varying regularly we impose on Ω a kind of "piecewise almost uniform triangulation" which can be constructed piecewisely by the vertices of a smoothly transformed uniform mesh. For any node z in the interior of each piece there exist two triangles e and e' such that $e \cap e' = \{z\}$. Then, the average gradient

$$\bar{\nabla} u^h(z) = \frac{1}{2}(\nabla u^h|_e + \nabla u^h|_{e'})$$

has not only the usual type of superconvergence

$$(\bar{\nabla} u^h - \nabla u)(z) = O(h^2)$$

but also an extrapolation type of superconvergence

$$\frac{1}{3}\bar{\nabla}(4u^{h/2} - u^h)(z) - \nabla u(z) = O(h^4 \log \frac{1}{h}).$$

We are concerned in this paper with the superconvergence for the singular solution due to re-entrant corners or changing the boundary conditions.

For simplicity we suppose that Ω is composed of rectangles and the boundary $\partial\Omega$ is parallel to the x-and y-axis and has only one re-entrant corner at the origin 0. Let α be the interior angle at 0 and $\beta = \pi/\alpha$.

It is easy to see that

$$u \in H^3_{(\tau+1)} \quad \text{for } \tau > 1 - \beta,$$

where the Sobolev space $H^3_{(\tau+1)}$ is defined using the weighted norm

$$\|u\|_{3,(\tau+1)} = \Big[\sum_{|j|\le 3} \int_\Omega (|X|^{\tau-2+|j|}|\partial^j u|)^2 dX\Big]^{1/2}$$

with $X = (x, y)$.

We now introduce a rectangular mesh $T^h = \{e\}$, where (x_e, y_e) denotes the center of the element e and $2h_e$ and $2k_e$ are its widths in the x- and y-direction, respectively. Further, we set

$$d_e = \max(h_e, k_e), \quad h = \max\{d_e, e \in T^h\},$$

$$d_0 = \max\{d_e, e \in T^h, 0 \in e\}, \quad r_e = \min\{|X|, X \in e\}.$$

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Let T^h be split into two parts,

$$\Omega_0 = \{e \in T^h, r_e < d_0\}, \quad \Omega_1 = \{e \in T^h, \quad d_0 \le r_e\},$$

where the local meshes are assumed to satisfy the grading conditions

$$d_0 \le ch^q, \quad q > \frac{t}{\beta}, \quad t \le 2;$$

$$c_1 h r_e^p \le d_e \le c h r_e^p, \quad \forall d_e \le r_e, \quad p = 1 - \frac{1}{q},$$

where q is the grading parameter and t is the superconvergence parameter. For example, if $\Omega = (-1,1)\times(-1,1)\setminus[0,1]\times\{0\}$ (a slit domain), such meshes can be constructed by taking nodes

$$(\pm(i/n)^q, \pm(j/n)^q)(1 \le i, j \le n), \quad q > 2t.$$

Since a larger t will lead to a larger q, the user has to make up his choice between a higher accuracy and a less graded mesh. We note that the total number of nodes of the graded meshes is the same as for a uniform mesh of size h, and that the size of the largest element is of the order h.

Let

$$\Omega_2 = \{X \in \Omega, \quad |X| \ge \rho > 0\},$$

z be the interior node of Ω_2 and N the number of all interior nodes of Ω_2:

$$N = O(h^{-2}).$$

For such z there exist two elements e and e' such that $e \cap e' = \{z\}$ and we can define, for $\nu \in S^h$ the piecewise bilinear finite element space, the average gradient

$$\bar\partial_x \nu(z) = \frac{h_e}{h_e + h_{e'}}\partial_x \nu|_{e'}(z) + \frac{h_{e'}}{h_e + h_{e'}}\partial_x \nu|_e(z),$$

$$\bar\partial_y \nu(z) = \frac{k_e}{k_e + k_{e'}}\partial_y \nu|_{e'}(z) + \frac{k_{e'}}{k_e + k_{e'}}\partial_y \nu|_e(z).$$

Let $u^I \in S^h$ be the interpolation of u and $u^R \in S^h$ the Ritz projection of u. It is easy to see from Taylor expansion the superconvergence of u^I after averaging:

Lemma 1.

$$|(\bar\partial u^I - \partial u)(z)| \le ch^t \|u\|_{3,\infty,\Omega_2},$$

where the notation $\bar\partial$ means $\bar\partial_x$ or $\bar\partial_y$.

Our purpose is to prove the superconvergence of u^R after averaging:

Theorem. *The grading parameter q increases the gradient accuracy from β-order to nearly $q\beta$-order:*

$$\left[\frac{1}{N}\sum_{z\in\Omega_2} |(\bar\partial u^R - \partial u)(z)|^2\right]^{1/2} \le ch^t, \quad t < q\beta.$$

The proof of our theorem is based on the lemmas as follows (c.f. [1]–[2]).

Lemma 2. *For the function $F(x)$ satisfying $F(x_e \pm h_e) = 0$, we have*

$$\int_{x_e-h_e}^{x_e+h_e} F\,dx = \frac{1}{2}\int_{x_e-h_e}^{x_e+h_e} PF''\,dx,$$

where $P(x) = (x - x_e + h_e)(x - x_e - h_e)$.

Proof. Note that

$$P(x_e \pm h_e) = 0, \quad P''(x) = 2.$$

Integrating by parts leads to Lemma 2:

$$\int_{x_e-h_e}^{x_e+h_e} PF''dx = -\int_{x_e-h_e}^{x_e+h_e} P'F'dx = \int_{x_e-h_e}^{x_e+h_e} 2Fdx.$$

Lemma 3. *For $u \in H^3(e)$ and $\nu \in S^h$, there holds*

$$a_e(u^I - u, \nu) = -\frac{1}{2}\int_e (P\partial_y \partial_x^2 u \partial_y \nu + Q\partial_x \partial_y^2 u \partial_x \nu) + \int_e (P+Q)\partial_y \partial_x (u^I - u)\partial_y \partial_x \nu,$$

where $P(x)$ is defined as in Lemma 2 and $Q(y) = (y - y_e + k_e)(y - y_e - k_e)$.

Proof. Since

$$a_e(u^I - u, \nu) = \int_e \partial_x (u^I - u)\partial_x \nu + \int_e \partial_y (u^I - u)\partial_y \nu$$

we need only to expand, say, the second integral. Let l_1 and l_2 stand for the two edges of element e parallel to the x-direction. Integrating by parts we obtain

$$\int_e \partial_y (u^I - u)\partial_y \nu = \Big[\int_{l_1} - \int_{l_2}\Big](u^I - u)\partial_y \nu dx.$$

Setting $F = (u^I - u)\partial_y \nu$ in Lemma 2 we have

$$\partial_x^2 F = -\partial_x^2 u \partial_y \nu + 2\partial_x (u^I - u)\partial_y \partial_x \nu$$

and hence, by Lemma 2,

$$\int_{l_i} (u^I - u)\partial_y \nu dx = -\frac{1}{2}\int_{l_i} P(x)\partial_x^2 u \partial_y \nu dx + \int_{l_i} P(x)\partial_x (u^I - u)\partial_y \partial_x \nu.$$

Thus

$$\int_e \partial_y (u^I - u)\partial_y \nu = -\frac{1}{2}\int_e P(x)\partial_y (\partial_x^2 u \partial_y \nu) + \int_e P(x)\partial_y (\partial_x (u^I - u)\partial_y \partial_x \nu)$$

and Lemma 3 follows.

Since u becomes singular in Ω_0 we need the following

Lemma 4. *For $\tau = 1 - t/q$,*

$$|a_{\Omega_0}(u^I - u, \nu)| \le ch^t \|u\|_{2,(\tau)} \|\nu\|_1.$$

Proof. Since $u \in W^{1,a} \cap W^{2,b}$ for

$$2 < a = \frac{2b}{2-b} < \frac{2}{\tau}, \quad b < \frac{2}{1+\tau}$$

we have, by the Holder inequality,

$$|u - u^I|_{1,\Omega_0} \le cd_0^{1-2/a}|u - u^I|_{1,a,\Omega_0} \le cd_0^{1-2/a}|u|_{1,a,\Omega_0}.$$

An imbedding theorem implies that

$$|u|_{1,a,\Omega_0} \le c\|u\|_{2,b,\Omega_0} + cd_0^{-1}|u|_{1,b,\Omega_0}$$

and the Holder inequality implies that

$$\|u\|_{2,b,\Omega_0} \le cd_0^{2/b-1-\tau}\|u\|_{2,(\tau)}, \quad |u|_{1,b,\Omega_0} \le cd_0^{2/b-1-(\tau-1)}|u|_{1,(\tau-1)}$$

and hence,

$$|u - u^I|_{1,\Omega_0} \le cd_0^{1-\tau}\|u\|_{2,(\tau)} \le ch^t \|u\|_{2,(\tau)}.$$

Lemma 5. $\|u^I - u^R\|_1 \le ch^t \|u\|_{3,(\tau+1)}$.

Proof. For $\nu \in S^h$

$$a(u^I - u^R, \nu) = a(u^I - u, \nu) = \sum_{e\in\Omega_1} a_e(u^I - u, \nu) + a_{\Omega_0}(u^I - u, \nu).$$

For $e \in \Omega_1$ we use Lemma 3 and the following estimates:

$$|\partial_x\partial_y(u^I - u)|_{0,e} \le cd_e|u|_{3,e}, \quad |\partial_x\partial_y \nu|_{0,e} \le cd_e^{-1}\|\nu\|_{1,e}.$$

Since

$$d_e^2 \le cd_e^t r_e^{2-t} \le ch^t r_e^{1+\tau},$$

we obtain

$$|a_e(u^I - u, \nu)| \le ch^t \|u\|_{3,(\tau+1),e}\|\nu\|_{1,e}.$$

Thus, combining with Lemma 4 we obtain Lemma 5.

We now prove our theorem as follows.

Using an inverse inequality and noting $r_e \ge \rho$ we have

$$|\bar\partial(u^I - u^R)(z)| \le cd_e^{-1}\|u^I - u^R\|_{1,e} \le ch^{-1}r_e^{-\nu}\|u^I - u^R\|_{1,e} \le ch^{-1}\rho^{-\nu}\|u^I - u^R\|_{1,e}.$$

Thus,

$$\left[\frac{1}{N}\Sigma|\bar\partial(u^I - u^R)(z)|^2\right]^{1/2} \le c\|u^I - u^R\|_1.$$

Combining with Lemma 1 we obtain our theorem.

A $2t$-order accuracy can be achieved for the finite element solution after extrapolation (see [2], [5]-[6]). A similar result holds true also for the finite element gradient (see [3]).

References

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