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# NONLINEAR STABILITY OF GENERAL LINEAR METHODS*[1)]

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**Abstract**

This paper is devoted to a study of stability of general linear methods for the numerical solution of nonlinear stiff initial value problems in a Hilbert space. New stability concepts are introduced. A criterion of weak algebraic stability is established, which is an improvement and extension of the existing criteria of algebraic stability.

## §1. Introduction

The main goal of this paper is to make further advances on the theories of algebraic stability for general linear methods (cf. [1–4]). In Section 2, we introduce a family of classes of nonlinear test problems, $\{K_{\sigma,\tau} : \sigma\tau < 1\}$, in a Hilbert space. Section 3 is concerned with general linear methods in brief. In Section 4 a series of new stability concepts is introduced. In Section 5 we establish the criterion of $(k,p,q)$-weak algebraic stability, which is an essential improvement and extension of the criteria of algebraic stability presented by Burrage and Butcher[4].

## §2. Test Problems

Let $X$ be a real or complex Hilbert space with the inner product $\langle \cdot\ ,\ \cdot \rangle$ and the corresponding norm $\|\cdot\|$, $D$ an infinite subset of $X$, and $f : [0,+\infty) \times D \to X$ a given mapping. Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \ge 0, & (2.1) \\ y(0) = \iota, \qquad \iota \in D, & (2.2) \end{cases}$$

which is assumed to have a unique solution $y(t)$ on the interval $[0,+\infty)$.

**Definition 1.** *Let $\sigma, \tau$ be real constants with $\sigma\tau < 1$. Then the class of all problems* (2.1)–(2.2) *with*

$$2Re\ \langle u - v, f(t,u) - f(t,v)\rangle \le \sigma\|u-v\|^2 + \tau\|f(t,u) - f(t,v)\|^2 \quad \forall u, v \in D, \quad t \ge 0 \quad (2.3)$$

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*is called the class* $K_{\sigma,\tau}$.

From Definition 1 we obtain the following propositions:

**Proposition 1.** If $\sigma < 0, \varepsilon \geq 0$, then $K_{\sigma,\tau} \subset K_{\sigma-\varepsilon,\tau+\varepsilon M(\sigma,\tau)}$, where

$$M(\sigma,\tau) = [(1+\sqrt{1-\sigma\tau})/\sigma]^2. \tag{2.4}$$

**Proposition 2.** If $\tau < 0, \varepsilon \geq 0$, then $K_{\sigma,\tau} \subset K_{\sigma+\varepsilon N(\sigma,\tau),\tau-\varepsilon}$, where

$$N(\sigma,\tau) = [(1+\sqrt{1-\sigma\tau})/\tau]^2. \tag{2.5}$$

**Proposition 3.** If $X$ is the usual $N$-dimensional complex $U$-space and for fixed $t$, $\psi(t) \in X, A(t)$ is an $N \times N$ complex matrix, then the linear system

$$y'(t) = A(t)y(t) + \psi(t), \quad t \geq 0; \quad y(0) = \iota, \quad \iota \in D = X \tag{2.6}$$

belongs to the class $K_{\sigma,\tau}$ if and only if $\sigma \geq \sup_{t\geq 0} \lambda_{\max}$, where $\lambda_{\max}$ denotes the greatest eigenvalue of the Hermite matrix $A^* + A - \tau A^* A$.

In the literature only the special cases $K_{0,0}, K_{\sigma,0}$ and $K_{0,\tau}$ have been used respectively as the test problem class (cf. [1–5]).

## §3. General Linear Methods

Consider the general linear method for solving (2.1)–(2.2):

$$\begin{cases} Y_i^{(n)} = h\sum_{j=1}^{s} c_{ij}^{11} f(T_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} c_{ij}^{12} y_j^{(n-1)}, & i = 1,2,\cdots,s; \\ y_i^{(n)} = h\sum_{j=1}^{s} c_{ij}^{21} f(T_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} c_{ij}^{22} y_j^{(n-1)}, & i = 1,2,\cdots,r; \\ y_n = \sum_{j=1}^{r} \beta_j y_j^{(n)}, \end{cases} \tag{3.1}$$

where $h > 0$ is the stepsize, $c_{ij}^{IJ}$ and $\beta_j$ are constants in the base field of $X, Y_i^{(n)}, y_i^{(n)}$ and $y_n$ are approximations to $y(T_i^{(n)}), H_i(t_i^{(n)})$ and $y(t_n + \eta h)$ respectively, $T_i^{(n)} = t_{n-1} + \mu_i h, t_i^{(n)} = t_n + \nu_i h, t_n = nh, \mu_i, \nu_i, \eta$ are nonnegative constants, and each $H_i(t_i^{(n)})$ is a piece of information about $y(t)$.

Throughout this paper we always assume that each $y_i^{(n)}$ is an approximation to $y(t_i^{(n)})$ for $i \leq l$ and the following equalities hold exactly:

$$y_{l+i}^{(n)} = hf(t_i^{(n)}, y_i^{(n)}), \quad i = 1,2,\cdots,l; \quad n = 0,1,2,\cdots, \tag{3.2}$$

where $l$ is a fixed nonnegative integer not greater than $r/2$.

For any given $M \times N$ matrix $A = [a_{ij}]$ we define a linear mapping $\widetilde{A} : X^N \to X^M$ such that for any $U = (u_1, u_2, \cdots, u_N) \in X^N$ with each $u_i \in X$,

$$\widetilde{A}U = V = (v_1, v_2, \cdots, v_M) \in X^M,$$

with

$$v_i = \sum_{j=1}^{N} a_{ij} u_j, \quad i = 1, 2, \cdots, M.$$

For simplicity, we shall always use the same letter "$A$" to denote the linear mapping $\tilde{A}$ corresponding to the matrix $A$. Thus, the method (3.1) may be written equivalently in the more compact form

$$\begin{cases} Y^{(n)} = hC_{11}F(Y^{(n)}) + C_{12}y^{(n-1)}, \\ y^{(n)} = hC_{21}F(Y^{(n)}) + C_{22}y^{(n-1)}, \\ y_n = \beta y^{(n)} \end{cases} \tag{3.3}$$

with the following notational conventions:

$$y^{(n)} := (y_1^{(n)}, y_2^{(n)}, \cdots, y_r^{(n)}) \in X^r, \quad Y^{(n)} := (Y_1^{(n)}, Y_2^{(n)}, \cdots, Y_s^{(n)}) \in X^s,$$
$$F(Y^{(n)}) := (f(T_1^{(n)}, Y_1^{(n)}), f(T_2^{(n)}, Y_2^{(n)}), \cdots, f(T_s^{(n)}, Y_s^{(n)})) \in X^s,$$

and $C_{IJ}(I, J = 1, 2), \beta$ are linear mappings corresponding to the matrices $C_{IJ} = [c_{ij}^{IJ}]$ and $\beta = [\beta_1, \beta_2, \cdots, \beta_r]$ respectively. For further details we refer to [6–8].

## §4. $(k, p, q)$-Stability

In this section and the next one, the base field of $X$ is assumed to be complex (the case of real field can be discussed similarly). Let $\{(Y^{(n)}, y^{(n)}, y_n)\}$ and $\{(Z^{(n)}, z^{(n)}, z_n)\}$ be two sequences of approximations computed by the method (3.1) with stepsize $h$ for the same initial value problem (2.1)–(2.2). The following notational conventions will be made:

$$W^{(n)} = (W_1^{(n)}, W_2^{(n)}, \cdots, W_s^{(n)}) := Y^{(n)} - Z^{(n)} \in X^s;$$
$$w^{(n)} = (w_1^{(n)}, w_2^{(n)}, \cdots, w_r^{(n)}) := y^{(n)} - z^{(n)} \in X^r;$$
$$w_n := y_n - z_n \in X; \quad Q^{(n)} = (Q_1^{(n)}, Q_2^{(n)}, \cdots, Q_s^{(n)}) := h[F(Y^{(n)}) - F(Z^{(n)})] \in X^s.$$

**Definition 2.** *Let $k, p, q$ be real constants with $k > 0$ and $pq < 1$. The method (3.1) is said to be $(k, p, q)$-stable if for any given problem (2.1)–(2.2) of the class $K_{\sigma,\tau}$ with $\sigma h \le p$ and $\tau/h \le q$ there exists a nonnegative mapping $\varphi : X^r \to R_+$, which satisfies*

$$\varphi(U) \le c_\varphi \sum_{i=1}^{r} \|u_i\|, \quad U = (u_1, u_2, \cdots, u_r) \in X^r$$

*with each $u_i \in X$ and constant $c_\varphi$ independent of $U$, such that the following inequalities hold:*

$$\|w_n\| \le \varphi(w^{(n)}) \le k^{1/2}\varphi(w^{(n-1)}), \quad n = 1, 2, 3, \cdots. \tag{4.1}$$

As an important special case, a $(1, 0, 0)$-stable method is said to be $AH$-stable. Furthermore, for the given space $X$, if the mapping $\varphi$ depends only on the method (not on the initial value problem and stepsize), then we say the method is $(k, p, q)$-stable for the mapping $\varphi$.

Note that from (4.1) we can deduce

$$\|w_n\| \le k^{n/2}\varphi(w^{(0)}). \tag{4.2}$$

For $k \le 1$ inequalities (4.1), (4.2) describe the contractivity and stability behaviour of the method. From Definition 2 we directly obtain

**Proposition 4.** For $AH$-stable methods, the numerical solutions of initial value problems of the class $K_{0,0}$ are always stable without restriction on the steplength.

**Definition 3.** *Let $\alpha \in \left(0, \frac{\pi}{2}\right]$. The method (3.1) is said to be $AH(\alpha)$-stable if it is $(1, \cos^2\alpha/q, q)$-stable for all $q < 0$.*

**Definition 4.** *The method (3.1) is said to be $AH(0)$-stable if it is $AH(\alpha)$-stable for some $\alpha \in \left(0, \frac{\pi}{2}\right)$.*

**Definition 5.** *The method (3.1) is said to be stiff H-stable if it is $AH(0)$-stable and there exist $p, q < 0$ such that it is both $(1, p, 0)$-stable and $(1, 0, q)$-stable.*

**Definition 6.** *Suppose for $p \le 0$ the method (3.1) is $(k(p), p, 0)$-stable, where $k(p)$ is an increasing function with $k(0) = 1$. Then the method (3.1) is said to be strongly AH-stable if $\lim_{p\to-\infty} k(p) < 1$ and LH-stable if $\lim_{p\to-\infty} k(p) = 0$.*

**Theorem 1.** *The stability region $S$ of a $(1, p, q)$-stable method contains the region*

$$S_{p,q} = \{z \in \mathbb{C}\colon\ q|z|^2 - 2\mathrm{Re} z + p \ge 0\}. \tag{4.3}$$

*Proof.* Suppose the method (3.1) is $(1, p, q)$-stable and $S$ denotes its stability region. For any $\lambda \in S_{p,q}$, it can be seen from Proposition 3 that the scalar test problem $y' = \lambda y, y(0) = \iota$ belongs to the class $K_{p,q}$. We now apply (3.1) with stepsize $h = 1$ to this problem. Let $\{y_n\}$ denote the solution sequence and put $z_n = 0$. From (4.2) we have $\|y_n\| \le \varphi(y^{(0)})$. This means that the sequence $\{y_n\}$ is bounded. Thus $h\lambda = \lambda \in S$ and therefore $S_{p,q} \subset S$.

**Corollary 1.** For $q \ge 0, (1, 0, q)$-stable methods are $A$-stable; in particular, $AH$-stable methods are $A$-stable.

**Corollary 2.** $AH(\alpha)$-stable methods are $A(\alpha)$-stable for $\alpha \in \left(0, \frac{\pi}{2}\right)$, and $A$-stable for $\alpha = \frac{\pi}{2}$.

**Corollary 3.** $AH(0)$-stable methods are $A(0)$-stable.

**Corollary 4.** A stiff $H$-stable method is stiff stable in the following sense: (i) there exists a positive constant $a$ such that its stability region contains the half plane $\{z \in \mathbb{C}\colon \mathrm{Re} z \le -a\}$; (ii) there exists a positive constant $\gamma$ such that its stability region contains the disk $\{z \in \mathbb{C}\colon |z + \gamma| \le \gamma\}$; (iii) it is $A(0)$-stable.

**Corollary 5.** When $r = 1$, $LH$-stable methods are $L$-stable.

**Corollary 6.** For $AH(\alpha)$-stable methods, the numerical solutions of initial value problems of the class $K_{\sigma,\tau}$ with $\sigma, \tau < 0$ and $\sigma\tau \ge \cos^2\alpha$ are always stable without restriction on the steplength.

Corollaries 1, 2 can be derived from Theorem 1; Corollary 3 can be obtained from Corollary 2; Corollary 4 can be proved by Corollary 3 and Theorem 1; Corollary 6 can be deduced from Definitions 2 and 3. We now prove Corollary 5. Suppose (3.1) is $LH$-stable and $r = 1$. Then it is equivalent to a one-step method and is $AH$-stable and therefore $A$-stable. Applying it to the model problem $y' = \lambda y(\mathrm{Re}\lambda \leq 0), y(0) = 1$, we have

$$y_n = R^n(\bar{h})y_0, \tag{4.4}$$

where $R(\bar{h}) = \bar{h}C_{21}(I - \bar{h}C_{11})^{-1}C_{12} + C_{22}, \bar{h} = h\lambda, y_0 = 1$, and $I$ is a unit matrix. Because this problem belongs to the class $K_{2\mathrm{Re}\lambda,0}$ and the method is $(k(2\mathrm{Re}\bar{h}), 2\mathrm{Re}\bar{h}, 0)$-stable, there exists a mapping $\varphi : \mathbb{C} \to R_+$ such that

$$|y_n| \leq k^{n/2}(2\mathrm{Re}\bar{h})\varphi(y^{(0)}). \tag{4.5}$$

The relations (4.4) and (4.5) yield

$$|R(\bar{h})| \leq k^{1/2}(2\mathrm{Re}\bar{h})\varphi^{1/n}(y^{(0)}).$$

Setting $n \to +\infty, \mathrm{Re}\bar{h} \to -\infty$, we see that $|R(\bar{h})| \to 0$ and the method is $L$-stable.

**Theorem 2.** *Let $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $p, q < 0$. The method (3.1) is both $AH(\alpha)$-stable and stiffly H-stable if it is both $(1, p, 0)$-stable and $(1, 0, q)$-stable, and satisfies one of the following two conditions:*

(i) $pq \leq 4(1 - \sin\alpha)/(1 + \sin\alpha)$;

(ii) $pq > 4(1 - \sin\alpha)/(1 + \sin\alpha)$, *and for all* $\gamma \in (q(1 + \sin\alpha)/2, 2(1 - \sin\alpha)/p)$ *the method is* $(1, \cos^2\alpha/\gamma, \gamma)$*-stable.*

*Proof.* The conclusion follows directly from the following lemma.

**Lemma 1.** *If $p, q \leq 0$ and the method (3.1) is $(k, p, q)$-stable, then for any positive number $c \in [pq, 1)$ and negative numbers $\gamma \in [(c\beta_c)/p, q/\beta_c]$ and $\delta \in [(c\beta_c)/q, p/\beta_c]$, the method is both $(k, c/\gamma, \gamma)$-stable and $(k, \delta, c/\delta)$-stable. Here $\beta_c = (1+\sqrt{1-pq})/(1+\sqrt{1-c})$, and we define $(c\beta_c)/0 = -\infty$.*

The conclusion of this lemma follows from Definition 2 and Propositions 1, 2. We, however, have to leave out the details of the proof owing to the limitation of space.

**Corollary 7.** If $p, q < 0$ and $pq < 4$, then a method is both stiffly $H$-stable and $AH(\arcsin \frac{4 - pq}{4 + pq})$-stable provided that it is both $(1, p, 0)$-stable and $(1, 0, q)$-stable.

*Proof.* The conclusion follows directly from Theorem 2.

## §5. Stability Criteria

We define the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on the space $X^N (N \geq 1)$ as follows:

$$\langle\langle U, V \rangle\rangle = \sum_{i=1}^{N} \langle u_i, v_i \rangle, \tag{5.1}$$

where $U = (u_1, u_2, \cdots, u_N) \in X^N, V = (v_1, v_2, \cdots, v_N) \in X^N$ and $u_i, v_i \in X$ $(i = 1, 2, \cdots, N)$.

**Proposition 5.** If $A$ is an $N_1 \times N_2$ matrix, $U \in X^{N_1}$ and $V \in X^{N_2}$, then

$$\langle\langle U, AV\rangle\rangle = \langle\langle A^* U, V\rangle\rangle. \tag{5.2}$$

**Proposition 6.** If $A$ is a Hermite nonnegative definite $N \times N$ matrix and $U \in X^N$, then

$$\langle\langle U, AU\rangle\rangle \geq 0. \tag{5.3}$$

These two propositions hold true obviously, but note that the symbols $A$ and $A^*$ in (5.2) and (5.3) denote linear mappings corresponding to the matrix $A$ and its conjugate transpose matrix $A^*$ respectively.

**Definition 7.** *Let $k, p, q$ be real constants with $k > 0$ and $pq < 1$, $G$ a Hermite nonnegative definite $r \times r$ matrix, $D$ an $s \times s$ nonnegative diagonal matrix, and furthermore, for $l > 0$, $\tilde{D}$ and $\tilde{\tilde{D}}$ denote $l \times l$ nonnegative diagonal matrices. The method* (3.1) *is said to be $(k,p,q)$-weakly algebraically stable (for the matrices $G, D, \psi(\tilde{D}), \psi(\tilde{\tilde{D}})$) if the matrix*

$$M = M(k,p,q) = \begin{bmatrix} k\tilde{G} - C_{22}^*\tilde{G}C_{22} & C_{12}^*D - C_{22}^*\tilde{G}C_{21} \\ -pC_{12}^*DC_{12} + \psi(\tilde{D}) & -pC_{12}^*DC_{11} \\ & \\ DC_{12} - C_{21}^*\tilde{G}C_{22} & C_{11}^*D + DC_{11} - C_{21}^*\tilde{G}C_{21} \\ -pC_{11}^*DC_{12} & -pC_{11}^*DC_{11} - qD \end{bmatrix} \tag{5.4}$$

*is nonnegative definite. Here $\psi(\tilde{D})$ and $\tilde{G}$ are $r \times r$ matrices given by*

$$\psi(\tilde{D}) = \begin{bmatrix} -p\tilde{D} & \tilde{D} & 0 \\ \tilde{D} & -p\tilde{D} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{if } l > 0; \quad \psi(\tilde{D}) = 0 \quad \text{if} \quad l = 0 \tag{5.5}$$

*and*

$$\tilde{G} = G + \beta^*\beta - \psi(\tilde{\tilde{D}}). \tag{5.6}$$

As an important special case, a $(1,0,0)$-weakly algebraically stable method is called weakly algebraically stable for short. When $\psi(\tilde{D}) = \psi(\tilde{\tilde{D}}) = 0$, a $(k,p,q)$-weakly algebraically stable method is said to be $(k,p,q)$-algebraically stable.

**Theorem 3.** *If the method* (3.1) *is $(k,p,q)$-weakly algebraically stable for the matrices $G, D, \psi(\tilde{D})$ and $\psi(\tilde{\tilde{D}})$, then it is $(k,p,q)$-stable for the mapping $\varphi$ defined by*

$$\varphi(U) = (\max\{0; \langle\langle U, \tilde{G}U\rangle\rangle\})^{1/2} \quad \forall\, U \in X^r. \tag{5.7}$$

*Proof.* Suppose the method is applied to a problem (2.1)–(2.2) of the class $K_{\sigma,\tau}$ and $\sigma h \leq p$, $\tau/h \leq q$. Then it follows from the assumption about $G, D, \tilde{D}$ and $\tilde{\tilde{D}}$ that

$$\langle\langle w^{(n)}, \psi(\tilde{D})w^{(n)}\rangle\rangle \leq 0, \quad \langle\langle w^{(n)}, \psi(\tilde{\tilde{D}})w^{(n)}\rangle\rangle \leq 0, \quad n = 0,1,2,\cdots \tag{5.8}$$

and

$$2\mathrm{Re}\langle\langle W^{(n)}, DQ^{(n)}\rangle\rangle - p\langle\langle W^{(n)}, DW^{(n)}\rangle\rangle - q\langle\langle Q^{(n)}, DQ^{(n)}\rangle\rangle \leq 0, \quad n = 1,2,3,\cdots. \tag{5.9}$$

Using Propositions 5, 6 and relations (5.6), (5.7) and (5.8), we get

$$\varphi(w^{(n)}) = \langle\langle w^{(n)}, \tilde{G}w^{(n)}\rangle\rangle^{1/2}, \quad n = 0, 1, 2, \cdots,$$

and furthermore

$$\varphi^2(w^{(n)}) = \langle\langle w^{(n)}, Gw^{(n)}\rangle\rangle + \langle\langle w^{(n)}, \beta^*\beta w^{(n)}\rangle\rangle - \langle\langle w^{(n)}, \psi(\tilde{\tilde{D}})w^{(n)}\rangle\rangle \geq \|w_n\|^2 \tag{5.10}$$

and

$$\begin{aligned}
&\varphi^2\,(w^{(n)}) - k\varphi^2(w^{(n-1)}) - \langle\langle w^{(n-1)}, \psi(\tilde{D})w^{(n-1)}\rangle\rangle - 2\mathrm{Re}\langle\langle W^{(n)}, DQ^{(n)}\rangle\rangle \\
&+p\langle\langle W^{(n)}, DW^{(n)}\rangle\rangle + q\langle\langle Q^{(n)}, DQ^{(n)}\rangle\rangle = \langle\langle C_{21}Q^{(n)} + C_{22}w^{(n-1)}, \tilde{G}(C_{21}Q^{(n)} \\
&+C_{22}w^{(n-1)})\rangle\rangle + \langle\langle w^{(n-1)}, -k\tilde{G}w^{(n-1)}\rangle\rangle + \langle\langle w^{(n-1)}, -\psi(\tilde{D})w^{(n-1)}\rangle\rangle \\
&+2\mathrm{Re}\langle\langle C_{11}Q^{(n)} + C_{12}w^{(n-1)}, -DQ^{(n)}\rangle\rangle + \langle\langle C_{11}Q^{(n)} \\
&+C_{12}w^{(n-1)}, pD(C_{11}Q^{(n)} + C_{12}w^{(n-1)})\rangle\rangle + \langle\langle Q^{(n)}, qDQ^{(n)}\rangle\rangle \\
&= -\langle\langle (w^{(n-1)}, Q^{(n)}), M(w^{(n-1)}, Q^{(n)})\rangle\rangle \leq 0.
\end{aligned} \tag{5.11}$$

Inequalities (5.8)–(5.10) and (5.11) lead to inequality (4.1) immediately and therefore Theorem 3 follows.

Choose $k = 1$ and $p = q = 0$ in Theorem 3 so that we have

**Corollary 8.** Weakly algebraically stable methods are $AH$-stable.

It should be pointed out that the theory of weak algebraic stability introduced in this paper is an essential improvement on that of algebraic stability presented in [4]. On the one hand, if we use the definition of algebraic stability in [4], then not only is it impossible to obtain a result similar to Corollary 8, but also we may claim that even some explicit methods, such as the Euler method, are algebraically stable. In fact, the Euler method is equivalent to the general linear method

$$C = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right].$$

Let

$$G = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = 0.$$

Then it is easy to verify that this method is algebraically stable in the sense of [4]. On the other hand, the matrices $G$ used in the examples of [4] are actually restricted to the case that they are all positive definite. As a result of this, though the aforementioned drawback seems to have been overcome, it has to be claimed that even some $AH$-stable methods, such as trapezoidal method, are not algebraically stable. Using the theory of weak algebraic stability introduced in this paper, we can overcome all the disadvantages. On the one hand, weakly algebraically stable methods are necessarily $AH$-stable; on the other hand, it is easily seen that the trapezoidal method and some other $AH$-stable methods are weakly algebraically stable though they are not algebraically stable in the sense of [4].

**Corollary 9.** If the method (3.1) is $AH$-stable and $C_{11}$ nonsingular, then it is strongly $AH$-stable for $\rho(\tilde{C}_{22}) < 1$ and $LH$-stable for $\rho(\tilde{C}_{22}) = 0$, where $\rho(\tilde{C}_{22})$ denotes the spectral radius of the matrix $\tilde{C}_{22} := C_{22} - C_{21}C_{11}^{-1}C_{12}$.

This corollary is parallel to the relevant result in [4].

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