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AN UNCONDITIONALLY STABLE DIFFERENCE APPROXIMATION FOR A CLASS OF NONLINEAR DISPERSIVE EQUATIONS*

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Abstract

An unconditionally stable leap-frog finite difference scheme for a class of nonlinear dispersive equations is presented and analyzed. The solvability of the difference equation which is a tridiagonal circular linear system is discussed. Moreover, the convergence and stability of the difference scheme are also investigated by a standard argument so that more difficult priori estimations are avoided. Finally, numerical examples are given.

§1. Introduction

We shall consider a leap-frog finite difference approximation of the nonlinear dispersive equation given by

$$u_t + (a(x,t,u))_x + b(x,t,u) - u_{xxt} = 0, \quad (x,t) \in R \times I, \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in R, \tag{1.1b}$$

$$u(x+1,t) = u(x,t), \quad (x,t) \in R \times I, \tag{1.1c}$$

where $I = [0,T](T > 0)$, and R is the real line. The coefficients a and b in (1.1a) will be C^1 functions defined on $R \times I \times R$, 1-periodic with respect to their first argument. u_0 in (1.1b) is also g given 1-periodic function.

Theoretical results about the existence, uniqueness and regularity for (1.1) can be found in [1,2] and the references contained therein. Numerical approximations of (1.1) based on the finite element method[1], the finite difference method[3] and the spectral method[4] have also been considered. W.H. Ford and T.W. Ting[3] have studied the convergence and stability of the Crank-Nicolson scheme, but the Crank-Nicolson scheme is a nonlinear system and is hard to solve. Some physicists and engineers proposed some finite element and finite difference schemes, but they did not get the proof of convergence and stability (see [5, 6]).

In this paper we devote a leap-frog finite difference scheme which is a tridiagonal circular linear system and can be easily solved by the Seidel iteration method. Using the standard argument[1,7], we prove its convergence and stability. Therefore we can avoid quite difficult priori estimations.

Throughout this paper, we assume that (1.1) has a unique smooth solution U defined in $R \times I$. The letter C will be used to indicate generic constants, and the usual functional notation will be employed to specify dependence.

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§2. Some Symbols and Finite Difference Scheme

We introduce a grid $x_j = jh = (0 \le j \le J), h = 1/J, t^n = nk(0 \le n \le N)$, where J and N are positive integers, h and k are space-step length and time-step length respectively.

In the following we often use superscript n to denote the n-th time level. We set the following difference operators:

$$(u_j^n)_x = (u_{j+1}^n - u_j^n)/h, \ (u_j^n)_{\bar{x}} = (u_j^n - u_{j-1}^n)/h, \ (u_j^n)_{\hat{x}} = (u_{j+1}^n - u_{j-1}^n)/(2h),$$

where u_j^n is the approximation of $U(jh, nk)$. Similarly, we can define difference operators $(u_j^n)_t, (u_j^n)_{\bar{t}}$ and $(u_j^n)_{\hat{t}}$.

If V and W are 1-periodic grid functions, we denote by v_j and w_j their values at x_j respectively. Set

$$(V, W) = h\sum_{j=1}^{J} v_j \bar{w}_j, \quad (V, W)_1 = (V, W) + (V_x, W_x),$$

$$\|V\|^2 = (V, V), \qquad \|V\|_1^2 = (V, V)_1,$$

where $V_x = (v_{jx})$ and $W = (w_{jx})$ are difference grid functions.

With these notations we can consider the following leap-frog difference scheme of (1.1):

$$u_{j\hat{t}}^n + (a_j^n(u_j^n))_{\hat{x}} + b_j^n(u_j^n) - u_{jx\bar{x}\hat{t}}^n = 0; \quad 1 \le j \le J, 1 \le n \le N, \tag{2.1a}$$

$$u_{jt}^0 + (a_j^0(u_j^0))_{\hat{x}} + b_j^0(u_j^0) - u_{jx\bar{x}t}^0 = 0, \quad 1 \le j \le J, \tag{2.1b}$$

$$u_j^0 = u_0(jh), \quad 1 \le j \le J, \tag{2.1c}$$

$$u_{j+rJ}^n = u_j^n, \quad 1 \le j \le J, \ \ 0 \le n \le N, \ r = \pm 1, \pm 2, \cdots, \tag{2.1d}$$

where $a_j^n(u_j^n) = a(jh, nk, u_j^n)$ and $b_j^n(u_j^n) = b(jh, nk, u_j^n)$.

§3. Solvability, Convergence and Stability of the Difference Solution

First of all, we discuss the solvability of (2.1). Note that (2.1a) and (2.1b) can be rewritten as

$$-u_{j-1}^{n+1} + (2 + h^2)u_j^{n+1} - u_{j+1}^{n+1} = b_j^n, \quad 1 \le j \le J, \ 0 \le n \le N - 1, \tag{3.1}$$

where $b_j^n = h^2 u_j^{n-1} - h^2(u_j^{n-1})_{x\bar{x}} - 2kh^2[(a_j^n(u_j^n))_{\hat{x}} + b_j^n(u_j^n)]$ $(1 \le n \le N-1)$ and $b_j^0 = h^2 u_j^0 - h^2(u_j^0)_{x\bar{x}} - kh^2[(a_j^0(u_j^0))_{\hat{x}} + b_j^0(u_j^0)]$.

The coefficient matrix of the tridiagonal circular linear system (3.1) is nonsingular because it is strictly row-wise diagonally dominant. So the following result holds:

Theorem 3.1. *Difference equation* (2.1) *is always solvable.*

On the convergence and stability of the solution of (2.1), we have

Theorem 3.2. *Assume that solution $U(x,t)$ of equation (1.1) is in $C^3(I, H_p^4(0,1))$. If $\lim_{h\to 0}\|U^0 - u^0\|_1 = 0$, then there exist positive constants $C_j (j = 1,2,3)$, independent of h and k, such that for $h \le C_1$ and $k \le C_2$ the following error estimate holds:*

$$\sup_{1\le n\le N} \|u^n - U^n\|_1 \le C_3(h^2 + k^2 + \|u^0 - U^0\|_1), \tag{3.2}$$

where u^n is the solution of (2.1).

Proof. Set $e^n = U^n - u^n$. The truncation error defined by

$$U^n_{j\hat{t}} + (a^n_j(U^n_j))_{\hat{x}} + b^n_j(U^n_j) - U^n_{jx\bar{x}\hat{t}} = \tau^n_j, \quad 1 \le j \le J,\ 1 \le n \le N \tag{3.3}$$

is easily seen to be $O(h^2 + k^2)$.

Subtraction of (2.1a) from (3.3), multiplication by $h(\bar{e}^{n+1}_j + \bar{e}^{n-1}_j)$, and summation with respect to j from 1 to J yield

$$(e^n_{\hat{t}}, e^{n+1} + e^{n-1}) + ((a^n(U^n) - a^n(u^n))_{\hat{x}}, e^{n+1} + e^{n-1}) + (b^n(U^n) - b^n(u^n),$$
$$e^{n+1} + e^{n-1}) - (e^n_{x\bar{x}\hat{t}}, e^{n+1} + e^{n-1}) = (\tau^n, e^{n+1} + e^{n-1}). \tag{3.4}$$

Note that $(u^n_{\hat{x}}, v^n) = -(u^n, v^n_{\hat{x}})$ and $(u^n_{x\bar{x}}, v^n) = -(u^n_x, v^n_x)$. Then by (3.4) and the definition of $(\cdot,\cdot)_1$, we have

$$(\|e^{n+1}\|_1^2 - \|e^{n-1}\|_1^2)/(2k) = (\bar{a}^n_u e^n, (e^{n+1} + e^{n-1})_{\hat{x}}) - (\bar{b}^n_u e^n, e^{n+1} + e^{n-1}) + (\tau^n, e^{n+1} + e^{n-1}) \tag{3.5}$$

where $\bar{\alpha}^n_u = \int_0^1 \partial\alpha(u^n + \mu e^n)/\partial u d\mu$, for $\alpha = a$ or b.

We first estimate (3.5) to a^* and b^* respectively instead of a and b where a^* and b^* are 1-periodic functions in $C_b^1(R \times I \times R)$ with respect to their first argument; a^* and b^* extend respectively a and b from ε-neighborhood $S(\varepsilon)$ of solution surface S (the definitions of S and $S(\varepsilon)$ can be found in [7]). For convenience sake, here we still use a and b to denote a^* and b^* respectively. By (3.5) we have

$$\|e^{n+1}\|_1^2 - \|e^{n-1}\|_1^2 \le Ck(\|e^{n-1}\|_1^2 + \|e^n\|_1^2 + \|e^{n+1}\|_1^2 + \|\tau^n\|^2), \tag{3.6}$$

where we have used the fact $\|u^n_{\hat{x}}\| \le 2\|u^n_x\|$.

Summation of (3.6) with respect to n from 1 to $m (m < N)$ yields

$$\|e^{m+1}\|_1^2 + \|e^m\|_1^2 \le \|e^0\|_1^2 + \|e^1\|_1^2 + TC \max_{1\le n\le N} \|\tau^n\|^2 + kC\sum_{n=1}^{m}(\|e^n\|_1^2 + \|e^{n+1}\|_1^2). \tag{3.7}$$

Application of Gronwall's inequality for (3.7) implies

$$\|e^n\|_1 \le C(\|e^0\|_1 + \|e^1\|_1 + h^2 + k^2), \tag{3.8}$$

where C is a constant dependent on ε, a, b, T and $U(x,t)$.

Similarly to the above proof, by (2.1b) we have

$$\|e^1\|_1 \le \|e^0\|_1 + kC(\|\tau^0\| + \|e^0\|). \tag{3.9}$$

Because $\tau_j^0 = O(h^2 + k)$, we have $\|\tau^0\| \le C(h^2+k)$. By (3.8), (3.9) and the fact that $k \le T$, we have

$$\|e^n\|_1 \le C(\|e^0\|_1 + h^2 + k^2), \tag{3.10}$$

where C depends on ε, T, a, b and $U(x,t)$.

Thus we complete the proof of the theorem in the case that a^* and $b^* \in C_b^1(R \times I \times R)$.

Finally, similarly to the standard argument in [1, 7], we can remove the hypothesis that the coefficients and their first derivatives are bounded. This completes the proof of Theorem 3.2.

By Theorem 3.2 and the Sobolev inequality $\|U^n - u^n\|_\infty \le C\|U^n - u^n\|_1$, we can get the uniform norm convergence of the difference solution

Corollary 3.1. Under the assumptions of Theorem 3.2, we have

$$\|U^n - u^n\|_{L^\infty \times L^\infty} \le C(h^2 + k^2 + \|U^0 - u^0\|_1), \tag{3.11}$$

where

$$\|U^n - u^n\|_{L^\infty \times L^\infty} = \sup_{1 \le n \le N, 1 \le j \le J} |U_j^n - u_j^n|.$$

Similarly to the proof of Theorem 3.2, we have

Theorem 3.3. *The difference solution of* (1.1) *is unconditionally stable in the norm of* $H_p^1(0,1)$.

§4. Numerical Results

We consider the dispersive equations (1.1) with $a(x,t,u) = e^t u^3$ and $b(x,t,u) = (1 + 4\pi^2 - 3\cos(2\pi x)u/(2\pi))u$. It is not difficult to check that $u = e^{-t}\sin(2\pi x)$ is an exact solution of (1.1).

We compare the finite difference solution of (1.1) among the leap-frog scheme (L-F) and the Euler scheme and the Crank-Nicolson scheme (C-N). The calculations of the three schemes are performed by using Turbo Pascal Version 5.0 on IBM PC XT/286 computer. We use the Seidel iterative method to calculate the first two difference schemes and the iterative method to calculate the C-N scheme. The stopping criterion is

$$\|u^{(l+1)} - u^{(l)}\| \le \text{ eps}, \tag{4.1}$$

where $u^{(l)}$ is the l-th iterative approximation and epx$=10^{-8}$.

Errors, time-step and space-step lengths and computer times are listed in the following table, where the error is with respect to L^2-norm. When $h = k = 0.1$, the first two numbers of iteration are at most 43 and computer times are almost the same, but the C-N scheme needs more computer time than first two schemes. When $h = k = 0.01$, the number of iterations of the Euler scheme is at most 64, the leap-frog is at most 69 and the C-N is at least 209. So the C-N scheme takes more operating time. But its accuracy is the same as the L-F scheme.

Remark 1. Our leap-frog difference scheme (2.1) is squarely convergent just like the Crank-Nicolson scheme. Moreover, it is a tridiagonal circular linear system and can be solved by the Seidel iterative method more easily than the Crank-Nicolson scheme which is a nonlinear algebraic system.

Remark 2. The leap-frog difference scheme can be also used to calculate the solution of the initial-valus problem of (1.1) (cf. [5, 6]). Moreover, theoretical results in the above section are also available for the initial-value problem. By the way, we point out that the Causs elimination method to solve (3.1) is not stable because $h^2 << 1$. When $h = k = 0.1$, $e(0, 8) = 128065109.8$ When $k = h = 0, 001$, $e(0.5) = 93823694976$.

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t_n	$h = 0.1,\ k = 0.1$			t_n	$h = 0.01,\ k = 0.01$		
	L-F	Euler	C-N		L-F	Euler	C-N
0.10	0.0013	0.0080	0.0016	0.10	0.000014	0.01066	0.000016
0.20	0.0021	0.0150	0.0024	0.20	0.000025	0.02031	0.000030
0.30	0.0028	0.0211	0.0031	0.30	0.000035	0.02903	0.000046
0.40	0.0035	0.0266	0.0038	0.40	0.000043	0.03692	0.000059
0.50	0.0040	0.0313	0.0042	0.50	0.000051	0.04406	0.000082
0.60	0.0045	0.0354	0.0046	0.60	0.000058	0.05050	0.000096
0.70	0.0049	0.0389	0.0052	0.70	0.000064	0.05634	0.000102
0.80	0.0052	0.0420	0.0057	0.80	0.000069	0.06161	0.000107
0.90	0.0055	0.0446	0.0059	0.90	0.000075	0.06638	0.000119
1.00	0,0057	0.0468	0.0065	1.00	0.000079	0.07069	0.000126
time	2.511	2.511	5.311	time	$6^5 1211$	$6^5 0511$	$14^5 4711$

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