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A SEQUENTIAL ALGORITHM FOR SOLVING A SYSTEM OF NONLINEAR EQUATIONS*1)

Fang Kai-tai
(*Institute of Applied Mathematics, Academia Sinica, Beijing, China*)

Wang Yuan
(*Institute of Mathematics, Academia Sinica, Beijing, China*)

Abstract

A sequential algorithm for solving a system of nonlinear equations based on the number-theoretic method is proposed. In order to illustrate the effectiveness of the method, the following two problems are discussed in detail: the problems for finding out a representative point of a continuous univariate distribution, and a fixed point of a continuous mapping of a closed bounded domain into itself.

§1. Introduction

Suppose D is a domain of $\mathbf{R}^s$. We want to solve the system of equations

$$\begin{cases} f_1(\mathbf{x}) = f_1(x_1, \cdots, x_s) = 0, \\ \qquad \cdots\cdots \\ f_t(\mathbf{x}) = f_t(x_1, \cdots, x_s) = 0. \end{cases} \tag{1.1}$$

There are many well-known methods for solving (1.1) if f_i's are all linear, but it is difficult to find out an analytic expression for the solutions of (1.1) in usual if f_i's are not all linear functions, so that (1.1) can be solved only by numerical methods, for example (see [5]), the iteration method (see [1]), Newton's method (see [6]), Brown's method[3], Brent's method[2], quasi Newton's method (see [5]), etc. However, the above methods are contained in detail in a book of Feng [6]. These methods require that f_i's have continuous derivatives of first order or even higher orders, or satisfy certain properties of convexity in order that the convergences of these methods are ensured. It is difficult to obtain the explicit formulas of derivatives of the functions f_i's, and sometimes f_i's even do not satisfy the required conditions, for instance, max, min and $|x|$ appear in the expressions of f_i's.

In fact the problem for solving the system of equations (1.1) can be reduced to a problem of optimization. Let

$$L(\mathbf{x}) = \sum_{i=1}^{t} |f_i(\mathbf{x})|, \quad \mathbf{x} \in D \tag{1.2}$$

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or

$$\tilde{L}(x) = \sum_{i=1}^{t} f_i^2(x), \quad x \in D. \tag{1.3}$$

Then the problem of finding out a solution $x_0 = (x_{01}, \cdots, x_{0s})$ of (1.1) is equivalent to the problem of finding out a point such that $L(x)$ (or $\tilde{L}(x)$) attains its minimum. Notice that such points that $L(x)$ (or $\tilde{L}(x)$) attains its minimum $M = 0$ are not unique in general, and our aim is to find out at least one among them.

We have proposed a sequential algorithm for optimization based on the number-theoretic method, and it is denoted by SNTO [9]. The continuities are required only for the functions f_i's in SNTO such that the convergences of the approximate minimum M^* and the maximum point x^* to the respective M and x_0 are ensured. Besides, it is easy to work out a program in SNTO, and more precisely the programs are almost the same for distinct sets of f_i's. It is the aim of our paper to recommend SNTO for finding an approximate minimum point of L (or $\tilde{L}$), that is, an approximate solution of (1.1). In Section 2, we give SNTO in detail. In order to illustrate the effectiveness and also universality of SNTO, we apply SNTO to treat two problems. The first is the so-called quantization problem. Let X be a random variable with a continuous cumulative distribution function $F(x)$ with a standard deviation 1 and n be a given positive integer. For any given numbers $-\infty < x_1 < x_2 < \cdots < x_n < \infty$, an n-level quantizer Q_n is defined by

$$Q_n(x) = x_k, \quad \text{if } a_k < x \leq a_{k+1}, \quad k = 1, \cdots, n,$$

where

$$a_1 = -\infty, \quad a_{n+1} = \infty, \quad a_k = (x_k + x_{k-1})/2, \quad k = 2, \cdots, n.$$

We use the mean square error (MSE)

$$\text{MSE}(x) = E(X - Q_n(X))^2 = \int_{-\infty}^{\infty} \min(x - x_i)^2 p(x) dx \tag{1.4}$$

to measure the distortion between X and $Q_n(X)$, where $x = (x_1, \cdots, x_n)$ and $p(x)$ denotes the probability density function (pdf) of $F(x)$. We shall call x^* a representative point of $F(x)$ if it has the least MSE, i.e. $\text{MSE}(x^*) = \min_x \text{MSE}(x)$. The problem of finding out a representative point appears in many fields, such as information theory, clustering analysis, theory of quantization and theory of stochastic simulation. Max[12], Lloyd[11], and Fang and He[7] proposed independently numerical methods for finding out the representative points. Their methods are the same in essence, where as Fang and He's method is to reduce the above problem to a problem of solving a system of nonlinear equations (cf. (2.2)). In this paper, we shall give a numerical method for finding out a representative point based on the number-theoretic method.

Let f be a continuous mapping which maps a closed bounded domain D into itself. We shall call x_0 a fixed point of f if $f(x_0) = x_0$. Our second problem is to find out a fixed point of f. This problem is close to the problem of solving a system of nonlinear equations, and there appeared several related monographs in recent years, for instances, [13], [14] and [15].

Let

$$L(x) = ||x - f(x)||, \quad x \in D, \tag{1.5}$$

where $||\cdot||$ denotes l_1 or l_2 modulus. Then the problem of finding out a fixed point of $f(x)$ is reduced to the problem of finding out a point $x = x_0$ of D such that $L(x)$ attains its minimum at x_0. Hence we may use SNTO also to find out an approximate fixed point.

These two problems will be discussed in detail in Section 3 and 4.

§2. SNTO

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors of $\boldsymbol{R}^s$, where $a_i < b_i$, $i = 1, \cdots, s$. We use $[\boldsymbol{a}, \boldsymbol{b}]$ to denote the rectangle $[a_1, b_1] \times \cdots \times [a_s, b_s]$. First of all, we give the sequential algorithm for solving (1.1) as follows:

1) Take n_1 points $P^{(1)} = \{\boldsymbol{y}_k^{(1)} = (y_{k1}^{(1)}, \cdots, y_{ks}^{(1)}),\ k = 1, \cdots, n_1\}$ which are uniformly scattered on $D^{(1)} = [\boldsymbol{a}, \boldsymbol{b}]$ by the methods of Konobov, Hua and Wang (K-H-W) (cf. [9]). Find out the minimum $M^{(1)}$ and a mininum point $\boldsymbol{x}^{(1)}$ of $L(\boldsymbol{x})$ on $P^{(1)}$, i.e,

$$L(\boldsymbol{x}^{(1)}) = \min_{1 \le k \le n_1} L(\boldsymbol{y}_k^{(1)}).$$

2) The domain $D^{(1)}$ is contracted to $D^{(2)} = [\boldsymbol{a}^{(2)}, \boldsymbol{b}^{(2)}]$, where $\boldsymbol{a}^{(2)} = (a_1^{(2)}, \cdots, a_s^{(2)})$, $\boldsymbol{b}^{(2)} = (b_1^{(2)}, \cdots, b_s^{(2)})$, $a_i^{(2)} = \max(x_i^{(1)} - c_i^{(1)}/2, a_i)$, $b_i^{(2)} = \min(x_i^{(1)} + c_i^{(1)}/2, b_i)$, $i = 1, \cdots, s$, and $\boldsymbol{c}^{(1)} = (\boldsymbol{b} - \boldsymbol{a})/2 = (c_1^{(1)}, \cdots, c_s^{(1)})$. Then, take n_2 points $P^{(2)} = \{\boldsymbol{y}_k^{(2)},\ k = 1, \cdots, n_2\}$ which are uniformly scattered on $D^{(2)}$, and find out the minimum $M^{(2)}$ and a minimum point of $L(\boldsymbol{x})$ on $P^{(1)} \cup P^{(2)}$.

3) Suppose that the domain in the t th step is $D^{(t)} = [\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)}]$ and the corresponding set of points on $D^{(t)}$ is $P^{(t)} = \{\boldsymbol{y}_k^{(t)},\ k = 1, \cdots, n_t\}$, and that the minimum and a minimum point of $L(\boldsymbol{x})$ on $P^{(1)} \cup \cdots \cup P^{(t)}$ are $M^{(t)}$ and $\boldsymbol{x}^{(t)}$ respectively. Let δ be a pre-assigned positive number which is used to control the process of the algorithm: If $\max_{1 \le i \le s} c_i^{(t)} = \max_{1 \le i \le s} \frac{1}{2}(b_i^{(t)} - a_i^{(t)}) < \delta$, then the process is stoped, and $M^{(t)}$ and $x^{(t)}$ are considered to be the approximations of $M = 0$ and $\boldsymbol{x}_0$. Otherwise, it enters into the $(t+1)$ th step:

Let $a_i^{(t+1)} = \max(x_i^{(t)} - c_i^{(t)}/2, a_i)$, $b_i^{(t+1)} = \min(x_i^{(t)} + c_i^{(t)}/2, b_i)$, $i = 1, \cdots, s$, $\boldsymbol{a}^{(t+1)} = (a_1^{(t+1)}, \cdots, a_s^{(t+1)})$, $\boldsymbol{b}^{(t+1)} = (b_1^{(t+1)}, \cdots, b_s^{(t+1)})$ and $D^{(t+1)} = [\boldsymbol{a}^{(t+1)}, \boldsymbol{b}^{(t+1)}]$. Take n_{t+1} points $P^{(t+1)} = \{\boldsymbol{y}_k^{(t+1)},\ k = 1, \cdots, n_{t+1}\}$ which are uniformly scattered on $D^{(t+1)}$. Then find out the minimum $M^{(t+1)}$ and a minimum point $\boldsymbol{x}^{(t+1)}$ of $L(\boldsymbol{x})$ on $P^{(1)} \cup \cdots \cup P^{(t+1)}$, and return to 3) by using $t+1$ instead of t.

Now suppose D has a parameter representation

$$x_i = x_i(\phi_1, \cdots, \phi_t) = x_i(\boldsymbol{\phi}),\ i = 1, \cdots, s,$$

where $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_t) \in [0,1]^t$, $t \le s$ and ϕ_i's are independent in the sense of statistics (cf. [16]). Given a set of points $\{\boldsymbol{b}_k,\ k = 1, \cdots, n\}$ which are uniformly scattered on $[0,1]^t$, we can obtain a set of points $P = \{\boldsymbol{y}_k,\ k = 1, \cdots, n\}$ uniformly scattered on D (cf. [16]). Let $\boldsymbol{x}$ be a point of P such that $P(\boldsymbol{x})$ attains its minimum at $\boldsymbol{x}$ among the points of P, i.e, $L(\boldsymbol{x}) = \min_{1 \le k \le n} L(\boldsymbol{y}_k)$. Suppose that $\boldsymbol{x}$ corresponds to $\boldsymbol{b}^*$ of $\{\boldsymbol{b}_k\}$. Then a rectangle $[\boldsymbol{a}, \boldsymbol{b}]$ with centre $\boldsymbol{b}^*$ corresponds a domain of D which includes $\boldsymbol{x}$. Using this correspondence, we can define the process for contraction of D by means of the contractions on $[0,1]^t$ stated above.

Since the method mentioned here for optimization is just the SNTO, we call SNTO the method for solving the system of nonlinear equations.

§3. Representative Points of a Continuous Univariate Distribution

Let $F(x)$ be a given distribution function which has pdf $p(x)$. We may assume without loss of generality that it has variance 1. Given $\boldsymbol{x} = (x_1, \cdots, x_n)$, where $x_1 < \cdots < x_n$, we have a quantizer $Q_n(x)$ and use MSE (cf. (1.4)) to measure the distortion between X and $Q_n(X)$. By (1.4) we have

$$\mathrm{MSE}(\boldsymbol{x}) = \int_{-\infty}^{\infty} \min_{1 \le k \le n} (x - x_k)^2 p(x) dx. \tag{3.1}$$

Let

$$TR_n = \{(x_1, \cdots, x_n) : x_1 < \cdots < x_n\}.$$

The so-called representative point $\boldsymbol{x}^* = (x_1^*, \cdots, x_n^*)$ is defined such that $\mathrm{MSE}(\boldsymbol{x}^*)$ attains its minimum on TR_n at $\boldsymbol{x}^*$, i.e,

$$\mathrm{MSE}(\boldsymbol{x}^*) = \min_{\boldsymbol{x} \in TR_n} \mathrm{MSE}(\boldsymbol{x}).$$

It is obvious that

$$MSE(\boldsymbol{x}) = \int_{-\infty}^{(x_1+x_2)/2} (x - x_1)^2 p(x) dx + \int_{(x_1+x_2)/2}^{(x_2+x_3)/2} (x - x_2)^2 p(x) dx + \cdots$$
$$+ \int_{(x_{n-1}+x_n)/2}^{\infty} (x - x_n)^2 p(x) dx.$$

Using the ralations ∂ MSE $(\boldsymbol{x})/\partial x_j = 0$, $j = 1, \cdots, n$, for minimization of MSE we have a system of equations

$$\begin{cases} f_1(\boldsymbol{x}) = \int_{-\infty}^{(x_1+x_2)/2} (x - x_1) p(x) dx = 0, \\ f_2(\boldsymbol{x}) = \int_{(x_1+x_2)/2}^{(x_2+x_3)/2} (x - x_2) p(x) dx = 0, \\ \quad \cdots\cdots \\ f_n(\boldsymbol{x}) = \int_{(x_{n-1}+x_n)/2}^{\infty} (x - x_n) p(x) dx = 0. \end{cases} \tag{3.2}$$

Hence the problem of finding out a representative point is reduced to the problem of finding out a solution of (3.2).

Suppose $F(x)$ is the standard normal distribution. We have $x_i^* = -x_{n+1-i}^*$, $i = 1, \cdots, \frac{n+1}{2}$, by the symmetry of the distribution density with respective to the origin, and therefore $x_{(n+1)/2}^* = 0$ if n is an odd number. So we need only to find out the nonnegative coordinates $x_1^*, \cdots, x_s^*$ of $\boldsymbol{x}^*$, where $0 \le x_1^* < \cdots < x_s^*$ and $s = [n/2]$ in which $[x]$ denotes the integral part of x. Consequently, (3.2) is reduced to

$$\begin{cases} f_1(\boldsymbol{x}) = \phi(v) - \phi(\frac{x_1+x_2}{2}) - x_1(\Phi(\frac{x_1+x_2}{2}) - \Phi(v)) = 0, \\ f_2(\boldsymbol{x}) = \phi(\frac{x_1+x_2}{2}) - \phi(\frac{x_2+x_3}{2}) - x_2(\Phi(\frac{x_2+x_3}{2}) - \Phi(\frac{x_1+x_2}{2})) = 0, \\ \cdots\cdots \\ f_s(\boldsymbol{x}) = \phi(\frac{x_{s-1}+x_s}{2}) - x_s(1 - \Phi(\frac{x_{s-1}+x_s}{2})) = 0, \end{cases}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \Phi(x) = \int_{-\infty}^{x} \phi(t)dt$$

and

$$v = \begin{cases} 0, & \text{if } n \text{ even}, \\ x_1/2, & \text{otherwise}. \end{cases}$$

The following results show that the advantages of SNTO are not only on its simple algorithm but also on its precision compared with some other known methods (cf. [12], [11] and [7]). Let

$$L(\boldsymbol{x}) = \sum_{i=1}^{s} |f_i(\boldsymbol{x})|, \quad \boldsymbol{x} \in TR_s^+, \tag{3.4}$$

where

$$TR_s^+ = \{(x_1, \cdots, x_s) : \ 0 \leq x_1 < \cdots < x_s\}.$$

The problem of finding out a representative point $\boldsymbol{x}^*$ is equivalent to the problem of finding out a minimum point of $L(\boldsymbol{x})$ on TR_s^+. It is known by the properties of standard normal distribution that $\boldsymbol{x}^*$ must fall in the region

$$TR_s^+(B) = \{(x_1, \cdots, x_s) : \ 0 \leq x_1 < \cdots < x_s < B\},$$

where B is a certain constant, for example, we may take $B = 3.5$ if $10 < n \leq 30$ and $B = 3.0$ if $n \leq 10$. Let $D = TR_s^+(B)$. We can obtain the approximation of a representative point $\boldsymbol{x}^*$ by SNTO. The results are given in Table 1, where the calculations are made on an IBM PC/XT, and we always take $n_2 = n_3 = \cdots$. In Table 1, we use N to denote the total number of points for calculation, T the number of contractions for the domain $DR_s^+(B)$, $\boldsymbol{x}^{**}$ a representative point obtained by SNTO, $\boldsymbol{x}^*$ a representative point given in [7] which is slightly better than the result in [12], and $L(\boldsymbol{x}^{**})$ and $L(\boldsymbol{x}^*)$ the respective values of the function $L(\boldsymbol{x})$ at $\boldsymbol{x}^{**}$ and $\boldsymbol{x}^*$. Table 1 shows that $\boldsymbol{x}^{**}$ has higher precision than $\boldsymbol{x}^*$ given by [7] and [12]. Table 2 gives the values of $\boldsymbol{x}^{**}$ for $n \leq 10$.

Table 1. Comparison between SNTO and Fang-He's methods

n	n_1	n_2	T	N	$L(x^{**})$	$L(x^*)$
4	144	89	20	1835	1.490E-8	3.308E-6
5	144	89	20	1835	1.490E-8	6.124E-6
6	701	199	22	4880	1.490E-8	1.305E-5
7	701	199	23	5079	1.490E-8	1.882E-5
8	1069	523	22	12052	7.078E-8	2.424E-5
9	1069	523	22	12052	2.235E-8	7.964E-3
10	4001	1069	22	26450	1.576E-6	3.400E-5

Table 2. Representative points of standard normal distribution

n	x_1	x_2	x_3	x_4	x_5	$L(x^*)$
4	.452780	1.51042				
5	.764567	1.72415				
6	.317717	1.00011	1.89360			
7	.560579	1.18814	2.03336			
8	.245096	.756014	1.34392	2.15195		
9	.443636	.918791	1.47639	2.25465		
10	.199614	.609828	1.05777	1.59125	2.34488	

Remark. 1) If

$$\tilde{L}(x) = \sum_{i=1}^{s} f_i^2(x)$$

is used instead of (3.4), the results for the same n_1, n_2, T are all the same as given in Tables 1 and 2. It is worth mentioning that the results given by in these two functions are not always the same; the results are the same if $N \to \infty$ and differences may appear if $N(n_1, n_2)$ is not large.

2) Substituting the expressions of f_i's in (3.2) into (3.4), we know that the algorithm given here can apply also to any one-dimensional continuous distribution.

3) Although the algorithm is used for finding out a minimum point on TR_n, the problem is really to find out a minimum point on a rectangle by the properties of representative points. More precisely, we use $x_1^{(n)} < \cdots < x_n^{(n)}$ $(n = 1, 2, \cdots)$ to denote representative points of the distribution function $F(x)$. Therefore we have

$$-B < x_1^{(n)} < x_1^{(n-1)} < x_2^{(n)} < x_2^{(n-1)} < \cdots < x_{n-1}^{(n-1)} < x_n^{(n)} < B,$$

where B is a bound of $\{x_j^{(n)}\}$. $x_1^{(1)}$ is easy to find out in general. Next, find out $(x_1^{(2)}, x_2^{(2)})$ in $(-B, x_1^{(1)}) \times (x_1^{(1)}, B)$, and then $(x_1^{(3)}, x_2^{(3)}, x_3^{(3)})$ in $(-B, x_1^{(2)}) \times (x_1^{(2)}, x_2^{(2)}) \times (x_2^{(2)}, B)$ and so on. Since the points we use for calculation are scattered uniformly on a rectangle, there is no need to transform them to a TR_n and thus the amount of calculations can be further decreased. Besides, the scope of solutions is reduced.

§4. Fixed Points

Let f be a continuous mapping which maps D into D. We want to find out a fixed point x_0 of f, i.e.,

$$x_0 = f(x_0). \tag{4.1}$$

Suppose $L(x)$ is defined by (1.5). Then, the problem of finding out x_0 is equivalent to the problem of finding out a minimum point x such that $L(x)$ attains its minimum $M = 0$. We shall give an example to illustrate the efficiency of SNTO for finding out x_0.

Example. Let

$$D = \{(x_1, x_2, x_3) : \ x_i \geq 0, \ i = 1, 2, 3, \ x_1 + x_2 + x_3 = 1\}$$

be a simplex. Let

$$\begin{cases} y_1 = (x_1 + 2x_2 + 3x_3)/S, \\ y_2 = (4x_1 + 5x_2 + 6x_3)/S, \\ y_3 = (7x_1 + 8x_2 + 9x_3)/S, \end{cases}$$

where

$$S = 12x_1 + 15x_2 + 18x_3.$$

This is a continuous mapping which maps D into D with $y_2 \equiv 1/3$. We want to find out a fixed point of this mapping. We use l_1 modulus in (1.5). Take $n_1 = 233$ and $n_2 = 144$. By SNTO with 18 domain contractions, we have the approximations of M and x_0 as follows: $M^{(18)} = 2.98 \times 10^{-7}$, and $x^{(18)} = (x_1^{(18)}, x_2^{(18)}, x_3^{(18)})$, where $x_1^{(18)} = 0.1471927$, $x_2^{(18)} = 0.3333332$, $x_3^{(18)} = 0.5194746$, which are very close to $M = 0$ and $x_0 = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ where $x_1^{(0)} = 0.1471927$, $x_2^{(0)} = 0.3333333$, $x_3^{(0)} = 0.5194740$.

The design of domain contraction in SNTO is to shorten each edge length of the original rectangle by one-half time, and thus the volume of the resulting rectangle is 2^{-s} times the original one; here 1/2 is called the contraction ratio. Is it possible to contract the domain still faster? i.e, a small number ε less than 1/2 is used instead of 1/2 in the contraction. The answer is confirmative in some cases. Now we shall use contraction ratios 1/2, 1/4, 1/8, 1/16 in our example for comparisons, and we see that the amount of calculations can be reduced at least one-half time to obtain a result with still higher precision if a number $\varepsilon \leq 1/4$ is used instead of 1/2 in the contraction. The results are given in Table 3, where T denotes also the number of contractions. In general, n_2 cannot be chosen too small if the contraction ratio ε is small.

Table 3. Comparison of the results for different contraction ratios

contraction ratio	n_1	n_2	T	$L(x)$
1/2	233	144	18	2.98E-7
1/2	233	144	9	1.98E-4
1/4	233	144	9	9.54E-7
1/8	233	144	9	9.54E-7
1/16	233	144	9	5.96E-8

References

[1] E. Allgower and K. Georg, Simplical and continuation method for approximating fixed points and solutions to systems of equations, *SIMA Review*, **22** (1980), 28-84.

[2] R.P. Brent, Some efficient algorithms for solving systems of nonlinear equations, *SIAM, J.Numer.Anal.*, **10** (1973), 327-344.

[3] K.M. Brown, Solution of simultaneous nonlinear equations, *Comm. ACM*, **10** (1967), 728-729.

[4] S. Cambanis and N.L. Gerr, A simple class of asymptotically optimal quantizers, *IEEE Trans. Inform Theory*, IT-**29** (1983), 664-676.

[5] J.E. Dennis and J.J. Moré, Quasi-Newton methods, motivation and theory, *SIAM Review*, **19** (1977), 46-89.

[6] G.C. Feng, Iterative Methods of a System of Nonlinear Equations, Shanghai Press of Science and Technology, 1989.

[7] K.T. Fang and S.D. He, The problem of selecting a given number of representative points in a normal population and a generalized Mill's ratio, Technical Report, No.5, U.S. Army Research Office Contract DAAG 29-82-K-0156, Department of Stanford University, California, 1982.

[8] K.T. Fang and S.D. He, The problem of selecting a specified number of representative points from a noumal population, *Acta Math. Appl. Sinica*, **7** (1984), 293-306.

[9] K.T. Fang and Y. Wang, A sequential algorithm for optimization and its applications to regression analysis, Technical Report, 002, Institute of Applied Mathematics, Academia Sinica, Beijing, 1989.

[10] L.K. Hua and Y. Wang, Applications of Number Theory to Numerical Analysis, Springer-Verlag and Science Press, 1981.

[11] S.P. Lloyd, Least squares quantization in PCM, *IEEE Trans. Inform. Theory*, IT-**28** (1982), 129-137.

[12] J. Max, Quantizing for minimum distortion, *IRE Trans. Trans. Inform. Theory*, IT-**6**, (1960), 7-12.

[13] M. Todd, The Computation of Fixed Points and Applications, Springer Lecture Note in Econ. and Math. Systems, Berlin, 1976.

[14] Z.K. Wang, Basic Algorithm for Fixed Points, Publishing House of Zhongshan University, 1986.

[15] Z.K. Wang, Algorithms for Fixed Points, Shanghai Press of Science and Technology, 1987.

[16] Y. Wang and K.T. Fang, Number theory methods in applied statistics, *Chinese Anal. Math*, 1990.