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A NOTE ON CONVERGENCE OF SYMPLECTIC SCHEMES FOR HAMILTONIAN SYSTEMS*1)

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Abstract

In this note we prove that all canonical (or symplectic) schemes for Hamiltonian systems constructed in [1-3] are convergent.

In [1-3] Feng and his colleagues proposed a systematical method for the construction of canonical schemes with arbitrary order of accuracy for the Hamiltonian system

$$\frac{dz}{dt} = JH_z, \quad z \in U \subset R^{2n}. \tag{1}$$

In this note we shall prove, using the method in [4], that all these canonical schemes are convergent.

A normal Darboux matrix

$$\begin{aligned} \alpha &= \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix} = \begin{pmatrix} J & -J \\ \frac{1}{2}(I+JB) & \frac{1}{2}(I-JB) \end{pmatrix}, \quad B' = B, \\ \alpha^{-1} &= \begin{pmatrix} A^\alpha & B^\alpha \\ C^\alpha & D^\alpha \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(JBJ-J) & I \\ \frac{1}{2}(JBJ+J) & I \end{pmatrix} \end{aligned} \tag{2}$$

define a linear transformation in the product space $R^{2n} \times R^{2n}$ by

$$\begin{pmatrix} \hat{w} \\ w \end{pmatrix} = \alpha \begin{pmatrix} \hat{z} \\ z \end{pmatrix}, \quad \begin{pmatrix} \hat{z} \\ z \end{pmatrix} = \alpha^{-1} \begin{pmatrix} \hat{w} \\ w \end{pmatrix},$$

i.e.

$$\hat{w} = J\hat{z} - Jz, \quad w = \frac{1}{2}(I+JB)\hat{z} + \frac{1}{2}(I-JB)z \quad B' = B. \tag{3}$$

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Let $z \to \hat{z} = g(z,t)$ be the phase flow of the Hamiltonian system (1); it is a time-dependent canonical map. There exist , for sufficiently small $|t|$ and in (some neighborhood of) R^{2n}, a time-dependent gradient map $w \to \hat{w} = f(w,t)$ with Jacobian $f_w(w,t) \in sm(2n)$ (i.e. everywhere symmetric) and a time-dependent generating function $\phi = \phi_{\alpha,H}(w,t)$ such that

$$f(w,t) = \nabla\phi_{\alpha,H}(w,t), \quad A_\alpha g(z,t) + B_\alpha z \equiv (\nabla\phi)(C_\alpha g(z,t) + D_\alpha z, t). \tag{4}$$

On the other hand, for a given time-dependent scalar function $\psi(w,t) : R^{2n} \times R \to R$, we can get a time-dependent canonical map $\tilde{g}(z,t)$. If $\psi(w,t)$ approximates the generating function $\phi_{\alpha,H}(w,t)$ of the Hamiltonian system (1), then $\tilde{g}(z,t)$ approximates the phase flow $g(z,t)$.

For sufficiently small $\tau > 0$ as the time-step, define

$$\psi^{(m)} = \sum_{k=1}^{m} \phi^{(k)}(w)\tau^k, \tag{5}$$

where $\phi^{(1)}(w) = -H(w)$, and for $k \geq 0$, $A^\alpha = \frac{1}{2}(JBJ - J)$,

$$\begin{aligned} \phi^{(k+1)}(w) = \frac{-1}{k+1} \sum_{m=1}^{k} \frac{1}{m!} \sum_{i_1,\cdots,i_n=1}^{2n} H_{z_{i_1}\cdots z_{i_m}}(w) \\ \times \sum_{\substack{j_1+\cdots+j_m=k \\ j_l \geq 1}} (A^\alpha\nabla\phi^{(j_1)}(w))_{i_1} \cdots (A^\alpha\nabla\phi^{(j_m)}(w))_{i_m}. \end{aligned} \tag{6}$$

Then, $\psi^{(m)}(w,\tau)$ is the m-th approximant of $\phi_{\alpha,H}(w,\tau)$, and the gradient map

$$w \to \hat{w} = \tilde{f}(w,\tau) = \nabla\psi^{(m)}(w,\tau) \tag{7}$$

defines a canonical map $z \to \hat{z} = \tilde{g}(z,\tau)$ implicitly by equation

$$A_\alpha\hat{z} + B_\alpha z = (\nabla\psi^{(m)})(C_\alpha\hat{z} + D_\alpha z, \tau). \tag{8}$$

An implicit canonical difference scheme

$$z = z^k \to \hat{z} = z^{k+1} = \tilde{g}(z^k, \tau) \tag{9}$$

for system (1) is obtained in [1-3], and this scheme is of m-th order of accuracy [5]. For the sake of simplicity, we denote $\tilde{g}_\tau(z) = \tilde{g}(z,\tau)$. Then,

$$\tilde{g}_0(z) = z, \qquad \left.\frac{d^i\tilde{g}_\tau(z)}{d\tau^i}\right|_{\tau=0} = \left.\frac{d^i g_\tau(z)}{d\tau^i}\right|_{\tau=0}, \tag{10}$$

where $g_\tau(z)$ is the phase flow $g(z,\tau)$.

Theorem. *If H is analytical in $U \subset R^{2n}$, then scheme (9) is convergent with m-th order of accuracy.*

Proof. For the step-forward operator $\tilde{g}_\tau$ we set

$$z_1 = \tilde{g}_\tau(z), \quad z_2 = \tilde{g}_\tau(z_1), \quad \cdots z_k = \tilde{g}_\tau(z_{k-1}),$$

we have $z^k = \tilde{g}_\tau^k$.

First, we prove that the convergence holds locally. We begin by showing that for any z_0 the iterates $\tilde{g}^n_{t/k}, n \leq k$, are defined if t is sufficiently small. Indeed, on a neighborhood of z_0, $\tilde{g}_\tau(z) = z + o(\tau)$. Thus, if $\tilde{g}^l_{t/k}(z)$ is defined for z in the neighborhood of z_0, $l = 1, 2, \cdots, n-1$,

$$\begin{aligned}\tilde{g}^n_{t/k}(z) - z &= (\tilde{g}^n_{t/k}(z) - g^{n-1}_{t/k}(z)) + (\tilde{g}^{n-1}_{t/k}(z) - g^{n-2}_{t/k}(z)) + \cdots + (\tilde{g}_{t/k}(z) - z) \\ &= \underbrace{o(t/k) + \cdots + o(t/k)}_{n} = o(t).\end{aligned}$$

This is small which independent of k for sufficiently small t. So, $\tilde{g}^n_{t/k}$, $n \leq k$, is defined and remains in U_{z_0} for z near z_0.

Since to H is analytical, for any z_1, $z_2 \in U_{z_0}$, there exists a constant C such that

$$\| JH_z(z_1) - JH_z(z_2) \| \leq \| J \| \| H_z(z_1) - H_z(z_2) \| \leq C \| z_1 - z_2 \| .$$

Let $F(t) = \| g(z_1, t) - g(z_2, t) \|$, where $g(z_i, t) = z_i + \int_0^t JH_z(g(z_i, s))ds, i = 1, 2,$

$$\begin{aligned}F(t) &= \left\| \int_0^t JH_z(g(z_1, s))ds - \int_0^t JH_z(g(z_2, s))ds + z_1 - z_2 \right\| \\ &\leq \| z_1 - z_2 \| + C \int_0^t F(s)ds.\end{aligned}$$

Using Gronwall's inequality, we have $F(t) = \| g(z_1, t) - g(z_2, t) \| \leq e^{C|t|} \| z_1 - z_2 \|, g_t(z) - \tilde{g}^k_{t/k}(z) = g^k_{t/k}(z) - \tilde{g}^k_{t/k}(z) = g^{k-1}_{t/k} g_{t/k}(z) - g^{k-1}_{t/k} \tilde{g}_{t/k}(z) + g^{k-2}_{t/k} g_{t/k}(Y_1) - g^{k-2}_{t/k} \tilde{g}_{t/k}(Y_1) + \cdots + g^{k-l}_{t/k} g_{t/k}(Y_{l-1}) - g^{k-l}_{t/k} \tilde{g}_{t/k}(Y_{l-1}) + \cdots + g_{t/k}(Y_{k-1}) - \tilde{g}_{t/k}(Y_{k-1})$ where $Y_l = \tilde{g}^l_{t/k}(z)$. Noting (11), we have

$$\begin{aligned}\| g_t(z) - \tilde{g}^k_{t/k}(z) \| &\leq \sum_{l=1}^{k} \exp\{\frac{C|k-l||t|}{k}\} \| g_{t/k}(Y_{l-1}) - \tilde{g}_{t/k}(Y_{l-1}) \| \\ &\leq k \exp\{C|t|\} o(t/k)^m \to 0 \quad \text{as} \quad k \to \infty\end{aligned}$$

since $g_\tau(z) - \tilde{g}_\tau(z) = o(\tau)^m$ by (10).

Now suppose $g(z, t)$ is defined for $0 \leq t \leq T$. We shall show that $\tilde{g}^k_{t/k}(z)$ converges to $g(z, t)$. By the above proof and compactness, if N is large enough, $g_{t/N} = \lim_{k \to \infty} \tilde{g}^k_{t/kN}$ uniformly on a neighborhood of the curve $t \to g_t(z)$. Thus, for $0 \leq t \leq T$, $g_t(z) = g^N_{t/N} = \lim_{k \to \infty} (\tilde{g}^k_{t/kN})^N(z)$. By the uniformity of t, $g_t(z) = \lim_{k \to \infty} \tilde{g}^k_{t/k}(z)$. From the proof one can see that, if H is not analytical but H_z satisfies the local Lipschitz condition, then scheme (9) with order $m = 1$ is convergent .

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