Journal of Computational Mathematics, Vol.10, No.4, 1992, 376–387.

SECOND-ORDER METHODS FOR SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

In this paper we discuss the numerical methods with second-order accuracy for solving stochastic differential equations. An unbiased sample approximation method for $I_n = \int_{t_n}^{t_{n+1}} (B_u - B_{t_n})^2 du$ is proposed, where $\{B_u\}$ is a Brownian motion. Then second-order schemes are derived both for scalar cases and for system cases. The errors are measured in the mean square sense. Several numerical examples are included, and numerical results indicate that second-order schemes compare favorably with Euler's schemes and 1.5th-order schemes.

§1. Introduction

In this paper we discuss an approach of numerical solution for stochastic differential equations (abbreviated SDE) with second-order accuracy.

Assume that $\underline{B}_t$ is an m-dimensional Brownian motion on $(\Omega, \Im, P)$, and $\Im_t = \sigma(\underline{B}_s, s \le t)$ is an increasing family of sub-sigma-algebra of $\Im$.

Consider a SDE on $(\Omega, \Im_t, \Im, P)$, as in [1] or [15]:

$$dX(t) = b(X(t), t)dt + \sigma(X(t), t)dB_t, \quad \underline{X}(0) = X_0, \tag{1.1}$$

where $\underline{b}$ and σ are two sufficiently smooth functions satisfying the Lipshitz condition with respect to t, x. For simplicity, we only consider σ independent of t and $\underline{x}$, but the proof given here is valid for the general case without any more essential difficulties.

SDE (1.1) has exerted a profound impact on the modeling and analysis of problems in physics [2], chemistry [3], biology [4] and other fields [5, 6]. So more and more authors pay their attention to the numerical method for solving the SDE, such as H.J. Kushner, J.M.C. Clark, C.C. Chang [7,8,9,10,16]. Since solutions of the PDE can be expressed as functionals of solutions of the SDE, numerical solutions of the SDE can also be applied

* Received May 12, 1989.

to numerical calculations of solutions of the PDE. H.J. Kushner, N.J. Rao succeeded in obtaining numerical solutions of the PDE by computing functionals of the numerical solutions of the SDE [7,12]. However, the numerical methods of highest order accuracy for solving the SDE used by the authors above are of order 1.5 (a numerical solution X_n^t of equation (1.1) is said to be of order α if

$$E\Big(\frac{\sum_{i=1}^N \psi(X_n^i)}{N} - E\psi(X(t_n))\Big)^2 \le \frac{C_1}{N} + C_2 h^{2\alpha},$$

where $X(t_n)$ is the solution of equation (1.1); see [16] or Section 3.). In order to get higher order accuracy, as pointed out by C.C.Chang and W.Rumelin [11,13], the main difficulty is how to simulate $I_n = \int_{t_n}^{t_{n+1}} (B_u - B_{t_n})^2 du$, where B_u is a one-dimensional Brownian motion. Because of the complexity of the distribution of I_n [14], it seems difficult to directly sample I_n. In this paper an unbiased sample approximation for I_n is proposed. Then second-order accuracy numerical methods both for the scalar case and the system case are drived. Numerical results also show that the second-order method produces errors smaller than the 1.5th-order method does.

§2. An Unbiased Sample Approximation for I_n

First, we outline the probability background for later use.

Definition 2.1 (martingale). *Suppose that the real valued stochastic process $Y(t)$ defined on $(\Omega, \Im, P)$ is adaptable to $\{\Im_t\}$ which is an increasing family of sub-sigma-algebra of $t \ge 0$. $\{Y(t), \Im_t, +\infty > t \ge 0\}$ is called a martingale, if $\forall t \ge 0$, $s \ge 0$, with probability one*

$$E|Y(t)| < +\infty, \qquad E(Y(t+s)|\Im_t) = Y(t).$$

Itô Formula. Let

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

and $F(x,t)$ be a continuous function on $R^n \times R^1$ together with $F(\cdot,t) \in C^1$ and $F(x,\cdot) \in C^2$. Then $F(x(t),t)$ satisfies

$$dF(X_t,t) = F_x(X_t,t)dX_t + F_t(X_t,t)dt + \frac{1}{2}F_{xx}(X_t,t)\sigma^2 dt.$$

For example, when $F(x) = X^2, dx(t) = dB_t$,

$$B_t^2 - B_s^2 = 2\int_s^t B_u dB_u + (t-s), \quad t > s > 0. \tag{2.1}$$

As stated earlier, to sample I_n directly seems difficult. Nevertheless, if we have a sufficiently good approximation of I_n which can be easily sampled, then the second-order scheme can still be obtained. Proposition 2.1 suggests a method for this sample approximation of I_n.

Suppose that

$$\Pi = (t_0 = 0, \ldots, t_{i+1} = t_i + h, \ldots, t_N = T)$$

is a partition of $[0,T]$, where $h = T/N$. $B_t^{(1)}, B_t^{(2)}$ are two independent Brownian motions. We define

$$I_n^{(k)} = \int_{t_n}^{t_{n+1}} (B_u^{(k)} - B_{t_n}^{(k)})^2 du, \quad k = 1,2,$$

$$J_n^{12} = \int_{t_n}^{t_{n+1}} (B_u^{(1)} - B_{t_n}^{(1)})(B_u^{(2)} - B_{t_n}^{(2)}) du.$$

Let $\{S_0^n, \dots, S_m^n\}$ be a subset of $[t_n, t_{n+1}]$, with $t_n = S_0^n < S_1^n < \dots < S_m^n = t_{n+1}$, $S_{t+1}^n - S_t^n = h/m$. (For notational convenience, denote $S_t^n = S_t$.)

$$\Theta_n^{12} = \sum_{i=1}^m \Big((B_{s_{i-1}}^{(1)} - B_{t_n}^{(1)}) \int_{s_{i-1}}^{s_i} (B_u^{(2)} - B_{s_{i-1}}^{(2)}) du + (B_{s_{i-1}}^{(2)} - B_{t_n}^{(2)}) \int_{s_{i-1}}^{s_i} (B_u^{(1)} - B_{s_{i-1}}^{(1)}) du + \frac{h}{m}(B_{s_{i-1}}^{(2)} - B_{t_n}^{(2)})(B_{s_{i-1}}^{(1)} - B_{t_n}^{(1)}) \Big)$$

$$\Gamma_n^{(k)} = 2\sum_{i=1}^m (B_{s_{i-1}}^{(k)} - B_{t_n}^{(k)}) \int_{s_{i-1}}^{s_i} (B_u^{(k)} - B_{s_{i-1}}^{(k)}) du + \frac{h}{m}\Big(\sum_{i=1}^m (B_{s_{i-1}}^{(k)} - B_{t_n}^{(k)})^2\Big) + \frac{h^2}{2m}, \quad k = 1,2.$$

Proposition 2.1. For any n, we have

i) $$EI_n^{(k)} = E\Gamma_n^{(k)} = \frac{1}{2}h^2, \quad k = 1,2, \tag{2.2}$$

ii) $$E(I_n^{(k)} - \Gamma_n^{(k)})^2 \le \frac{h^4}{m^3}, \quad k = 1,2, \tag{2.3}$$

iii) $$E\Theta_n^{12} = EJ_n^{12} = 0, \tag{2.4}$$

iv) $$E(\Theta_n^{12} - J_n^{12})^2 \le \frac{h^4}{m^3}. \tag{2.5}$$

Proof. In the proof of i) and ii) we omit the dimensional index k.

i)

$$\begin{aligned}\Gamma_n &= 2\sum_{i=1}^m (B_{s_{i-1}} - B_{t_n}) \int_{s_{i-1}}^{s_i} (B_u - B_{s_{i-1}}) du + \frac{h}{m}\Big(\sum_{i=1}^m (B_{s_{i-1}} - B_{t_n})^2\Big) + \frac{h^2}{2m} \\ &= 2\sum_{i=1}^m (B_{s_{i-1}} - B_{t_n}) \int_{s_{i-1}}^{s_i} (B_u - B_{s_{i-1}}) du \\ &\quad + \frac{h}{m}\Big(\sum_{i=1}^m \Big((B_{s_{i-1}} - B_{t_n})^2 - (S_{s_{i-1}} - S_0)\Big)\Big) + 0.5h^2. \end{aligned} \tag{2.6}$$

Equation (2.6) implies that $E\Gamma_n = 0.5h^2$.

ii) Since

$$\int_{t_n}^{t_{n+1}} (B_u - B_{t_n}) du = \sum_{i=1}^m \int_{s_{i-1}}^{s_i} (B_u - B_{t_n}) du$$

$$= \sum_{i=1}^{m} \left(\int_{s_{i-1}}^{s_i} (B_u - B_{s_{i-1}}) du + \int_{s_{i-1}}^{s_i} (B_{s_{i-1}} - B_{t_n}) du \right); \tag{2.7}$$

the substitution of equation (2.7) into equation (2.6) gives

$$\Gamma_n = 2\sum_{i=1}^{m} B_{s_{i-1}} \int_{s_{i-1}}^{s_i} (B_u - B_{s_{i-1}}) du + \frac{h}{m}\left(\sum_{i=1}^{m}(B_{s_{i-1}}^2 - B_{s_0}^2 - S_{s_{i-1}} + S_0)\right) + \frac{1}{2}h^2 - 2B_{t_n}\int_{t_n}^{t_{n+1}} (B_u - B_{t_n}) du. \tag{2.8}$$

On the other hand,

$$I_n = \int_{t_n}^{t_{n+1}} (B_u - B_{t_n})^2 du = \int_{t_n}^{t_{n+1}} (B_u^2 - B_{t_n}^2) du - 2B_{t_n}\int_{t_n}^{t_{n+1}} (B_u - B_{t_n}) du. \tag{2.9}$$

Using the Itô Formula, we have

$$I_n = 2\int_{t_n}^{t_{n+1}} \int_{t_n}^{u} B_v dB_v du + \frac{1}{2}h^2 - 2B_{t_n}\int_{t_n}^{t_{n+1}} (B_u - B_{t_n}) du \tag{2.10}$$

$$= 2\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \int_{s_{i-1}}^{u} B_v dB_v du + 2\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \int_{t_n}^{s_{i-1}} B_v dB_v du + \frac{1}{2}h^2 - 2B_{t_n}\int_{t_n}^{t_{n+1}} (B_u - B_{t_n}) du \tag{2.11}$$

$$= 2\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \int_{s_{i-1}}^{u} B_v dB_v du + \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} (B_{s_{i-1}}^2 - B_{s_0}^2 - S_{s_{i-1}} + S_0) du + 0.5h^2 - 2B_{t_n}\int_{t_n}^{t_{n+1}} (B_u - B_{t_n}) du. \tag{2.12}$$

Subtracting equation (2.8) from equation (2.12), we have

$$I_n - \Gamma_n = 2\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \int_{s_{i-1}}^{u} (B_u - B_{s_{i-1}}) dB_v du.$$

In view of the properties of the martingale, we deduce that

$$E(I_n - \Gamma_n)^2 = 4\sum_{i=1}^{m} E\left(\int_{s_{i-1}}^{s_i} \int_{s_{i-1}}^{u} (B_u - B_{s_{i-1}}) dB_v du \right)^2$$

$$\le 4\frac{h}{m}\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \int_{s_{i-1}}^{u} (v - s_{i-1}) dv du \le \frac{h^4}{m^3}.$$

iii) The proof is nothing but a simple calculation of expectations of stochastic integrals.

iv)

$$J_n^{12} = \int_{t_n}^{t_{n+1}} (B_u^{(1)} - B_{t_n}^{(1)})(B_u^{(2)} - B_{t_n}^{(2)}) du = \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} (B_u^{(1)} - B_{t_n}^{(1)})(B_u^{(2)} - B_{t_n}^{(2)}) du$$

$$=\sum_{i=1}^{m}\int_{s_{i-1}}^{s_i} B_u^{(1)}B_u^{(2)}du - B_{t_n}^{(1)}\sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{t_n}^{(2)})du - B_{t_n}^{(2)}\int_{t_n}^{t_{n+1}} B_u^{(1)}du$$

$$=\sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{s_{i-1}}^{(2)})B_u^{(1)}du + \sum_{i=1}^{m} B_{s_{i-1}}^{(2)}\int_{s_{i-1}}^{s_i} B_u^{(1)}du$$

$$- B_{t_n}^{(1)}\sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{t_n}^{(2)})du - B_{t_n}^{(2)}\int_{t_n}^{t_{n+1}} B_u^{(1)}du$$

$$=\sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{s_{i-1}}^{(2)})(B_u^{(1)}-B_{s_{i-1}}^{(1)})du + \sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{s_{i-1}}^{(2)})B_{s_{i-1}}^{(1)}du$$

$$+\sum_{i=1}^{m}(B_{s_{i-1}}-B_{t_n}^{(2)})\int_{s_{i-1}}^{s_i} B_u^{(1)}du - B_{t_n}^{(1)}\sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{t_n}^{(2)})du.$$

Rearranging the terms in the equation above, we find that

$$J_n^{12} = \sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{s_{i-1}}^{(2)})(B_u^{(1)}-B_{s_{i-1}}^{(1)})du + \Theta_n^{12},$$

namely,

$$J_n^{12} - \Theta_n^{12} = \sum_{i=1}^{m}\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{s_{i-1}}^{(2)})(B_u^{(1)}-B_{s_{i-1}}^{(1)})du.$$

Hence

$$E(J_n^{12}-\Theta_n^{12})^2 = \sum_{i=1}^{m} E\left(\int_{s_{i-1}}^{s_i}(B_u^{(2)}-B_{s_{i-1}}^{(2)})(B_u^{(1)}-B_{s_{i-1}}^{(1)})du\right)^2 \le \frac{h^4}{m^3}.$$

Assume that $\{\Delta_n^\alpha W_j^{(k)}\}$, $\{\Delta_n^\alpha \xi_j^{(k)}\}$ are independent random variables and have normal distribution $N(O,h/m)$, where α represents the α-th sample, and k is the dimensional index. Let

$$\Delta_n^\alpha W^{(k)} = \sum_{j=1}^{m}\Delta_n^\alpha W_j^{(k)}, \quad k=1,2, \tag{2.13}$$

$$\Delta_n^\alpha \beta_j^{(k)} = \frac{1}{2}h\Delta_n^\alpha W_j^{(k)} + \frac{\sqrt{3}}{6}h\Delta_n^\alpha \xi_j^{(k)}, \quad k=1,2, \tag{2.14}$$

$$\Delta_n^\alpha \beta^{(k)} = \sum_{j=1}^{m}\Delta_n^\alpha \beta_j^{(k)}, \quad k=1,2, \tag{2.15}$$

$$\Delta_n^\alpha \theta_{12} = \sum_{i=2}^{m}\left(\left(\sum_{j=1}^{i-1}\Delta_n^\alpha W_j^{(1)}\right)\Delta_n^\alpha \beta_i^{(2)} + \left(\sum_{j=1}^{i-1}\Delta_n^\alpha W_j^{(2)}\right)\Delta_n^\alpha \beta_i^{(1)}\right.$$

$$\left. + \frac{h}{m}\left(\sum_{j=1}^{i-1}\Delta_n^\alpha W_j^{(1)}\right)\left(\sum_{j=1}^{i-1}\Delta_n^\alpha W_j^{(2)}\right)\right), \tag{2.16}$$

$$\Delta_n^\alpha \gamma^{(k)} = 2\sum_{i=2}^{m}\left(\left(\sum_{j=1}^{i-1}\Delta_n^\alpha W_j^{(k)}\right)\Delta_n^\alpha \beta_i^{(k)}\right) + \frac{h}{m}\sum_{i=2}^{m}\left(\sum_{j=1}^{i-1}\Delta_n^\alpha W_j^{(k)}\right)^2 + \frac{h^2}{2m},$$

$$k = 1,2. \qquad (2.17)$$

Proposition 2.2. For Borel measurable function f on R^n, the distribution of

$$f(\Delta_n^\alpha W^{(1)}, \Delta_n^\alpha \beta^{(1)}, \Delta_n^\alpha \beta^{(2)}, \Delta_n^\alpha \theta_{12}, \Delta_n^\alpha \gamma^{(1)}, \Delta_n^\alpha \gamma^{(2)}, h)$$

coincides with the distribution of

$$f(\Delta_n B^{(1)}, \Delta_n B^{(2)}, \beta_n^{(1)}, \beta_n^{(2)}, \theta_n^{12}, \Gamma_n^{(1)}, \Gamma_n^{(2)}, h)$$

where

$$\beta_n^{(k)} = \int_{t_n}^{t_{n+1}}\left(B_u^{(k)} - B_{t_n}^{(k)}\right)du, \quad k = 1,2.$$

Proof. see [16].

§3. A Second Order Scheme

First we consider the scalar case and still use the symbols in Sec. 2.

Theorem 3.1. *Assume* $|b^{(i)}| \le L, i = 0,1,2,3,4, m = [h^{-\frac{1}{2}}] + 1,$

$$X_{n+1}^\alpha = X_n^\alpha + b(X_n^\alpha)h + \frac{1}{2}b'(X_n^\alpha)b(X_n^\alpha)h^2 + b'(X_n^\alpha)\Delta_n^\alpha \beta$$
$$+ \frac{1}{2}b''(X_n^\alpha)\Delta_n^\alpha \gamma + \Delta_n^\alpha W, \qquad (3.1)$$

which represents α-th simulation, and $\alpha = 1,2,\dots,N$. Then for any $\psi \in Lip.$, we have

$$E\left(\frac{\sum_{i=1}^N \psi(X_n^\alpha)}{N} - E\psi(X(t_n))\right)^2 \le \frac{C_1}{N} + C_2 h^4, \qquad (3.2)$$

where C_1, C_2 are two constants independent of N, h, and $X(t_n)$ is the solution of equation (1.1).

Proof. The theoretical scheme corresponding to equation (3.1) is defined as

$$X_{t_{n+1}} = X_{t_n} + b(X_{t_n})h + \frac{1}{2}b'(X_{t_n})b(X_{t_n})h^2 + b'(X_{t_n})\beta_n + \frac{1}{2}b''(X_{t_n})\Gamma_n + \Delta_n B. \quad (3.3)$$

If the following assertion is true:

$$E(X_{t_n} - X(t_n))^2 \le ch^4 \qquad (3.4)$$

where c is a constant independent of h, then

$$E\left(\frac{\sum_{i=1}^N \psi(X_n^\alpha)}{N} - E\psi(X(t_n))\right)^2 \le 2E\left(\frac{\sum_{i=1}^N \psi(X_n^\alpha)}{N} - E\psi(X_{t_n})\right)^2$$
$$+ 2L_\psi^2 E(X(t_n) - X_{t_n})^2, \qquad (3.5)$$

where L_ψ is the Lipshitz constant of ψ. According to Proposition 2.2, equation (3.2) is obtained. So what remains to be proved is equation (3.4). From equation (1.1) we can write

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} b(X(u))du + \Delta_n B.$$

By means of the Taylor formula, we obtain

$$\begin{aligned}
X(t_{n+1}) &= X(t_n) + bh + b' \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{u} b(X(v))dv + \Delta_u B \right) du \\
&+ \frac{1}{2} b'' \int_{t_n}^{t_{n+1}} \left(\left(\int_{t_n}^{u} b(X(v))dv \right)^2 + 2\Delta_u B \int_{t_n}^{u} b(X(v))dv + (\Delta_u B)^2 \right) du \\
&+ \frac{1}{3!} b''' \int_{t_n}^{t_{n+1}} \left(\left(\int_{t_n}^{u} b(X(v))dv \right)^3 + 3\left(\int_{t_n}^{u} b(X(v))dv \right)^2 \Delta_n B \right. \\
&\left. +3\left(\int_{t_n}^{u} b(X(v))dv \right)(\Delta_u B)^2 + (\Delta_u B)^3 \right) du \\
&+ \frac{1}{4!} \int_{t_n}^{t_{n+1}} b^{(4)}(\eta_1)(X(u) - X(t_n))^4 du + \Delta_n B, \qquad (3.6)
\end{aligned}$$

where $b^{(i)} = b^{(i)}(X(t_n))$, $i = 0, 1, 2, 3$. Let $M_n = X(t_n) - X_{t_n}$. The subtraction of the above equation from equation (3.3) gives

$$\begin{aligned}
M_{n+1} &= M_n + (b - b(X_{t_n}))h + b' \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{u} b(X(v))dv \right) du \\
&- \frac{1}{2} b'(X_{t_n}) b(X_{t_n}) h^2 + (b' - b'(X_{t_n})\beta_n + \frac{1}{2} b'' I_n - \frac{1}{2} b''(X_{t_n})\Gamma_n \\
&+ \frac{1}{2} b'' \int_{t_n}^{t_{n+1}} \left(\left(\int_{t_n}^{u} b(X(v))dv \right)^2 + 2\Delta_u B \int_{t_n}^{u} b(X(v))dv \right) du \\
&+ \frac{1}{3!} b''' \int_{t_n}^{t_{n+1}} \left(\left(\int_{t_n}^{u} b(X(v))dv \right)^3 + 3\left(\int_{t_n}^{u} b(X(v))dv \right)^2 \Delta_u B \right. \\
&\left. +3\left(\int_{t_n}^{u} b(X(v))dv \right)(\Delta_u B)^2 \right) du \\
&+ \frac{1}{3!} b''' \int_{t_n}^{t_{n+1}} (\Delta_u B)^3 du + \frac{1}{4!} \int_{t_n}^{t_{n+1}} b^{(4)}(\eta_1)(X(u) - X(t_n))^4 du \\
&= M_n + (b - b(X_{t_n}))h + \frac{1}{2}(b'b - b'(X_{t_n})b(X_{t_n}))h^2 + (b' - b'(X_{t_n}))\beta_n \\
&+ \frac{1}{2} b'' I_n - \frac{1}{2} b''(X_{t_n})\Gamma_n + (b')^2 \int_{t_n}^{t_{n+1}} \int_{t_n}^{u} \Delta_v B dv du + bb'' \int_{t_n}^{t_{n+1}} (u - t_n)\Delta_u B du \\
&+ \frac{1}{3!} b''' \int_{t_n}^{t_{n+1}} (\Delta_u B)^3 du + (b')^2 \int_{t_n}^{t_{n+1}} \int_{t_n}^{u} \int_{t_n}^{v} b(X(w)) dw dv du
\end{aligned}$$

$$+\frac{1}{2!}b'\int_{t_n}^{t_{n+1}}\int_{t_n}^{u}b''(\eta_2)(X(v)-X(t_n))^2dvdu$$

$$+\frac{1}{2}b''\int_{t_n}^{t_{n+1}}\left(\left(\int_{t_n}^{u}b(X(v))dv\right)^2+2\Delta_uB\int_{t_n}^{u}b'(\eta_3)(X(v)-X(t_n))dv\right)du$$

$$+\frac{1}{3!}b'''\int_{t_n}^{t_{n+1}}\left(\left(\int_{t_n}^{u}b(X(v))dv\right)^3+3\left(\int_{t_n}^{ub}(X(v))dv\right)^2\Delta_uB\right.$$

$$\left.+3\left(\int_{t_n}^{u}b(X(v))dv\right)(\Delta_uB)^2\right)du+\frac{1}{4!}\int_{t_n}^{t_{n+1}}b^{(4)}(\eta_1)(X(u)-X(t_n))^4du. \quad (3.7)$$

Define

$$R_1=(b')^2\int_{t_n}^{t_{n+1}}\int_{t_n}^{u}\Delta_vBdvdu+bb''\int_{t_n}^{t_{n+1}}(u-t_n)\Delta_uBdu$$

$$+\frac{1}{3!}b'''\int_{t_n}^{t_{n+1}}(\Delta_uB)^3du \quad (3.8)$$

$$R_2=(b')^2\int_{t_n}^{t_{n+1}}\int_{t_n}^{u}\int_{t_n}^{v}b(X(w))dwdvdu+\frac{1}{2!}b'\int_{t_n}^{t_{n+1}}\int_{t_n}^{u}b''(\eta_2)(X(v)-X(t_n))^2dvdu$$

$$+\frac{1}{2}b''\int_{t_n}^{t_{n+1}}\left(\left(\int_{t_n}^{u}b(X(v))dv\right)^2+2\Delta_uB\int_{t_n}^{u}b'(\eta_3)(X(v)-X(t_n))dv\right)du$$

$$+\frac{1}{3!}b'''\int_{t_n}^{t_{n+1}}\left(\left(\int_{t_n}^{u}b(X(v))dv\right)^3+3\left(\int_{t_n}^{ub}(X(v))dv\right)^2\Delta_uB\right.$$

$$\left.+3\left(\int_{t_n}^{u}b(X(v))dv\right)(\Delta_uB)^2\right)du+\frac{1}{4!}\int_{t_n}^{t_{n+1}}b^{(4)}(\eta_1)(X(u)-X(t_n))^4du. \quad (3.9)$$

It is clear that

$$E(R_1|\Im_{t_n})=0, \quad (3.10)$$

$$E(R_1^2)=O(h^5), \quad (3.11)$$

$$E(R_2)=O(h^3), \quad (3.12)$$

$$E(R_2^2)=O(h^6), \quad (3.13)$$

and now (3.7), (3.8) and (3.9) combine to give

$$M_{n+1}=M_n+(b-b(X_{t_n}))h+\frac{1}{2}(b'b-b'(X_{t_n})b(X_{t_n}))h^2+(b'-b'(X_{t_n}))\beta_n$$

$$+\frac{1}{2}b''I_n-\frac{1}{2}b''(X_{t_n})\Gamma_n+R_1+R_2. \quad (3.14)$$

Taking expectations of squares of both sides of (3.14) and using (3.10)–(3.13) with $E(\beta_n|\Im_{t_n})=0$, $E(I_n-\Gamma_n|\Im_{t_n})=0$, we obtain

$$EM_{n+1}^2=E\left(M_n+(b-b(X_{t_n}))h+\frac{1}{2}(b'b-b'(X_{t_n}b(X_{t_n})h^2\right)^2$$

$$+ E(b' - b'(X_{t_n}))^2\beta_n + \frac{1}{4}E(b''I_n - b''(X_{t_n})\Gamma_n)^2$$

$$+ 2E\left(M_n + (b - b(X_{t_n}))h + \frac{1}{2}(b'b - b'(X_{t_n})b(X_{t_n})h^2\right)^2 R_2$$

$$+ E(b' - b'(X_{t_n}))\beta_n(b''I_n - b''(X_{t_n})\Gamma_n) + 2E(b' - b'(X_{t_n}))\beta_n(R_1 + R_2)$$

$$+ E(b''I_n - b''(X_{t_n})\Gamma_n)^2(R_1 + R_2) + 2ER_1R_2 + O(h^5).$$

By the conditions in the theorem, it follows that

$$EM_{n+1}^2 \leq (1 + Lh + L^2h^2)EM_n^2 + 3L^2E\beta_n^2M_n^2 + 3L^2E(I_n - \Gamma_n)^2$$
$$+ 3L^2EM_n^2I_n^2 + h(1 + Lh + L^2h^2)EM_n^2 + O(h^5).$$

From Proposition 2.1, we conclude that

$$EM_{n+1}^2 \leq (1 + L_1h)EM_n^2 + O\left(\frac{h^4}{m^3}\right) + O(h^5),$$

where L_1 is a constant independent of h and m. Thus Theorem 3.1 follows.

In the system case, we refer to Theorem 3.2. The proof is similar to the proof of Theorem 3.1. Now equation (1.1) can be rewritten as

$$dx(t) = b(x(t), y(t))dt + dB_t^{(1)}, \quad x(0) = x \quad \text{(const.)},$$
$$dy(t) = f(x(t), y(t))dt + dB_t^{(2)}, \quad y(0) = y \quad \text{(const.)}. \tag{3.15}$$

Theorem 3.2. *If* $\sup_{x,y}\left|\frac{\partial^{\mu_1+\mu_2}}{\partial x^{\mu_1}\partial y^{\mu_2}}f(x,y)\right|$ *and* $\sup_{x,y}\left|\frac{\partial^{\mu_1+\mu_2}}{\partial x^{\mu_1}\partial y^{\mu_2}}g(x,y)\right|$ *are finite for* $0 \leq \mu_1 + \mu_2 \leq 4$ *in the following system of equations:*

$$X_{n+1}^\alpha = X_n^\alpha + b(X_n^\alpha, Y_n^\alpha)h + \frac{1}{2}b'_x(X_n^\alpha, Y_n^\alpha)b(X_n^\alpha, Y_n^\alpha)h^2$$
$$+ \frac{1}{2}b'_y(X_n, Y_n)f(X_n, Y_n)h^2 + b'_x(X_n, Y_n)\Delta_n^\alpha\beta^{(1)} + b'_y(X_n, Y_n)\Delta_n^\alpha\beta^{(2)}$$
$$+ \frac{1}{2}b''_{xx}(X_n, Y_n)\Delta_n^\alpha\gamma^{(1)} + \frac{1}{2}b''_{yy}(X_n, Y_n)\Delta_n^\alpha\gamma^{(2)}$$
$$+ b''_{xy}(X_n, Y_n)\Delta_n^\alpha\theta_{12} + \Delta_n^\alpha W^{(1)}, \tag{3.16}$$

$$Y_{n+1}^\alpha = Y_n^\alpha + f(X_n^\alpha, Y_n^\alpha)h + \frac{1}{2}f'_x(X_n^\alpha, Y_n^\alpha)b(X_n, Y_n)h^2$$
$$+ \frac{1}{2}f'_y(X_n, Y_n)f(X_n, Y_n)h^2 + f'_x(X_n, Y_n)\Delta_n^\alpha\beta^{(1)} + f'_y(X_n, Y_n)\Delta_n^\alpha\beta^{(2)}$$
$$+ \frac{1}{2}f''_{xx}(X_n, Y_n)\Delta_n^\alpha\gamma^{(1)} + \frac{1}{2}f''_{yy}(X_n, Y_n)\Delta_n^\alpha\gamma^{(2)}$$
$$+ f''_{xy}(X_n, Y_n)\Delta_n^\alpha\theta_{12} + \Delta_n^\alpha W^{(2)}.$$

then $\forall\psi \in$ Lip., *and we have*

$$E\left(\frac{\sum_{i=1}^{N}\psi(X_n^\alpha, Y_n^\alpha)}{N} - E\psi(X(t_n), Y(t_n))\right)^2 \le \frac{C_1}{N} + C_2 h^4, \tag{3.17}$$

where C_1, C_2 *are two constants independent of* N, h, *and* $(X(t_n), Y(t_n))$ *is the solution of equation* (3.13).

It should also be pointed out that, although the second-order scheme needs some more arithmetic operations in each step, for a given tolerated error, this scheme allows a larger time step size than Euler's or 1.5th-order scheme, and the total computational work of the second-order scheme can still be of smallamount. Thus, besides its high accuracy, the second-order scheme can also be more efficient in computation.

§4. Numerical Results

Example 1. One-dimensional non-linear equation

$$x(0) = 0, \quad dx(t) = e^{-x(t)}dt + dB_t. \tag{4.1}$$

We consider the expectation $E(\exp(x(t)))$, which has the exact value

$$E(e^{x(t)}) = 3e^{\frac{1}{2}t} - 2.$$

Taking $h = 0.1$, we have

Euler's scheme.

$$X_{n+1}^\alpha = X_n^\alpha + e^{-X_n^\alpha}h + \Delta_n^\alpha W; \tag{4.2}$$

1.5th-order scheme:

$$X_{n+1}^\alpha = X_n^\alpha + e^{-X_n^\alpha}h - 0.5e^{-2X_n^\alpha}h^2 - e^{-X_n^\alpha}\Delta_n^\alpha\beta + \Delta_n^\alpha W; \tag{4.3}$$

second-order scheme:

$$X_{n+1}^\alpha = X_n^\alpha + e^{-X_n^\alpha}h - 0.5e^{-2X_n^\alpha}h^2 - e^{-X_n^\alpha}\Delta_n^\alpha\beta + \Delta_n^\alpha W + 0.5e^{X_n^\alpha}\Delta_n^\alpha\gamma. \tag{4.4}$$

According to Theorem 3.1, $m = [0.1^{-\frac{1}{2}}] + 1 = 3$. We use the random numbers generated by the Box-Muller method, which are normally distributed, with means 0 and variances 0.1/3. Hence

$$\Delta_n^\alpha W = \Delta_n^\alpha W_1 + \Delta_n^\alpha W_2 + \Delta_n^\alpha W_3, \tag{4.5}$$

$$\Delta_n^\alpha \beta_j = \frac{0.1}{2}\Delta_n^\alpha W_j + \frac{0.1\sqrt{3}}{6}\Delta_n^\alpha \xi_j, \quad j = 1,2,3, \tag{4.6}$$

$$\Delta_n^\alpha \beta = \Delta_n^\alpha \beta_1 + \Delta_n^\alpha \beta_2 + \Delta_n^\alpha \beta_3, \tag{4.7}$$

$$\Delta_n^\alpha \gamma = 2\Delta_n^\alpha W_n^\alpha W_1(\Delta_n^\alpha \beta_2 + \Delta_n^\alpha \beta_3) + 2\Delta_n^\alpha W_2 \Delta_n^\alpha \beta_3 + \frac{0.1}{3}((\Delta_n^\alpha W_1)^2 + (\Delta_n^\alpha W_1 + \Delta_n^\alpha W_2)^2) + \frac{(0.1)^2}{6}. \tag{4.8}$$

In Table 1, we list the computation errors of equation (4.1) by using equations (4.2), (4.3), (4.4).

Table 1. Numerical results by using three schemes

Number of simulations: $N = 10000$ step size: $t = 0.1$					
Number of nodes	0.1	0.2	0.3	0.4	0.5
EULER scheme	0.007013	0.012872	0.012614	0.023975	0.033028
Order one & half scheme	0.004048	0.009846	0.022007	0.023118	0.026635
Second order scheme	0.001000	0.003424	0.012165	0.009580	0.009295
Real value	1.153813	1.315513	1.485503	1.664208	1.852076

Example 2. 2×2 system of non-linear equations

$$dx(t) = e^{-x(t)-y(t)}dt + dB_t^{(1)}, \quad x(0) = 0,$$

$$dy(t) = e^{-x(t)-y(t)}dt + dB_t^{(2)}, \quad y(0) = 0. \tag{4.9}$$

We consider the expectation $E(\exp(x(t) + y(t)))$, which has the exact value

$$E(e^{x(t)+y(t)}) = 3e^t - 2.$$

Let $h = 0.1$ as in example 1. We can generate $\Delta_n^\alpha W^{(k)}$, $\Delta_n^\alpha \beta^{(k)}$, $\Delta_n^\alpha \gamma^{(k)}$. In addition,

$$\begin{aligned}\Delta_n^\alpha \theta_{12} = {} & \Delta_n^\alpha W_1^{(1)} \Delta_n^\alpha \beta_2^{(2)} + (\Delta_n^\alpha W_1^{(1)} + \Delta_n^\alpha W_2^{(1)}) \Delta_n^\alpha \beta_3^{(2)} \\ & + \Delta_n^\alpha W_1^{(2)} \Delta_n^\alpha \beta_2^{(1)} + (\Delta_n^\alpha W_1^{(2)} + \Delta_n^\alpha W_2^{(2)}) \Delta_n^\alpha \beta_3^{(1)} \\ & + \frac{0.1}{3}(\Delta_n^\alpha W_1^{(1)} \Delta_n^\alpha W_1^{(2)} + (\Delta_n^\alpha W_1^{(1)} + \Delta_n^\alpha W_2^{(1)})(\Delta_n^\alpha W_1^{(2)} + \Delta_n^\alpha W_2^{(2)})).\end{aligned}$$

In Table 2, the errors of the computational results are listed.

Table 2. Numerical results by using three schemes

Number of simulations: $N = 10000$ step size: $t = 0.1$					
Number of nodes	0.1	0.2	0.3	0.4	0.5
EULER scheme	0.027574	0.068065	0.108164	0.154767	0.199851
Order one & half scheme	0.022175	0.036650	0.055429	0.069115	0.091179
Second order scheme	0.007543	0.005092	0.004032	0.002320	0.004634
Real value	1.315513	1.664208	2.049577	2.475474	2.946164

From the above two examples, we can see that the second order scheme is better than the order one and half scheme. The errors are reduced by one power of the step size. This result is consistent with our theoretical analysis.

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