Journal of Computational Mathematics, Vol.10, No.4, 1992, 366–375.

A QUASI-PROJECTION ANALYSIS FOR ELASTIC WAVE PROPAGATION IN FLUID-SATURATED POROUS MEDIA*

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Abstract

This paper deals with the superconvergence phenomena for Galerkin approximations of solutions of Biot's dynamic equations describing elastic wave propagation in fluid-saturated porous media. An asymptotic expansion to high order of Galerkin solutions is used to derive these results.

§1. Introduction

An isotropic, elastic porous solid saturated by a compressible viscous fluid can be described by the system of partial differential equations [1], [6],

$$\mathcal{A}\frac{\partial^2 u}{\partial t^2}+\mathcal{C}\frac{\partial u}{\partial t}-L(u)=F(x,t),(x,t)\in\Omega\times[0,T], \tag{1.1}$$

where Ω is a bounded domain in R^2, $u(x,t)=(u_1,u_2)$ is the displacement vector on Ω, $u_1=(u_{11},u_{12})$ and $u_2(x,t)=(x_{21},u_{22})$ are the displacement of the solid and the average fluid displacement, respectively, and $F(x,t)$ is the force applied to the system. The differential operator $L(u)$ is defined by

$$L(u)=(\nabla\cdot\theta_1(u),\nabla\cdot\theta_2(u),\nabla s(u)),$$

where the vectors $\theta_i(u), i=1,2$, and the scalar $s(u)$ are

$$\theta_i(u)=(\theta_{i1},\theta_{i2}),\ i=1,2;\quad s(u)=Q\nabla\cdot u_1+R\nabla\cdot u_2,$$

and

$$\theta_{ij}(u)=\sigma_{ij}(u_1)+Q\delta_{ij}\nabla\cdot u_2,\quad i,j=1,2.$$

Here δ_{ij} denotes the Kronecker symbol, and the stress tensors σ_{ij} and the strain tensors ϵ_{ij} for Ω are related by

$$\varepsilon_{ij}(u_1)\frac{1}{2}\Big(\frac{\partial u_{1i}}{\partial x_j}+\frac{\partial u_{1j}}{\partial x_i}\Big),\ \ 1\le i,\ j\le 2,$$

$$\sigma_{ij}(u_1)=A\delta_{ij}\sum_{k=1}^{2}\varepsilon_{kk}(u_1)+2N\varepsilon_{ij}(u_1),\ \ 1\le i,\ j\le 2.$$

* Received May 4, 1989.

$A = A(x), N = N(x), Q = Q(x)$, and $R = R(x)$ are the elastic coefficients for Ω. They will be assumed to satisfy the constraints

$$0 < N_* \leq N(x) \leq N^* < \infty, \ x \in \bar{\Omega} = \Omega \cup \partial\Omega,$$

$$0 < A_* \leq A(x) \leq A^* < \infty, \ x \in \bar{\Omega},$$

$$0 < Q_* \leq Q(x) \leq Q^* < \infty, \ x \in \bar{\Omega},$$

$$0 < R_* \leq R(x) \leq R^* < \infty, \ x \in \bar{\Omega},$$

$$R(A+N) - Q^2 > 0, \ x \in \bar{\Omega}.$$

In (1.1), $\mathcal{A} \in \boldsymbol{R}^{4\times 4}$ and $\mathcal{C} \in \boldsymbol{R}^{4\times 4}$ denote the density matrix and the dissipative matrix given by

$$\mathcal{A} = \begin{bmatrix} \rho_{11} & 0 & \rho_{12} & 0 \\ 0 & \rho_{11} & 0 & \rho_{12} \\ \rho_{12} & 0 & \rho_{22} & 0 \\ 0 & \rho_{12} & 0 & \rho_{22} \end{bmatrix}, \ \mathcal{C} = b(x) \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

where $\rho_{11} = \rho_1 - \rho_{12}$, $\rho_{22} = \rho_2 - \rho_{12}$, $\rho_1 = \rho_1(x)$ (respectively, $\rho_2 = \rho_2(x)$) is the mass of solid (respectively, fluid) per unit of the aggregate, and $\rho_{12} = \rho_{12}(x)$ is a mass coupling parameter between fluid and solid; $b = b(x)$ is the dissipation coefficient for Ω.

From physical consideration, it will be assumed that

$$\rho_{11}\rho_{22} - \rho_{12}^2 > 0, \quad x \in \bar{\Omega}, \tag{1.2}$$

$$0 < b_* \leq b(x) \leq b^* < \infty, \quad x \in \bar{\Omega}. \tag{1.3}$$

Then, it follows that $\mathcal{A}$ is positive-definite and $\mathcal{C}$ is nonnegative.

We shall impose initial conditions

$$u(x,0) = u^0, \ x \in \Omega, \quad \frac{\partial u}{\partial t}(x,0) = v^0, \ x \in \Omega, \tag{1.4}$$

and the homogeneous boundary conditions

$$(\theta_1(u) \cdot n, \theta_2(u) \cdot n, s(u)) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{1.5}$$

where $n = n(x)$ is the outward unit normal along $\partial\Omega$.

In this paper we shall analyze superconvergence phenomena for the numerical solution of (1.1). The analysis is based on a regularity assumption on the solution of (1.1) [6], [7], and the general method of an asymptotic expansion, called a quasi-projection, of the approximate solution [4]. We shall also rely on some earlier results on the subject [6], [7], which analyzed the existence and uniqueness of solution of (1.1) and the Galerkin procedure for the approximate solution of such equations.

The paper is organized as follows. In §2 we give some notation and quote some known results. In §3 we present the Galerkin procedure of (1.1). Then in §4 we develop the quasi-projection for the procedure. Finally, in §5 we give the superconvergence results of the same type as those of Bramble and Schatz [2] for Galerkin methods.

§2. Notation and Preliminaries

Let $\Omega \subset \boldsymbol{R}^2$ be a bounded domain. For m a nonnegative integer let $H^m(\Omega) = W^{m,2}(\Omega)$ be the usual Sobolev space with norm

$$\|v\|_m^2 = \sum_{|\alpha|\le m} \int_\Omega |D^\alpha v(x)|^2 dx.$$

For $n \ge 1$ the norm of $v = (v_1, \cdots, v_n)$ in $[H^m(\Omega)]^n$ will be given by

$$\|v\|_m^2 = \sum_{i=1}^{n} \|v_i\|_m^2.$$

The inner product and norm in $[L^2(\Omega)]^2$ will be denoted by

$$(v, w) = \sum_{i=1}^{2} \int_\Omega v_i w_i dx, \quad \|v\|_0^2 = (v, v).$$

We shall also use the norm on the dual space $H^{-m}(\Omega) = (H^m(\Omega))'$; let

$$\|v\|_{-m} = \sup\left\{\frac{(v,u)}{\|u\|_m} : u \in H^m(\Omega),\ \|u\|_m \ne 0\right\}.$$

Let $H(div, \Omega) = \{q \in [L^2(\Omega)]^2 : \nabla \cdot q \in L^2(\Omega)\}$ with the norm

$$\|q\|_{H(div,\Omega)}^2 = \|q\|_0^2 + \|\nabla \cdot q\|_0^2.$$

Set $V = [H^1(\Omega)]^2 \times H(div, \Omega)$ with the norm

$$\|v\|_V^2 = \|v_1\|_1^2 + \|v_2\|_{H(div,\Omega)}^2, \quad v = (v_1, v_2) \in V.$$

Finally, if X is a Banach space with norm $\| \ \|_X$ and if $v : [0,T] \to X$, we use

$$\|v\|_{L^2(X)}^2 = \int_0^T \|v(t)\|_X^2 dt, \quad \|v\|_{L^\infty(X)}^2 = \operatorname*{ess\ sup}_{0\le t\le T} \|v(t)\|_X.$$

Let B be the bilinear form associated with L in (1.1)

$$B(v, w) = M(v_1, w_1) + (Q\nabla \cdot v_2, \nabla \cdot w_1) + (Q\nabla \cdot v_1 + R\nabla \cdot v_2, \nabla \cdot w_2),$$

for $v = (v_1, v_1)$ and $w = (w_1, w_2) \in V$, with M being defined by

$$M(v_1, w_1) = \int_\Omega \left[A\nabla \cdot v_1 \nabla \cdot w_1 + 2N \sum_{i,j=1}^{2} \varepsilon_{ij}(v_1)\varepsilon_{ij}(w_1)\right] dx.$$

Test (1.1) against $v \in V$, integrate by parts over Ω, and apply (1.5) to the $(L(u), v)$-term. Then,

$$\left(A\frac{\partial^2 u}{\partial t^2}, v\right) + \left(C\frac{\partial u}{\partial t}, v\right) + B(u,v) = (F,v), v \in V, t \in [0,T]. \tag{2.1}$$

Let the matrix $E \in R^{4\times4}$ be given by

$$E = \begin{bmatrix} A+2N & A & 0 & Q \\ A & A+2N & 0 & Q \\ 0 & 0 & 4N & 0 \\ Q & Q & 0 & R \end{bmatrix}$$

Then it follows from the assumptions on the elastic coefficients that E is positive definite. If λ_* denotes the minimum eigenvalue of E, there exists a constant $c > 0$ such that [6]

$$B(v,v) \ge c\|v\|_V^2 - \lambda_*\|v\|_0^2, v \in V. \tag{2.2}$$

It is convenient to introduce the differential operator L^* by

$$L^*(u) = -L(u) + \lambda_* u. \tag{2.3}$$

Then, the bilinear form associated with L^* is given by

$$B_1(v,w) = B(v,w) + \lambda_*(v,w), v, w, \in V.$$

Thus the symmetric form B_1 satisfies

$$|B_1(v,w)| \le c\|v\|_V\|w\|_V, v, w \in V, B(v,v) \ge c\|v\|_V^2, v \in V. \tag{2.4}$$

Let $s \ge 0$ and assume that $\psi \in [H^s(\Omega)]^2$. Let φ be determined as the solution of the boundary problem

$$L^*(\varphi) = \psi, \quad x \in \Omega, \quad (\theta_1(\varphi)\cdot n, \theta_2(\varphi)\cdot n, s(\varphi)) = 0, \ x \in \partial\Omega.$$

To obtain the superconvergence estimate we shall use an asymptotic expansion to high order of Galerkin solutions for which the following regularity assumption is needed:

$$\|\varphi_1\|_{s+2} + \|\varphi_2\|_{s+1} + \|\nabla\cdot\varphi_2\|_{s+1} \le c\|\psi\|_s. \tag{2.6}$$

§3. The Galerkin Procedure

Let $r \ge 1$ be an integer and let $0 < h < 1$. Let $\mathcal{T}_h$ and T_h be quasiregular partitions of Ω into triangles or rectangles of diameter bounded by h. Let $M_h \subset [H^1(\Omega)]^2$ be a standard finite element space associated with $\mathcal{T}_h$ such that

$$\inf_{v\in M_h}\{\|\varphi - v\|_0 + h\|\varphi - v\|_1\} \le c\|v\|_s h^s, \ 1 \le s \le r+1. \tag{3.1}$$

Also, let W_h be a finite dimensional subspace of $H(div,\Omega)$ associated with T_h such that

$$\inf_{v\in W_h} \|w-v\|_0 \le c\|w\|_s h^s,\quad 1\le s\le r+1,$$

$$\inf_{v\in W_h} \|w-v\|_{H(div,\Omega)} \le c(\|w\|_s+\|\nabla\cdot w\|_s)h^s,\quad 1\le s\le r+1. \tag{3.2}$$

Here we can take W_h to be one of the Raviart-Thomas vector spaces [5] or the Brezzi-Douglas-Marini vector spaces [3] of index r associated with T_h.

Set $V_h = M_h \times W_h$. Then, by (3.1) and (3.2), we have

$$\inf_{v\in V_h} \|u-v\|_0 \le c\|u\|_s h^s,\quad 1\le s\le r+1,$$

$$\inf_{v\in V_h} \|u-v\|_0 \le c(\|u_1\|_{s+1}+\|u_2\|_s+\|\nabla\cdot u_2\|_s)h^s,\ u=(u_1,u_2),\quad 1\le s\le r. \tag{3.3}$$

The continuous-time Galerkin approximation to the solution of (1.1) is defined as the twice-differentiable map $u_h:[0,T]\to V_h$ such that

$$\Big(\mathcal{A}\frac{\partial^2 u_h}{\partial t^2},v\Big)+\Big(\mathcal{C}\frac{\partial u_h}{\partial t},v\Big)+B(u_h,v)=(F,v), v\in V_h, t\in[0,T]. \tag{3.4}$$

The initial conditions $u_h(0)$ and $\partial u_h(0)/\partial t$ will be specified later.

§4. The Analysis of Quasi-Projection

In this section we shall consider the quasi-projection for the Galerkin approximation (3.4). Set $J=(0,T)$ and let $\tilde u_h : J\to V_h$ be defined by

$$B_1(\tilde u_h-u,v)=0,\ v\in V_h,\ t\in J. \tag{4.1}$$

Let $u_0=\tilde u_h, z_0=u_0-u$, and $\theta_0=u_0-u_h$. Then it follows from (2.1), (3.4), and (4.1) that

$$\Big(\mathcal{A}\frac{\partial^2\theta_0}{\partial t^2},v\Big)+\Big(\mathcal{C}\frac{\partial\theta_0}{\partial t},v\Big)+B(\theta_0,v)=\Big(\mathcal{A}\frac{\partial^2 z_0}{\partial t^2},v\Big)+\Big(\mathcal{C}\frac{\partial z_0}{\partial t},v\Big)-\lambda_*(z_0,v),$$
$$\text{for } v\in V_h,\ t\in J. \tag{4.2}$$

Define maps $z_j : J\to V_h$ recursively by

$$B_1(z_j,v)=-\Big(\mathcal{A}\frac{\partial^2 z_{j-1}}{\partial t^2},v\Big)-\Big(\mathcal{C}\frac{\partial z_{j-1}}{\partial t},v\Big)+\lambda_*(z_{j-1},v),\ v\in V_h,\ t\in J, j=1,2,\cdots. \tag{4.3}$$

Set

$$u_j=u_0+z_1+\cdots+z_j,\ j\ge1, \tag{4.4}$$

and

$$\theta_j=u_j-u_h,\ j\ge1. \tag{4.5}$$

Then it is easy to see from (4.2) and (4.3) that

$$\Big(\mathcal{A}\frac{\partial^2\theta_j}{\partial t^2},v\Big)+\Big(\mathcal{C}\frac{\partial\theta_j}{\partial t},v\Big)+B(\theta_j,v)=\Big(\mathcal{A}\frac{\partial^2 z_j}{\partial t^2},v\Big)+\Big(\mathcal{C}\frac{\partial z_j}{\partial t},v\Big)-\lambda_*(z_j,v),$$
$$\text{for}\quad v\in V_h,\quad t\in J,\quad j=0,1,2,\cdots. \tag{4.6}$$

Let $t\in J$ and let $0\le j<r$, $0\le s<r-j-1$, and $k\ge 0$, an integer. Since the coefficients of L are independent of t, it follows from (4.3) that

$$B_1\Big(\frac{\partial^k z_j}{\partial t^k},v\Big)=-\Big(\mathcal{A}\frac{\partial^{k+2}z_{j-1}}{\partial t^{k+2}},v\Big)-\Big(\mathcal{C}\frac{\partial^{k+1}z_{j-1}}{\partial t^{k+1}},v\Big)+\lambda_*\Big(\frac{\partial^k z_{j-1}}{\partial t^k},v\Big),\ v\in V_h. \tag{4.7}$$

By setting $v=\frac{\partial^k z_j}{\partial t^k}$ in (4.7) and noting the properties of $\mathcal{A}$ and $\mathcal{C}$, we see that

$$\Big\|\frac{\partial^k z_j}{\partial t^k}\Big\|_V\le c\Big\{\Big\|\frac{\partial^{k+2}z_{j-1}}{\partial t^{k+2}}\Big\|_0+\Big\|\frac{\partial^{k+1}z_{j-1}}{\partial t^{k+1}}\Big\|_0+\Big\|\frac{\partial^k z_{j-1}}{\partial t^k}\Big\|_0\Big\}. \tag{4.8}$$

Let $s\ge 0$ and assume that $\psi\in[H^s(\Omega)]^2$; let φ be determined by (2.5). Then

$$\Big(\frac{\partial^k z_0}{\partial t^k},\psi\Big)=\Big(\frac{\partial^k z_0}{\partial t^k},L^*(\varphi)\Big)=B_1\Big(\frac{\partial^k z_0}{\partial t^k},\varphi\Big)=B_1\Big(\frac{\partial^k z_0}{\partial t^k},\varphi-v\Big),\ v\in V_h.$$

Hence, (3.3) and (2.6) imply that

$$\Big|\Big(\frac{\partial^k z_0}{\partial t^k},\psi\Big)\Big|\le c\Big\|\frac{\partial^k z_0}{\partial t^k}\Big\|_V\|\psi\|_s h^{s+1},\ s\le r-1. \tag{4.9}$$

Let $1\le q\le r+1$. Then, it follows by the usual argument that

$$\Big\|\frac{\partial^k z_0}{\partial t^k}\Big\|_V\le c\Big\|\frac{\partial^k u}{\partial t^k}\Big\|_q h^{q-1},\quad\text{for}\quad \frac{\partial^k u}{\partial t^k}\in H^q(\Omega). \tag{4.10}$$

Hence (4.9) and (4.10) imply that

$$\Big\|\frac{\partial^k z_0}{\partial t^k}\Big\|_{-s}\le c\Big\|\frac{\partial^k u}{\partial t^k}\Big\|_q h^{q+s},s\le r-1,\ 1\le q\le r+1. \tag{4.11}$$

Next, it follows from (4.7) that

$$\begin{aligned}\Big(\frac{\partial^k z_j}{\partial t^k},\psi\Big)&=\Big(\frac{\partial^k z_j}{\partial t^k},L^*(\varphi)\Big)=B_1\Big(\frac{\partial^k z_j}{\partial t^k},\varphi\Big)=B_1\Big(\frac{\partial^k z_j}{\partial t^k},\varphi-v\Big)+B_1\Big(\frac{\partial^k z_j}{\partial t^k},v\Big)\\&=B_1\Big(\frac{\partial^k z_j}{\partial t^k},\varphi-v\Big)+\Big(\mathcal{A}\frac{\partial^{k+2}z_{j-1}}{\partial t^{k+1}},\varphi-v\Big)-\Big(\mathcal{A}\frac{\partial^{k+2}z_{j-1}}{\partial t^{k+2}},\varphi\Big)\\&\quad+\Big(\mathcal{C}\frac{\partial^{k+1}z_{j-1}}{\partial t^{k+1}},\varphi-v\Big)-\Big(\mathcal{C}\frac{\partial^{k+2}z_{j-1}}{\partial t^{k+1}},\varphi\Big)-\lambda_*\Big(\frac{\partial^k z_{j-1}}{\partial t^k},\varphi-v\Big)\\&\quad+\lambda_*\Big(\frac{\partial^k z_{j-1}}{\partial t^k},\varphi\Big).\end{aligned} \tag{4.12}$$

Thus, together with (3.3) and (4.8), this implies that

$$\Big\|\frac{\partial^k z_j}{\partial t^k}\Big\|_{-s} \le c\Big\{\Big(\Big\|\frac{\partial^{k+2} z_{j-1}}{\partial t^{k+2}}\Big\|_0 + \Big\|\frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}\Big\|_0 + \Big\|\frac{\partial^k z_{j-1}}{\partial t^k}\Big\|_0\Big)h^{s+1}$$
$$+ \Big\|\frac{\partial^{k+2} z_{j-1}}{\partial t^{k+2}}\Big\|_{s-1} + \Big\|\frac{\partial^{k+1} z_{j-1}}{\partial t^{k+1}}\Big\|_{-s-1} + \Big\|\frac{\partial^k z_{j-1}}{\partial t^k}\Big\|_{-s-1}\Big\}.$$

Then (4.11) and an induction on j imply that with the reduction on the range of s going from $j-1$ to j,

$$\Big\|\frac{\partial^k z_j}{\partial t^k}\Big\|_{-s} \le c\sum_{i=0}^{2j}\Big\|\frac{\partial^{i+k} u}{\partial t^{i+k}}\Big\|_q h^{s+q+j}, \ t \in J. \tag{4.13}$$

The results above can be summarized in the following lemma.

Lemma 4.1. *Let $1 \le q \le r+1$ and let $0 \le j < r$, $0 \le s < r-j-1$, and $k \ge 0$, an integer. If $\partial^{i+k}u/\partial t^{i+k} \in H^q(\Omega)$ for $t \in J$, $i = 1, \cdots, 2j$, then (4.13) holds.*

We now turn to the estimation of θ_j, which satisfies (4.6). First, we shall set up initial conditions for (3.4). Set

$$u_h(0) = u_k(0), \ \frac{\partial u_h}{\partial t}(0) = \frac{\partial u_k}{\partial t}(0), \ \ k \le r-1. \tag{4.14}$$

It follows immediately from (4.14) that

$$\theta_k(0) = 0, \ \frac{\partial \theta_k}{\partial t}(0) = 0, \ \ k \le r-1. \tag{4.15}$$

Take $v = \dfrac{\partial \theta_k}{\partial t}$ in (4.6). Then,

$$\frac{1}{2}\frac{d}{dt}\Big[\Big(\mathcal{A}\frac{\partial \theta_k}{\partial t}, \frac{\partial \theta_k}{\partial t}\Big) + B(\theta_k, \theta_k)\Big] + \Big(\mathcal{C}\frac{\partial \theta_k}{\partial t}, \frac{\partial \theta_k}{\partial t}\Big) = \Big(\mathcal{A}\frac{\partial^2 z_k}{\partial t^2}, \frac{\partial \theta_k}{\partial t}\Big) + \Big(\mathcal{C}\frac{\partial z_k}{\partial t}, \frac{\partial \theta_k}{\partial t}\Big)$$
$$- \lambda_*\Big(z_k, \frac{\partial \theta_k}{\partial t}\Big) \le c\Big\{\Big\|\frac{\partial^2 z_k}{\partial t^2}\Big\|_0^2 + \Big\|\frac{\partial z_k}{\partial t}\Big\|_0^2 + \|z_k\|_0^2 + \Big\|\frac{\partial \theta_k(t)}{\partial t}\Big\|_0^2\Big\}.$$

Adding

$$\frac{\lambda_*}{2}\frac{d}{dt}\|\theta_k\|_0^2 \le \frac{\lambda_*}{2}\Big(\|\theta_k\|_0^2 + \Big\|\frac{\partial \theta_k}{\partial t}\Big\|_0^2\Big)$$

to the inequality above and noting the nonnegativeness of C, we find that

$$\frac{1}{2}\frac{d}{dt}\Big[\Big(\mathcal{A}\frac{\partial \theta_k}{\partial t}, \frac{\partial \theta_k}{\partial t}\Big) + B_1(\theta_k, \theta_k)\Big] \le c\Big\{\Big\|\frac{\partial^2 z_k}{\partial t^2}\Big\|_0^2 + \Big\|\frac{\partial z_k}{\partial t}\Big\|_0^2 + \|z_k\|_0^2 + \|\theta_k\|_V^2 + \Big\|\frac{\partial \theta_k}{\partial t}\Big\|_0^2\Big\}. \tag{4.16}$$

Then, if we integrate (4.16) in time from 0 to t and use (4.5) and the Gronwall lemma, we derive that

$$\Big\|\frac{\partial \theta_k}{\partial t}\Big\|_{L^\infty([L^2(\Omega)]^2)} + \|\theta_k\|_{L^\infty(V)}$$
$$\le c\Big\{\Big\|\frac{\partial^2 z_k}{\partial t^2}\Big\|_{L^2([L^2(\Omega)]^2)} + \Big\|\frac{\partial z_k}{\partial t}\Big\|_{L^2([L^2(\Omega)]^2)} + \|z_k\|_{L^2([L^2(\Omega)]^2)}\Big\}. \tag{4.17}$$

Theorem 4.2. *If $k < r$, $r \le q \le r+1$, and $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$ are defined by (4.14), then*

$$\Big\|\frac{\partial \theta_k}{\partial t}\Big\|_{L^\infty([L^2(\Omega)]^2)} + \|\theta_k\|_{L^2(V)} \le c\sum_{i=0}^{2k+2}\Big\|\frac{\partial^i u}{\partial t^i}\Big\|_{L^2([H^q(\Omega)]^2)} h^{q+k}. \tag{4.18}$$

Proof. The theorem follows from (4.13) and (4.17).

Corollary 4.3. If $k \le r-1$, $1 \le q \le r+1$, and $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$ are defined by (4.14), then

$$\|u-u_h\|_{L^\infty([H^{-s}(\Omega)]^2)} \le c\Big\{\sum_{i=0}^{2k}\Big\|\frac{\partial^i u}{\partial t^i}\Big\|_{L^\infty([H^q(\Omega)]^2)} + \sum_{i=1}^{2}\Big\|\frac{\partial^{2k+i} u}{\partial t^{2k+i}}\Big\|_{L^2([H^q(\Omega)]^2)}\Big\}h^{q+s}, \tag{4.19}$$

for $0 \le s \le \min(k, r-k-1)$.

Proof. Note that $u - u_h = (u-u_0) + \theta_k - (z_1 + \cdots + z_k)$. Then, (4.19) follows from (4.13) and (4.18).

We now discuss the evaluation of $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$. Let $k \le r-1$. To evaluate $z_k(0)$ using (4.3), we must evaluate $\partial^2 z_{k-1}(0)/\partial t^2$, $\partial z_{k-1}(0)/\partial t$, and $z_{k-1}(0)$ first, which in turn require $\partial^2 z_{k-2}(0)/\partial t^2, \cdots, \partial^k z_0(0)/\partial t^k$ by (4.7). Also, it follows from (4.1) that

$$B_1\Big(\frac{\partial^s \tilde{u}_h}{\partial t^s}(0) - \frac{\partial^s u}{\partial t^s}(0), v\Big) = 0, \quad v \in V_h;$$

hence, $u_h(0)$ and $\frac{\partial u_h}{\partial t}(0)$ can be evaluated from (4.14) and using F, u^0, v^0, and time-derivatives of the differential equation.

§5. Superconvergence

In this section we shall consider a simple case which assumes that Ω is the unit square and that the problem (1.1) has a periodic solution of period Ω, in place of the previous boundary condition. To use the general argument of Bramble and Schatz [2] to obtain superconvergence by means of post-processing the computed approximate solution, we shall assume that $\mathcal{T}_h$ (and T_h) has a translation invariance. Let $h = (h_1, h_2)$, $h_i > 0$, and let $\alpha = (\alpha_1, \alpha_2), \alpha_i$ an integer. Define the translation operator G_h^α by

$$G_h^\alpha u(x) = u(x_1 + \alpha_1 h_1,\ x_2 + \alpha_2 h_2).$$

Assume that

$$G_h^\alpha u_h \in V_h, \ \forall u_h \in V_h, \ \forall_\alpha.$$

Set

$$\partial_j u(x) = [u(x + h_j e_j) - u(x)]/h_j,$$

where e_j is the jth unit vector, and let h_1 and h_2 be comparable in the following sense. Let h be the parameter of M_h. Then there exists a constant Q, independent of h, such that

$$h_i \in [h, Qh], \; i = 1, 2.$$

Assume that the coefficients of (1.1) are constant. Then $\partial^\alpha u$ and $\partial^\alpha u_h$ are the solutions of (1.1) and (3.4) with data $\{\partial^\alpha F, \partial^\alpha u^0, \partial^\alpha v^0\}$ and $\{\partial^\alpha F, \partial^\alpha u_h^0, \partial^\alpha v_h^0\}$, respectively. Hence, it follows from (4.9) that

$$\|\partial^\alpha u - u_h\|_{L^\infty([H^{-s}(\Omega)]^2)} \le c\Big\{\sum_{i=0}^{2k}\Big\|\frac{\partial^i u}{\partial t^i}\Big\|_{L^\infty([H^{q+|\alpha|}(\Omega)]^2)}$$

$$+\sum_{i=1}^{2}\Big\|\frac{\partial^{2k+i}u}{\partial t^{2k+i}}\Big\|_{L^2([H^{q+|\alpha|}(\Omega)]^2)}\Big\}h^{q+s}, \tag{5.1}$$

Let K_h be the Bramble-Schatz Kernel [2]

$$K_h(x) = K_{h,s}^{2s}(x).$$

Then, we define the post-processed approximation u_h^* to u by convolution with K_h:

$$u_h^*(x) = K_h \star u_h. \tag{5.2}$$

It follows from [2] that

$$\|K_h \star u - u\|_0 \le c\|u\|_q h^q, \quad q = 0, \cdots, 2r,$$

$$\|D^\alpha(K_h \star u)\|_m \le c\|\partial^\alpha u\|_m, \quad m = 0, \pm 1, \pm 2, \cdots,$$

$$\|u\|_0 \le c\sum_{|\alpha|\le m}\|D^\alpha u\|_{-m}, \quad m = 1, 2, \cdots. \tag{5.3}$$

Thus, by (5.1) and (5.3),

$$\|u - u_h^*\|_{L^\infty([L^2(\Omega)]^2)} \le \|u - K_h \star u\|_{L^\infty([L^2(\Omega)]^2)} + \|K_h \star (u - u_h)\|_{L^\infty([L^2(\Omega)]^2)}$$

$$\le c\Big\{\|u\|_{L^\infty([H^{q+s}(\Omega)]^2)}h^{q+s} + \sum_{|\alpha|\le s}\|D^\alpha(K_h \star (u - u_h))\|_{L^\infty([H^{-s}(\Omega)]^2)}\Big\}$$

$$\le c\Big\{\sum_{i=0}^{2k}\Big\|\frac{\partial^i u}{\partial t^i}\Big\|_{L^\infty([H^{q+s}(\Omega)]^2)} + \sum_{i=1}^{2}\Big\|\frac{\partial^{2k+i}u}{\partial t^{2k+i}}\Big\|_{L^2([H^{q+s}(\Omega)]^2)}\Big\}h^{q+s}.$$

Theorem 5.1. *Let $k \le r-1$, $1 \le q \le r+1$. Assume that u_h^* is defined by (5.2). Then*

$$\|u - u_h^*\|_{L^\infty([L^2(\Omega)]^2)} \le c\Big\{\sum_{i=0}^{2k}\Big\|\frac{\partial^i u}{\partial t^i}\Big\|_{L^\infty([H^{q+s}(\Omega)]^2)} + \sum_{i=1}^{2}\Big\|\frac{\partial^{2k+i}}{\partial t^{2k+i}}\Big\|_{L^2([H^{q+s}(\Omega)]^2)}\Big\}h^{q+s},$$

for $0 \le s \le \min(k, r-k-1)$.

Acknowledgments. The author should like to thank Professors Jim Douglas,Jr., and J. E. Santos for their help.

References

[1] M. A. Biot, Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range, *Journal of the Acoustical Society of America,* **28** : 2 (1956), 168–178.

[2] J. H. Bramble and A. H. Schatz, Higher order local accuracy by averaging in the finite element, *Math. Compt.*, **31** (1977), 94–111.

[3] F. Brezzi, J. Douglas Jr. and L.D. Marini, Two families of mixed finite elements for second order elliptic problems, *Numer. Math.*, **47** (1985), 217–235.

[4] J. Douglas, Jr., T. Dupont and M. F. Wheeler, A quasi-projection analysis of Galerkin methods for parabolic and hyperbolic equations, *Math. Comput.*, **32** (1978), 345–362.

[5] P. A. Raviart and J. M. Thomas, A mixed finite element method for 2nd order elliptic problems, Mathematical aspects of the finite element method, Lecture Notes in Math. 606, Springer-Verlag, Berlin, 1977.

[6] J. E. Santos, Elastic wave propagation in fluid-saturated porous media, Part I. The existence and uniqueness theorems, *Mathematical Modelling and Numerical Analysis,* **20** (1986), 113–128.

[7] J. E. Santos and E. J. Orena, Elastic wave propagation in fluid-saturated porous media, Part II. The Galerkin procedures, *Mathematical Modelling and Numerical Analysis,* **20** (1986), 129–139.