Journal of Computational Mathematics, Vol.10, No.4, 1992, 348–357.

ON DISCRETE SUPERCONVERGENCE PROPERTIES OF SPLINE COLLOCATION METHODS FOR NONLINEAR VOLTERRA INTEGRAL EQUATIONS*1)

Hermann Brunner
(*Department of Mathematics and Statistics, Meorial University, Newfoundland, Canada*)

Abstract

It is shown that the error corresponding to certain spline collocation approximations for nonlinear Volterra integral equations of the second kind is the solution of a nonlinearly perturbed linear Volterra integral equation. On the basis of this result it is possible to derive general estimates for the order of convergence of the spline solution at the underlying mesh points. Extensions of these techniques to other types of Volterra equations are indicated.

§1. Introduction

Consider the nonlinear Volterra integral equation of the second kind

$$y(t) = g(t) + \int_0^t k(t, s, y(s))ds, \quad t \in I := [0, T], \tag{1.1}$$

where $g: I \to R$ and $k : S \times R \to R$ (with $S := \{(t, s) : 0 \le s \le t \le T\}$) denote given continuous functions, which are assumed to be such that (1.1) has a unique solution $y \in C(I)$. Suppose that $u: I \to R$ is an approximation to y satisfying

$$\|y - u\|_\infty := \sup\{|y(t) - u(t)| : t \in I\} = \sigma(h^p) \quad p > 0, \tag{1.2}$$

as $h \to 0$. Here, $h = h^{(N)}$ is the diameter of the underlying mesh $\Pi_N : 0 = t_0 < t_1 < \cdots < t_N = T$ (with $t_n = t_n^{(N)}$): $h := \max\{t_{n+1} - t_n : 0 \le n \le N-1\}$. Often u converges faster to y on the mesh Π_N than on I, i.e. there exists a $p^* > p$ so that

$$\max\{|y(t) - u(t)| : t \in \Pi_N\} = \sigma(h^{p^*}). \tag{1.3}$$

We then say that u exhibits discrete (or local) superconvergence of order P^* at the mesh points.

This paper is concerned with the following question: assuming that we have established a global convergence result of the form (1.2), how can we verify if the approximation u (obtained, e.g., by collocation in some finite-dimensional function space)

* Received January 25, 1989.
1) This work was supported by the Natural Sciences and Engineering Research Council of Canada (Operating Grant A9406).

where

$$A(t,s) := \left.\frac{\partial k(t,s,y)}{\partial y}\right|_{y=y(s)}$$

and

$$(Be)(t) := -\frac{1}{2}\int_0^t \left.\frac{\partial^2 k(t,s,y)}{\partial y^2}\right|_{y=w(s)} \cdot e^2(s)ds,$$

with $w(s) := y(s) + \theta(s)e(s)$ *for some* $\theta \in (-1,0)$.

Proof. It follows from (1.1) and (2.7) that $e(t)$ satisfies

$$e(t) = r(t) + \int_0^t \{k(t,s,y(s)) - k(t,s,u(s))\}ds, \quad t \in I.$$

The integrand can be written as

$$k(t,s,y(s)) - k(t,s,y(s) - e(s)) = k(t,s,y(s)) - \left.\frac{\partial k(t,s,y)}{\partial y}\right|_{y=y(s)} \cdot e(s)$$
$$+ \frac{1}{2}\left.\frac{\partial^2 k(t,s,y)}{\partial y^2}\right|_{y=w(s)} \cdot e^2(s),$$

where, by Taylor's formula, $w(s) := y(s) + \theta(s)e(s)$, with $-1 < \theta < 0$. This yields (2.8).

In an analogous way we obtain an expression for $e_{it}(t) : +y(t) - u_{it}(t)$.

Lemma 2.2. *The iterated collocation error* $e_{it}(t)$ *corresponding to the iterated collocation solution* $u_{it}(t)$ *given by* (2.3) *is related to* $e(t)$ *by*

$$e_{it}(t) = \int_0^t A(t,s)e(s)ds + (Be)(t), \quad t \in I, \tag{2.9}$$

with $A(t,s)$ *and* $(Be)(t)$ *as in Lemma* 2.1.

Lemma 2.3. *Let* $R(t,s)$ *be the solution of the resolvent equation*

$$R(t,s) = -A(t,s) + \int_s^t A(t,v)R(v,s)dv, \quad (t,s) \in S, \tag{2.10}$$

where the kernel $A(t,s)$ *has been introduced in Lemma 2.1. Then* $e(t)$ *solves the non-linearly perturbed linear Volterra integral equation (2.8) if, and only if, it satisfies*

$$e(t) = r(t) - \int_0^t R(t,s)r(s)ds + (Be)(t) - \int_0^t R(t,s)(Be)(s)ds, \ t \in I. \tag{2.11}$$

Proof. Setting $F(t) := r(t) + (Be)(t)$ it follows from the classical Volterra theory (compare also [8,pp. 189-193]) that the solution of

$$e(t) = F(t) + \int_0^t A(t,s)e(s)ds, \quad t \in I,$$

is given by

$$e(t) = F(t) + \int_0^t R(t,s)F(s)ds, \quad t \in I,$$

where the resolvent kernel $R(t,s)$ associated with $A(t,s)$ is defined by (2.10). This yields (2.11). Obviously, the above steps are reversible.

Lemma 2.4. *The iterated collocation error can be expressed in the form*

$$e_{it}(t) = -\int_0^t R(t,s)r(s)ds + (Be)(t) - \int_0^t R(t,s)(Be)(s)ds, t \in I. \tag{2.12}$$

Proof. If we replace $e(s)$ in (2.9) by the expression (2.11), we obtain

$$e_{it}(t) = \int_0^t A(t,s)\left\{r(s) - \int_0^s R(s,v)r(v)dv + (Be)(s) - \int_0^s R(s,v)(Be)(v)dv\right\} ds + (Be)(t).$$

The resulting double integrals can be simplified by means of the resolvent equation (2.10), e.g.,

$$\int_0^t \left(\int_0^s A(t,s)R(s,v)r(v)\right) ds = \int_0^t \left(\int_v^t A(t,s)R(s,v)ds\right) r(v)dv = \int_0^t (A(t,s) + R(t,s))r(s)ds,$$

and hence (2.12) is readily verified.

The results of Lemma 2.3 and Lemma 2.4 are the keys to answering the question about discrete superconvergence on Π_N. Consider first (2.11) with $t = t_n$ $(l \le n \le N)$. We write it as

$$e(t_n) = r(t_n) - \sum_{i=0}^{n-1} h_i \int_0^1 Q_n(t_i + vh_i)dv + (Be)(t_n) - \int_0^{t_n} R(t_n,s)(Be)(s)ds,$$

where we have set

$$Q_n(t_i + vh_i) := R(t_n, t_i + vh_i)r(t_i + vh_i), \quad i < n.$$

Let

$$\int_0^1 Q_n(t_i + vh_i)dv = \sum_{k=1}^m w_k Q_n(t_i + c_k h_i) + E_{n,i},$$

where

$$w_k := \int_0^1 L_k(s)ds, \quad L_k(s) := \prod_{j \ne k}^m (s - c_j)/(c_k - c_j), \ k = 1, \cdots, m$$

are the weights of the m-point interpolatory quadrature formula based on the abscissas $t_i + c_k h_i$ (i.e. the collocation points $t_{i,k}$) and where $E_{n,i}$ denotes the corresponding quadrature error. Due to the factor $r(t_i + vh_i), Q_n(t_i + vh_i)$ vanishes for $v = c_k$, and hence we obtain

$$e(t_n) = r(t_n) - \sum_{i=0}^{n-1} h_i E_{n,i} + (Be)(t_n) - \int_0^{t_n} R(t_n,s)(Be)(s)ds n = 1, \cdots, N. \tag{2.13}$$

(Note that if the given integral equation (1.1) is linear, i.e., if $k(t,s,y) = K(t,s)y$, then we have $(Be)(t) \equiv 0$ and (2.13) reduces the expression derived in [2].)

Since by assumption the kernel $A(t,s)$ in (2.8) is continuous on S, the resolvent $R(t,s)$ (cf.(2.10)) has the same property. Hence,

$$\left|\int_0^t r(t,s)ds\right| \leq \int_0^t |R(t,s)|ds \leq M_0, \quad t \in I,$$

for some finite constant M_0, and (2.13) leads to the estimate

$$|e(t_n)| \leq |r(t_n)| + Nh \cdot \sup\{|E_{n,i}| : \ 0 \leq i < n \leq N\} + \|Be\|_{infty} \cdot (1 + M_0),$$

$$n = 1, \cdots, N, \qquad (2.14)$$

where $Nh \leq \gamma T$ for some constant $\gamma \geq 1$ characterizing the underlying quasi-uniform mesh sequence $\{\Pi_N\}$.

Comparison of (2.12) with (2.11) reveals that an almost identical estimate holds for the iterated collocation error:

$$|e_{it}(t_n)| \leq Nh \cdot \sup\{|E_{n,i}| : \ 0 \leq i < n \leq N\} + \|Be\|_{infty} \cdot (1 + M_0),$$

$$n = 1, \cdots, N. \quad (2.15)$$

We observe that, except for the term $r(t_n)$ in (2.14), the orders of $e(t_n)$ and $e_{it}(t_n)$ depend on those of the quadrature errors $E_{n,i}$ and the norms of the perturbation term $(Be)(t)$. The order of $E_{n,i}$ is governed by the choice of the parameters $\{c_k\}$. Since the orthogonality condition (2.4) implies that an m-point interpolatory quadrature formula based on the abscissas $\{t_i + c_k h_i\}$ has degree of precision $m + d \leq 2m$, it follows from Peano's theorem (see, e.g., [7] or [6, pp. 285-290]) that

$$|E_{n,i}| \leq Ch^{m+d}, \quad 0 \leq i < n \leq N \qquad (2.16)$$

for sufficiently smooth integrands $Q_n(t_i + vh_i)$.

In order to estimate $\|Be\|_\infty$, recall Lemma 2.1: assuming that $|\partial^2 k(t,s,y)/\partial y^2|$ is bounded by some constant K_2 in a suitable region containing $D := \{(t,s,y(s)) : \ 0 \leq s \leq t \leq T\}$ (where y(s) is the analytical solution of (1.1)), we find

$$\|Be\|_\infty \leq \frac{1}{2} K_2 T \|e\|_\infty^2 = \sigma(\|e\|_\infty^2).$$

It is known (see [4]) that for sufficiently smooth data g and k in (1.1) the collocation error e behaves globally (i.e. on I) like $\|e\|_\infty = \sigma(h^m)$. Hence,

$$\|Be\|_\infty = \sigma(h^{2m}), \qquad (2.17)$$

as $h \to 0$ (with $Nh \leq \gamma^T$).

We are now ready to derive the results of Theorem 2.1.

(a) For $c_m = 1$ we have $t_n \in X(N)$, and hence $r(t_n) = 0 (n = 1, \cdots, N)$. By (2.16) and (2.17) it follows from (2.14) that

$$\max\{|e(t)| : 1 \leq n \leq N\} = \sigma(h^{m+d}) \text{ with } 0 \leq d \leq m.$$

It was shown in [2] that if t_n is not a collocation point, then, in general, we only have $r(t_n) = \sigma(h^m)$. Thus, for $c_m < 1$, $t_n \notin X(N)$, implying (2.5b). This holds in particular if $\{c_k\}$ are the Gauss points in $(0, 1)$.

(b) Proposition (2.6) follows immediately from (2.15),(2.16) and (2.17): we have

$$\max\{|e_{it}(t)| : 1 \leq n \leq N\} = \sigma(h^{m+d}), \quad 0 \leq d \leq m,$$

whether $c_m = 1$ or $c_m < 1$. This concludes the proof of Theorem 2.1.

The optimal value for d in (2.5a), $d = m-1$, is attained if, and only if, the collocation parameters $\{c_j\}$ are the zeros of $P_m(2s-1) - P_{m-1}(2s-1)$ (with P_m denoting the Legendre polynomial of degree m), i.e. the Radau II points. In (2.6) the optimal value of d is $d = m$; it is attained if, and only if, $\{c_j\}$ are the zeros of $P_m(2s-1)$, i.e. the Gauss points for (0,1).

§3. Extensions to Other Volterra Equations

In this section we shall indicate, by means of a $\nu - th$ order Volterra integral differential equation (VIDE), how the techniques introduced in Section 2 can be extended to other types of nonlinear Volterra functional equations. Consider the initial-value problem

$$y^{\nu}(t) = f(t, y(t), \cdots, y^{(\nu-1)}(t)) + \int_0^t k(t, s, y(s), \cdots, y^{\nu}(s)ds), \ t \in I,$$

$$y^{(j)}(0) = y_0^{(j)} \quad (j = 0, \cdots, \nu - 1), \tag{3.1}$$

where $\nu \geq 1$. Assume that we solve (3.1) by collocation in the spline space

$$S^q_{m+\nu}(\Pi_N) := \{u \in C^q(I) : \ u|_{\sigma_n} =: u_n \in \Pi_{m+q} \ (0 \leq n \leq N-1),$$

where $q : \nu - 1$ (compare also [3] for the linear version of (3.1)). Since we have $\dim S^{(q)}_{m+\nu}(\Pi_N) = N_m + \nu$, the set of collocation points, $X(N)$, will be as in (2.2). Writing the collocation equation for $u \in S^{(q)}_{m+\nu}(\Pi_N)$ in the form

$$u^{(\nu)}(t) = f(t, u(t), \cdots, u^{(\nu-1)}(t)) - r(t) + \int_0^t k(t, s, u(s), \cdots, u^{(\nu)}(s))ds, \ t \in I, \tag{3.2}$$

with $u^{(j)}(0) = y_0^{(j)}$ $(j = 0, \cdots, q)$ and with the residual function $r(t)$ vanishing on the set $X(N)$, it follows from (3.1) and (3.2) that the collocation error $e(t) : = y(t) - u(t)$ solves

$$e^{(\nu)}(t) = f(t, y(t), \cdots, y^{(\nu-1)}(t)) - f(t, u(t), \cdots, u^{(\nu-1)}(t)) + r(t)$$
$$+ \int_0^t \{k(t, s, y(s), \cdots, y^{(\nu)}(s)) - k(t, s, u(s), \cdots, u^{(\nu)}(s))\}ds, \ t \in I, \tag{3.3}$$

with $e^{(j)}(0) = 0 (j = 0, \cdots, q)$. Since $u = y - e$, we may write

$$f(t, y(t), \cdots, y^{(\nu-1)}(t)) - f(t, u(t), \cdots, u^{(\nu-1)}(t))$$
$$= - \sum_{i=0}^{\nu-1} D_i f(t, y(t), \cdots, y^{(\nu-1)}(t)) \cdot e^{(i)}(t)$$
$$+ \frac{1}{2} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\nu-1} D_i D_j f(t, v_0(t), \cdots, v_{\nu-1}(t)) \cdot e^{(i)}(t) e^{(j)}(t)$$

(where, for $f = f(t, z_0, \cdots, z_{\nu-1})$, $D_i := \partial/\partial z_i$), and

$$k(t,s,y(s),\cdots,y^{(\nu)}(s)) - k(t,s,u(s),\cdots,u^{(\nu)}(s))$$
$$= -\sum_{i=0}^{\nu-1} D_i k(t,s,y(s),\cdots,y^{(\nu)}(s)) \cdot e^{(i)}(s)$$
$$+ \frac{1}{2}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\nu-1} D_i D_j k(t,s,w_0(s),\cdots,w_\nu(s)) \cdot e^{(i)}(s)e^{(j)}(s)$$

(where, for $k = k(t, s, z_0, \cdots, z_\nu)$, $D_i : \partial/\partial z_i$). Here, the functions v_i and w_i are given by the respective Taylor remainder terms. Introducing the functions

$$p_i(t) := D_i f(t, y(t), \cdots, y^{(\nu-1)}(t)) \ \ (i = 0, \cdots, \nu - 1)$$

and

$$K_i(t,s) := D_i k(t,s,y(s),\cdots,y^{(\nu)}(s)) \ \ (i = 0,\cdots,\nu),$$

as well as the functional

$$(\beta e)(t) := -\frac{1}{2}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\nu-1} D_i D_j f(t, v_0(t),\cdots,v_{\nu-1}(t)) \cdot e^{(e)}(t)e^{(j)}(t)$$
$$-\frac{1}{2}\sum_{i=0}^{\nu}\sum_{j=0}^{\nu} D_i D_j k(t,s,w_0(s),\cdots,w_\nu(s)) \cdot e^{(e)}(s)e^{(j)}(s)ds,$$

we can rewrite (3.3) in the form

$$e^{(\nu)}(t) = \sum_{i=0}^{\nu-1} p_i(t)e^{(i)}(t) + r(t) + \int_0^t \sum_{i=0}^{\nu} K_i(t,s)e^{(i)}(s)ds + (\beta e)(t), \quad t \in I, \qquad (3.4)$$

with

$$e^{(j)}(0) = 0 \ \ (j = 0,\cdots,\nu - 1).$$

Let $z: \ I \to R^{\nu+1}$ have components $z_i(t) := e^{(i)}(t)$ $(i = 0, \cdots, \nu)$. We then have, due to the above initial conditions,

$$z_i(t) = \int_0^t z_{i+1}(s)ds \ \ (i = 0,\cdots,\nu - 1), \qquad (3.5a)$$

and hence, by (3.4)

$$z^{(\nu)}(t) = \sum_{i=0}^{\nu-1} p_i(t) \cdot \int_0^t z_{i+1}(s)ds + r(t) + \int_0^t \sum_{i=0}^{\nu} K_i(t,s)z^{(i)}(s)ds + (\beta e)(t),$$
$$t \in I. \qquad (3.5b)$$

The above equations (3.5a)–(3.5b) represent a system of $\nu+1$ Volterra integral equations of the second kind for the components of $z(t)$. Setting

$$D(t) := (0,\cdots,0,r(t))^T \in R^{\nu+1},$$
$$(Bz)(t) := (0,\cdots,0,(\beta e)(t))^T \in R^{+1},$$

and

$$A(t,s) := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ K_0(t,s) & p_0(t) + K_1(t,s) & \cdots & p_{\nu-1}(t) + K_\nu(t,s) \end{pmatrix},$$

this system assumes the form

$$z(t) = D(t) + \int_0^t A(t,s)z(s)ds + (Bz)(t), \quad t \in I. \tag{3.6}$$

We observe that (3.6) is the analogue of the nonlinearly perturbed linear Volterra integral equation (2.8). Thus, if $R(t,s)$ is the resolvent associated with the matrix kernel $A(t,s)$ in (3.6), i.e. the solution of the resolvent equation

$$R(t,s) = -A(t,s) + \int_s^t A(t,v)R(v,s)dv, \quad (t,s) \in S,$$

then the solution of (3.6) must satisfy

$$z(t) = D(t) - \int_0^t R(t,s)D(s)ds + (Bz)(t) - \int_0^t R(t,s)(Bz)(s)ds, \quad t \in I, \tag{3.7}$$

in analogy to (2.11). In particular, if we denote the elements of the matrix $R(t,s)$ by $R_{i,j}(t,s)$ $(i,j = 0,\cdots,\nu)$, then (3.7) yields

$$z_0(t) = e(t) = -\int_0^t R_{0,\nu}(t,s)r(s)ds - \int_0^t R_{0,\nu}(t,s)(\beta e)(s)ds, \quad t \in I.$$

Note that here, in contrast to (2.11), the residual term $r(t)$ no longer occurs outside the integral sign this implies that the discrete superconvergence result (2.5a) of Theorem 2.1 holds whether we have $c_m = 1$ or $c_m < 1$.

Moreover, (2.5a) is also valid for the components $z_\nu(t) = e^{(\nu)}(t)$ $(i = 1,\cdots,\nu-1)$ of $z(t)$, due to the structure of the vector $D(t)$, both for $c_m = 1$ and $c_m < 1$. However, it follows from (3.7) that the last component of $z(t)$, $z_\nu(t) = e^{(\nu)}(t)$, is given by

$$e^{(\nu)}(t) = r(t) - \int_0^t R_{\nu,\nu}(t,s)r(s)ds + (\beta e)(t) - \int_0^t R_{\nu,\nu}(t,s)(\beta e)(s)ds.$$

Thus, $\max\{|e^{(\nu)}(t_n)| : 1 \le n \le N\} = \sigma(h^{m+d})$ if $c_m = 1$; for $c_m < 1$ (e.g. for the Gauss points $\{c_j\}$) we only obtain $\max\{|e^{(\nu)}(t_n)| : 1 \le n \le N\} = \sigma(h^m)$, in analogy to (2.5b). In order to generate a more accurate approximation to $y^{(\nu)}(t_n)$ (in the case where $c_m < 1$) one would have to compute

$$u_{it}^{(\nu)}(t) := f(t,u(t),\cdots,u^{(\nu-1)}(t)) + \int_0^t k(t,s,u(s),\cdots,u^{(\nu)}(s))ds,$$

corresponding to the collocation solution $u \in S_{m+v}^{(q)}(\Pi_N)$ determined from (3.2). It is then easy to show that $e_{it}^{(\nu)} := y^{(\nu)}(t) - u_{it}^{(\nu)}(t)$ satisfies an estimate analogous to the one in (2.6).

As in Section 2, these discrete superconvergence results hinge on the global convergence properties of the collocation approximation u on I (as well as on the orthogonality condition (2.4)):

(i) It can be shown (see [4] for the underlying techniques) that for the collocation solution given by (3.2) there holds the global error estimate $\|e^{(i)}\|_\infty = \sigma(h^m)(i = 0, \cdots, \nu)$, provided f and k in (1.1) are sufficiently smooth functions.

(ii) It then follows from the form of the functional $(\beta e)(t)$ in (3.4) that

$$\|\beta e\|_\infty \leq C_0 \sum_{i=0}^{\nu-1} \sum_{j=0}^{\nu-1} \|e^{(i)}\|_\infty \cdot \|e^{(j)}\|_\infty + C_1 \sum_{i=0}^{\nu-1} \sum_{j=0}^{\nu-1} \|e^{(i)}\|_\infty \cdot \|e^{(j)}\|_\infty = \sigma(h^{2m})$$

for suitable constants C_0 and C_1.

References

[1] J.G. Blom and H. Brunner, The numerical solution of nonlinear Volterra integral equations of the second kind by collocation and iterated collocation methods, *SIAM J. Sci. Statist. Comput.*, **8** (1987), 806–830.

[2] H. Brunner, Iterated collocation methods and their discretization for Volterra integral equations, *SIAM J. Numer. Anal.*, **21** (1984), 1132–1145.

[3] H. Brunner, The approximate solution of initial-value problems for general Volterra integro-differential equations, *Computing*, **40** (1988), 125–137.

[4] H. Brunner and P.J. van der Houwen, the Numerical Solution of Volterra Equations, CWI Monographs, Vol. 3, North-Holland, Amsterdam, 1986.

[5] H. Brunner and S.P. Nrsett, Superconvergence of collocation methods for Volterra and Abel integral equations of the second kind, *Numer. Math.*, **36** (1981), 347–358.

[6] P.J. Davis and R. Rabinowitz, Methods of Numerical Integration (2nd ed.), Academic Press, London, 1984.

[7] F. de Hoog and R. Weiss, Implicit Runge-Kutta methods for second-kind Volterra integral equations, *Numer. Math.*, **23** (1975), 199–213.

[8] R.K. Miller, Nonlinear Volterra Integral Equations, 1971.