Journal of Computational Mathematics, Vol.10, No.4, 1992, 339–347.

OPTIMAL INTERPOLATION OF SCATTERED DATA ON A CIRCULAR DOMAIN WITH BOUNDARY CONDITIONS *

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Abstract

Optimal interpolation problems of scattered data on a circular domain with two different types of boundary value conditions are studied in this paper. Closed-form optimal solutions, a new type of spline functions defined by partial differential operators, are obtained. This type of new splines is a generalization of the well-known L_g-splines and thin-plate splines. The standard reproducing kernel structure of the optimal solutions is demonstrated. The new idea and technique developed in this paper are finally generalized to solve the same interpolation problems involving a more general class of partial differential operators on a general region.

§1. Introduction

The "thin plate" splines were first introduced by Duchon[5] over an unbounded domain $\Omega = R^2$. A multivariable interpolation problem relating to this type of thin plate splines was also studied by Meingnet[13]. Based on basis functions including thin plate splines, a numerical method for the interpolation problem of scattered data over a finite domain without boundary conditions was presented by Dyn and Levin[6]. A closed-form solution to the interpolation problem of scattered data over a circular domain with boundary conditions, which leads to the so-called "biharmonic spline" (a second order thin plate spline), was given via the variational technique by Li[12]. Other related results may also be found in, for example, Dyn and Wahba [7], Freeden [9, 10], Li [11], Utreras [15, 16], and Wahba [19]. Some interesting applications of the biharmonic spline to state-constrained minimum-energy optimal control problems with steady-state distributed harmonic systems were shown in Chen [1, 2, 3], where explicit closed-form optimal control and state functions were obtained from the reproducing kernel Hilbert space approach.

In this paper, we will study two optimal interpolation problems of scattered data on a circular domain with two different types of boundary value conditions. Closed-form solutions to the two corresponding problems posed below will be obtained. The optimal solutions will be defined as, in a minimum-norm interpolation sense, a new

* Received January 13, 1989.

type of spline functions defined by certain partial differential operators with boundary value conditions, which is a natural generalization of the well-known L_g and thin plate splines. Moreover, the elegant reproducing kernel structure for these two different types of splines will also be demonstrated. Finally, the idea and technique will be generalized to solve the same interpolation problems involving a more general class of partial differential operators on a general region.

§2. Statement of Problems

Let Ω be the open disk of radius a and centered at the origin in R^2 with a boundary Γ. For a nonnegative integer k, let $H^k = H^k(\Omega)$ be the usual (real) Sobolev space of order k on Ω, endowed with the inner product and norm defined respectively by

$$\langle f,g\rangle_{H^k} = \sum_{\alpha_1+\alpha_2\leq k} \int_\Omega \Big(\frac{\partial^{\alpha_1+\alpha_2}}{\partial^{\alpha_1}x\partial^{\alpha_2}y}f\Big)\Big(\frac{\partial^{\alpha_1+\alpha_2}}{\partial^{\alpha_1}x\partial^{\alpha_2}y}g\Big)dxdy$$

and

$$\|\cdot\|_{H^k} = (\langle\cdot,\cdot\rangle_{H^k})^{1/2},$$

where $\alpha_1, \alpha_2 \geq 0$. Denote by $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ the Laplacian operator as usual. Given m real-valued continuous functions $\{\phi_i\}_{i=0}^{m-1}$ defined on Γ, and a set of scattered data $\{z_i\}_{i=1}^{l}$, the main objective of this paper is to solve the following two constrained minimization problems:

$$\underset{w\in H^m(\Omega)}{\text{minimize}}: \int_\Omega (\Delta^m w)^2 dxdy \tag{2.1}$$

subject to

$$w(x_i, y_i) = z_i, \quad (x_i, y_i) \in \Omega, \quad i = 1, \cdots, l \tag{2.2}$$

and either

$$w|_\Gamma = \phi_0, \quad \Delta w|_\Gamma = \phi_1, \cdots, \Delta^{m-1} w|_\Gamma = \phi_{m-1}, \tag{2.3a}$$

or

$$w|_\Gamma = \phi_0, \quad \frac{\partial w}{\partial r}\Big|_\Gamma = \phi_1, \cdots, \frac{\partial^{m-1} w}{\partial r^{m-1}}\Big|_\Gamma = \phi_{m-1}, \tag{2.3b}$$

where $\{(x_i, y_i)\}_{i=1}^{l}$ are l distinct interior points in Ω and $\partial/\partial r$ is the outward normal derivative on Γ. For convenience, we will call the problem with boundary conditions (2.3a) Problem A and the one with (2.3b) Problem B.

§3. Construction of the Optimal Solutions

In this section, we will show that the optimal solutions to Problems A and B have the same form:

$$w^* = u^* + \sum_{k=0}^{m-1} w_k,$$

with the same u^*, where for each $k, k = 0, 1, \cdots, m-1, w_k$ is in $N(\Delta^{k+1})$, the null space of the operator Δ^{k+1}, and satisfies certain boundary conditions, and u^* is the optimal solution of a reformulated minimization problem. It will be shown in the next section that u^* has the standard structure consisting of the reproducing kernel of a Sobolev space.

To facilitate our discussion, we will use the polar coordinates. We will first discuss Problem A. For each $k, k = 0, 1, \cdots, m-1$, let w_k be the (unique) solution of the following boundary value problem:

$$\begin{cases} \Delta^{k+1} w_k = 0, \\ w_k|_\Gamma = \Delta w_k|_\Gamma = \cdots = \Delta^{k-1} w_k|_\Gamma = 0, \quad \Delta^k w_k|_\Gamma = \Phi_k. \end{cases} \tag{3.1}$$

It is clear that w_k is in $N(\Delta^{k+1}) \subset N(\Delta^m)$ and can be obtained by solving consecutively the following simple boundary value problems:

$$\begin{cases} \Delta w_0 = 0, \\ w_0|_\Gamma = \phi_0 \end{cases} \tag{3.2a}$$

and for $k = 1, \cdots, m-1$,

$$\begin{cases} \Delta v_1 = 0, \\ v_1|_\Gamma = \phi_k, \end{cases} \quad \begin{cases} \Delta v_2 = v_1, \\ v_2|_\Gamma = 0, \end{cases} \quad \cdots \quad \begin{cases} \Delta v_k = v_{k-1}, \\ v_k|_\Gamma = 0, \end{cases} \quad \begin{cases} \Delta w_k = v_k, \\ w_k|_\Gamma = 0. \end{cases} \tag{3.2b}$$

As is well known, each of these problems has a unique solution which can be explicitly expressed via the Poisson formula as follows: The first one is a Dirichlet problem with the solution

$$v_1 = \frac{1}{2\pi} \int_0^{2\pi} \phi_k \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - t) + r^2} dt;$$

by this result, the second one has the solution

$$v_2 = \int_\Omega H(r, \theta; s, t) v_1(s, t) ds dt$$

where

$$H(r, \theta; s, t) = \frac{1}{4\pi} \ln \left\{ \frac{r^2 - 2rs\cos(\theta - t) + s^2}{s^2(r^2 - 2ar\cos(\theta - t) + a^2)} \right\},$$

and so on.

On the other hand, let u^* be a solution of the following constrained minimization problem:

$$\begin{cases} \min\limits_{u \in H_0^m(\Omega)} \int_\Omega (\Delta^m u)^2, \\ u(r_i, \theta_i) = z_i - \sum\limits_{k=0}^{m-1} w_k(r_i, \theta_i), \quad (r_i, \theta_i) = (x_i, y_i), \quad i = 1, \cdots, l, \\ H_0^m(\Omega) = \{u : \Delta^m u \in L_2(\Omega), \quad u|_\Gamma = \Delta u|_\Gamma = \cdots = \Delta^{m-1} u|_\Gamma = 0\}. \end{cases} \tag{3.3}$$

It will be seen from the next section that u^* is unique and has a closed-form in the standard reproducing kernel structure. Finally, set

$$w^* = u^* + \sum_{k=0}^{m-1} w_k \tag{3.4}$$

as mentioned above. Then, it can be easily verified that w^* is the optimal solution to Problem A.

As to Problem B, we let w_k be the (unique) solution of the following boundary value problem instead of (3.1):

$$\begin{cases} \Delta^{k+1}w_k = 0, \\ w_k\Big|_\Gamma = \dfrac{\partial w_k}{\partial r}\Big|_\Gamma = \cdots = \dfrac{\partial^{k-1}w_k}{\partial r^{k-1}}\Big|_\Gamma = 0, \quad \dfrac{\partial^k w_k}{\partial r^k}\Big|_\Gamma = \phi_k - \sum_{i=0}^{k-1}\dfrac{\partial^k w_i}{\partial r^k}\Big|_\Gamma, \end{cases} \tag{3.5}$$

$k = 0,1,\cdots,m-1$. Clearly, $w_k \in N(\Delta^{k+1}) \subset N(\Delta^m)$. It can also be verified that the solutions of (3.5) may be obtained recursively as follows:

$$w_0 = \frac{1}{2\pi}\int_0^{2\pi} \phi_0 \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - t) + r^2}dt \tag{3.6}$$

and for $k = 1,\cdots,m-1$,

$$w_k = (r^2 - a^2)^k v_k$$

with v_k satisfying

$$\begin{cases} \Delta v_k = 0, \\ v_k|_\Gamma = \dfrac{1}{(2a)^k k!}\Big\{\phi_k - \displaystyle\sum_{i=0}^{k-1}\dfrac{\partial^k w_i}{\partial r^k}\Big|_\Gamma\Big\}, \quad k = 1,\cdots,m-1. \end{cases}$$

Again, this is a Dirichlet problem so that

$$w_k = \frac{(r^2 - a^2)^k}{2\pi(2a)^k k!}\int_0^{2\pi}\Big(\phi_k - \sum_{i=0}^{k-1}\frac{\partial^k w_i}{\partial r^k}\Big|_\Gamma\Big)\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - t) + r^2}dt, \tag{3.7}$$

$k = 1,\cdots,m-1$. It is also clear that the function w^* given by (3.4), with $\{w_k\}_{k=0}^{m-1}$ satisfying (3.5) and obtained in (3.6) and (3.7), is the optimal solution to Problem B.

Hence, what is left now is to solve the minimization problem (3.3).

§4. Reproducing Kernel Structure of the Optimal Solution u^*

In this section, we will obtain a closed-form optimal solution u^* for the minimization problem (3.3), and show that u^* has the standard structure consisting of the reproducing kernel of the Sobolev space $H_0^m(\Omega)$.

First, we recall the following old, yet useful, result which may be found in Vekua [17] or [18]:

Lemma 4.1. *The function defined by*

$$G_n(r) = \frac{r^{2(n-1)}}{2^{2n-1}\pi((n-1)!)^2}\ln(r) \tag{4.1}$$

is the fundamental solution of the equation $(-1)^n\Delta^n w = f$; namely,

$$(-1)^n\Delta^n G_n(r) = \delta(r). \tag{4.2}$$

Secondly, for the positive integer m defined in the previous sections, let $\tilde{G}(r)$ be the (unique) solution of the following boundary value problem:

$$\begin{cases} \Delta^{2m}\tilde{G} = 0, \\ \tilde{G}|_\Gamma = G_{2m}, \quad \Delta\tilde{G}|_\Gamma = \Delta G_{2m}, \cdots, \Delta^{2m-1}\tilde{G}|_\Gamma = \Delta^{2m-1}G_{2m}. \end{cases} \tag{4.3}$$

As has been seen in (3.1) and (3.2), this problem is equivalent to the following system of simple boundary value problems:

$$\begin{cases} \Delta g_1 = 0, \\ g_1|_\Gamma = \Delta^{2m-1}G_{2m}, \end{cases} \quad \cdots \quad \begin{cases} \Delta g_{2m-1} = g_{2m-2}, \\ g_{2m-1}|_\Gamma = \Delta G_{2m}, \end{cases} \quad \begin{cases} \Delta\tilde{G} = g_{2m-1}, \\ \tilde{G}|_\Gamma = G_{2m}, \end{cases} \tag{4.4}$$

and each of these problems has a unique solution which can be expressed via the Poisson formula as usual. Hence, $\tilde{G}(r)$ is uniquely and explicitly determined.

Furthermore, let the space $H_0^m(\Omega)$ defined in (3.3) be endowed with the inner product and (semi)-norm

$$\langle f, g\rangle_{H_0^m} = \int_\Omega (\Delta^m f)(\Delta^m g)$$

and

$$\|\cdot\|_{H_0^m} = \left(\int_\Omega (\Delta^m(\cdot))^2\right)^{1/2},$$

respectively. Set

$$K(r) = G_{2m}(r) - \tilde{G}(r) \tag{4.5}$$

with $G_{2m}(r)$ and $\tilde{G}(r)$ being defined in (4.1) and (4.3), respectively. Then, we have the following:

Theorem 4.1. *H_0^m is a Sobolev space with the reproducing kernel $K(r)$ given by* (4.5).

Proof. First, it follows from (4.3) that

$$\Delta^k K|_\Gamma = \Delta^k(G_{2m} - \tilde{G})|_\Gamma = 0, \quad k = 0, 1, \cdots, 2m-1.$$

Applying these homogeneous boundary conditions to the Green identity, i.e., integrating by parts m times, we have

$$\langle K(r-s), f(r,\theta)\rangle_{H_0^m} = \int_\Omega \Delta^m K(r-s)\Delta^m f(r,\theta) = \int_\Omega [\Delta^{2m}K(r-s)]f(r,\theta)$$

$$= \int_\Omega \delta(r-s)f(r,\theta) = f(s,\theta), \quad \forall\, f \in H_0^m(\Omega).$$

This implies that $K(r)$ is the reproducing kernel of $H_0^m(\Omega)$. Moreover, it can be easily seen that the semi-norm $\|\cdot\|_{H_0^m}$ defined above is actually a real norm since $\|f\|_{H_0^m} = 0$ implies

$$\begin{cases} \Delta^m f = 0, \\ f|_\Gamma = \Delta f|_\Gamma = \cdots = \Delta^{m-1}f|_\Gamma = 0 \end{cases}$$

which yields $f \equiv 0$ (cf. (3.1) and (3.2)). Hence, $H_0^m(\Omega)$ is a Sobolev space endowed with this norm. This completes the proof of the theorem.

Consequently, we have the following standard result. We include a proof for completeness.

Theorem 4.2. *The optimal solution u^* to the Problem* (3.3) *is uniquely determined by*

$$u^*(x,y) = \sum_{i=1}^{l} \beta_i K(x - x_i, y - y_i) \tag{4.6}$$

where $K(x,y) := K(r)$ *with* $r = \sqrt{x^2+y^2}$, *and*

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_l \end{bmatrix} = \begin{bmatrix} K(0) & K(x_1 - x_2, y_1 - y_2) & \cdots & K(x_1 - x_l, y_1 - y_l) \\ \vdots & & & \vdots \\ K(x_l - x_1, y_l - y_1) & K(x_l - x_2, y_l - y_2) & \cdots & K(0) \end{bmatrix}^{-1} \times \begin{bmatrix} z_1 - \sum_{k=0}^{m-1} w_k(x_1,y_1) \\ \vdots \\ z_l - \sum_{k=0}^{m-1} w_k(x_l,y_l) \end{bmatrix}. \tag{4.7}$$

Proof. Since by Theorem 4.1 $H_0^m(\Omega)$ is a reproducing kernel Hilbert space with

$$\|u\|_{H_0^m} = \left(\iint_\Omega (\Delta^m u)^2 dx dy \right)^{1/2},$$

the minimization problem (3.3) is equivalent to finding $u^* \in I_z$ such that

$$\|u^*\|^2_{H_0^m} = \min_{u \in I_z} \|u\|^2_{H_0^m},$$

where

$$I_z = \left\{ u \in H_0^m(\Omega) : \ u(x_i,y_i) = z_i - \sum_{k=0}^{m-1} w_k(x_i,y_i), \quad i = 1,2,\cdots,l \right\}.$$

Because the set I_z is closed and convex in $H_m^0(\Omega)$, the above minimization problem has a unique solution u^* in I_z.

It is clear that the functions $\{K(x-x_i, y-y_i)\}_{i=1}^l$ are linearly independent. Set

$$\mathcal{K} = \operatorname{span}\{K(x-x_1, y-y_1), \cdots, K(x-x_l, y-y_l)\}$$

and let $\mathcal{K}^\perp$ be the orthocomplement of $\mathcal{K}$ in $H_0^m(\Omega)$. Then, for any $v \in \mathcal{K}^\perp$, we have

$$0 = \langle v, K(x-x_i, y-y_i)\rangle_{H_0^m} = v(x_i,y_i), \quad i = 1,2,\cdots,l.$$

Hence, $u^* + v \in I_z$. However, by the minimum-norm property of u^*, we have

$$\|u^*\|^2_{H_0^m} \le \|u^* + v\|^2_{H_0^m} = \|u^*\|^2_{H_0^m} + 2\langle u^*, v\rangle_{H_0^m} + \|v\|^2_{H_0^m}.$$

This implies that $\langle u^*, v\rangle_{H_0^m} = 0$. Indeed, if $\langle u^*, v\rangle_{H_0^m} = \alpha \neq 0$, then

$$\|u^* - \beta v\|^2_{H_0^m} = \|u^*\|^2_{H_0^m} - 2\alpha\beta + \beta^2 \|v\|^2_{H_0^m}.$$

By choosing an appropriate nonzero β, it is possible to make $-2\alpha\beta + \beta^2\|v\|^2_{H_0^m} < 0$, so that $\|u^* - \beta v\|^2_{H_0^m} < \|u^*\|^2_{H_0^m}$ where $u^* - \beta v \in I_z$, contradicting the minimum-norm property of u^*. Hence, $\langle u^*, v\rangle_{H_0^m} = 0$; that is, u^* is orthogonal to $\mathcal{K}^\perp$, so that $u^* = \sum_{i=1}^{l} \beta_i K(x - x_i, y - y_i)$. The proof of the theorem is completed.

§5. Generalization of the Results

The idea and technique developed above can be generalized to the same problems with a more general class of partial differential operators in the form

$$\prod_{i=1}^{m} L_i := L_m L_{m-1} \cdots L_1,$$

where each L_i is a linear partial differential operator on a general region with boundary value conditions. More precisely, we will consider the following two constrained minimization problems:

$$\min_{w \in H^m(\Omega)} : \int_\Omega \Big(\prod_{i=1}^{m} L_i w\Big)^2 \tag{5.1}$$

subject to

$$w(r_i, \theta_i) = z_i, \quad (r_i, \theta_i) \in \Omega, \quad i = 1, \cdots, l, \tag{5.2}$$

and either

$$w|_\Gamma = \phi_0, \quad L_1 w|_\Gamma = \phi_1, \ \cdots, \ \prod_{i=1}^{m-1} L_i w\Big|_\Gamma = \phi_{m-1} \tag{5.3a}$$

or

$$w|_\Gamma = \phi_0, \quad \frac{\partial w}{\partial r}\Big|_\Gamma = \phi_1, \ \cdots, \ \frac{\partial^{m-1} w}{\partial r^{m-1}}\Big|_\Gamma = \phi_{m-1}. \tag{5.3b}$$

For a general region Ω, the second boundary value problem may not be well-defined and explicit closed-form optimal solutions may not exist. But the same idea and technique can nevertheless be carried out. In the following, we consider only the case that Ω is a disk. In this case, we will call the problem with boundary conditions (5.3a) Problem C and the one with (5.3b) Problem D. For the most general case where even the time variable is also taken into account, the reader is referred to de Figueiredo and Chen [4].

For Problem C, let w_k be a solution (if it exists) of the following boundary value problem:

$$\begin{cases} \prod_{i=1}^{k+1} L_i w_k = 0, \\ w_k|_\Gamma = L_1 w_k|_\Gamma = \cdots = \prod_{i=1}^{k-1} L_i w_k\Big|_\Gamma = 0, \quad \prod_{i=1}^{k} L_i w_k\Big|_\Gamma = \phi_k. \end{cases} \tag{5.4}$$

Then, for each $k, k = 0, 1, \cdots, m-1$, w_k is in $N(\prod_{i=1}^m L_i)$ and can be obtained by solving successively the following boundary value problems:

$$\begin{cases} L_1 w_0 = 0, \\ w_0|_\Gamma = \phi_0 \end{cases} \tag{5.5a}$$

and for $k = 1, \cdots, m-1$,

$$\begin{cases} L_{k+1} v_1 = 0, \\ v_1|_\Gamma = \phi_k, \end{cases} \quad \begin{cases} L_k v_2 = v_1, \\ v_2|_\Gamma = 0, \end{cases} \quad \cdots \quad \begin{cases} L_k v_k = v_{k-1}, \\ v_k|_\Gamma = 0, \end{cases} \quad \begin{cases} L_1 w_k = v_k, \\ w_k|_\Gamma = 0. \end{cases} \tag{5.5b}$$

For Problem D, we will let w_k be a solution (if it exists) of the following:

$$\begin{cases} \prod_{i=1}^{k+1} L_i w_k = 0, \\ w_k\Big|_\Gamma = \dfrac{\partial w_k}{\partial r}\Big|_\Gamma = \cdots = \dfrac{\partial^{k-1} w_k}{\partial r^{k-1}}\Big|_\Gamma = 0, \quad \dfrac{\partial^k w_k}{\partial r^k}\Big|_\Gamma = \phi_k - \sum_{i=0}^{k-1} \dfrac{\partial^k w_i}{\partial r^k}\Big|_\Gamma, \end{cases} \tag{5.6}$$

$k = 0, 1, \cdots, m-1$. It is also clear that $w_k \in N(\prod_{i=1}^m L_i)$. Similarly, for both Problems C and D, let u^* be a solution of the following constrained minimization problem:

$$\begin{cases} \min\limits_{u \in H_0^m(\Omega)} \int_\Omega \Big(\prod_{i=1}^m L_i u \Big)^2, \\ u(r_i, \theta_i) = z_i - \sum_{k=0}^{m-1} w_k(r_i, \theta_i), \quad (r_i, \theta_i) \in \Omega, \quad i = 1, \cdots, l, \\ H_0^m(\Omega) = \Big\{ u : \prod_{i=1}^m L_i u \in L_2(\Omega), \quad u\Big|_\Gamma = L_1 u\Big|_\Gamma = \cdots = \prod_{i=1}^{m-1} L_i u\Big|_\Gamma = 0 \Big\}, \end{cases} \tag{5.7}$$

and set

$$w^* = u^* + \sum_{k=0}^{m-1} w_k. \tag{5.8}$$

Then, it is easily seen that w^* will be a required optimal solution. Obviously, w^* generalizes in a natural way the well-known L_g-splines defined by ordinary differential operators.

Note that if the operator $(\prod_{i=1}^m L_i)$ is a general uniformly elliptic operator with a smooth (not necessarily circular) boundary $\partial\Omega$, under appropriate assumptions the above boundary value problems (5.4) and (5.6) always have solutions (cf. Fink [8]). We finally remark that a very general situation, where even the time variable is also considered, has been taken care of in a joint paper of the author [4].

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