Journal of Computational Mathematics, Vol.10, No.4, 1992, 321–338.

MINIMAX METHODS FOR OPEN-LOOP EQUILIBRA IN N-PERSON DIFFERENTIAL GAMES PART III: DUALITY AND PENALTY FINITE ELEMENT METHODS*1)

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Abstract

The equilibrium strategy for N-person differential games can be obtained from a min-max problem subject to differential constraints. The differential constraints can be treated by the duality and penalty methods and then an unconstrained problem can be obtained. In this paper we develop methods applying the finite element methods to compute solutions of linear-quadratic N-person games using duality and penalty formulations.

The calculations are efficient and accurate. When a (4,1)-system of Hermite cubic splines are used, our numerical results agree well with the theoretical predicted rate of convergence for the Lagrangian. Graphs and numerical data are included for illustration.

§1. Introduction

As in Part I and Part II, we consider an N-person differential game with the following dynamics:

$$(DE) \equiv \dot{x}(t) - A(t)x(t) - \sum_{i=1}^{N} B_i(t)u_i(t) - f(t) = 0, \qquad \text{on } [0,T],$$
$$x(0) = x_0 \in R^n. \tag{1.1}$$

The matrix and vector functions $A(t), f(t), B_i(t), u_i(t), i = 1, \cdots, N$, satisfy the same conditions as in Part I and II ([6] and [7]). Each player wants to minimize his cost

$$J_i(x,u) = J_i(x, u_1, \cdots, u_N), \quad i = 1, \cdots, N. \tag{1.2}$$

* Received December 31, 1988.
1) Supported in part by NSF Grant MCS 81-01892 and NASA contract No. NAS1-15810.

Let

$$F(x,u;X,v) = F(x,u_1,\cdots,u_N;x^1,\cdots,x^N,v_1,\cdots,v_N) = \sum_{i=1}^{N}\left[J_i(x,u) - J_i(x^i,v^i)\right], \tag{1.3}$$

where $X = (x^1,\cdots,x^N)$, $v^i = (u_1,\cdots,u_{i-1},v_i,u_{i+1},\cdots,u_N)$ and each x^i is the solution of

$$(DE)_i \equiv \dot{x}^i(t) - A(t)x^i(t) - \sum_{j\neq i} B_j(t)u_j(t) - B_i(t)v_i(t) - f(t) = 0, \quad \text{on } [0,T],$$

$$x^i(0) = x_0, \qquad i = 1,\cdots,N. \tag{1.4}$$

Following [6] and [7], we consider the primal and dual problems:

(P) $\inf_{x,u}\sup_{X,v}\{F(x,u;X,v)|(x,u)\in H_n^1\times U$ subject to (1.1), $(X,v)\in[H_n^1]^N\times U$ subject to (1.4), $i=1,\cdots,N\}$

(D) $\sup_{p_0\in L^2}\inf_{p\in[L^2]^N} L(p_0,p)$, where $L(p_0,p) = L(p_0,p_1,\cdots,p_N) = \inf_{x,u}\sup_{X,v} L(p_0,p;x,u;X,v)$ with the Lagrangian $L : L^2x[L^2]^N\times H_n^1\times U\times[H_n^1]^N\times U$ defined by

$$\mathbf{L}(p_0,p;x,u;X,v) \equiv F(x,u;X,v) + \left\langle p_0, \dot{x} - Ax - \sum_{j=1}^{N} B_ju_j - f\right\rangle + \sum_{i=1}^{N}\left\langle p_i, \dot{x}^i - Ax^i - \sum_{j\neq i} B_jU_j - B_iv_i - f\right\rangle \tag{1.5}$$

for x, X satisfying $x(0) = x_0, X(0) = X_0 = (x_0,\cdots,x_0)$. We inherit the notations $U = \prod_{i=1}^N U_i$ with $U_iL_m^2(0,T)$ from Part I, and the notations of L^2 and Sobolev spaces H_n^k, H_{0n}^1 and H_{n0}^1 are the same as in [6] and [7]. We sometimes denote $L^2 = L^2(0,T)$ without mention of dimensions.

In this paper, we consider the linear quadratic problem whose cost functionals are given by

$$J_i(x,u) = \frac{1}{2}\int_0^T [|C_i(t)x(t) - z_i(t)|^2_{R^{k_i}} + \langle M_i(t)u_i(t), u_i(t)\rangle_{R^{m_i}}]dt, \tag{1.6}$$
$$i = 1,\cdots,N, \quad (x,u) \text{ feasible}$$

just as in [6], [7]; here we assume that $C_i(t)$ and $M_i(t)$ are matrix-valued functions of appropriate sizes and smoothness, and $z_i(t)$ is a vector-valued function, Furthermore, $M_i(t)$ induces a linear operator $M_i : L^2_{m_i} \to L^2_{m_i}$ which is positive definite:

$$\langle M_iu_i, u_i\rangle_{L^2_{m_i}} \geq \mu\|u_i\|^2_{L^2_{m_i}}, \quad 1\leq i\leq N, \text{ for some } \mu > 0. \tag{1.7}$$

In §2, we formally derive the matrix Riccati equation from the duality point of view. §3 is devoted to error estimates and numerical computations. We prove sharp error bounds using the Aubin-Nitche trick. We finally present in §4 some numerical results obtained by dualty and penalty scheme briefly. These results agree well with the theoretical estimates.

§2. The Dual Max-Min Problem for Linear Quadratic Games

In this section, we give a formal derivation of the dual functional $L(p_0, p)$. This formal derivation will be justified later by assumptions (A3), (A4), and the Primal-Dual Equivalence Theorem.

Let the Lagrangian $\mathbf{L}$ be defined as in (1.5), using (1.6). We first study

$$\sup\{\mathbf{L}(p_0, p; x, u; X, v) | \text{ for } (X, v) \text{ such that } X(0) = X_0\}.$$

For given $p_0, p, x, u, \mathbf{L}(p_0, p; x, u; X, v)$ is strictly concave in v, and concave in X. Assume that this maximization problem has a solution $(\hat{X}, \hat{v})$, which depends on $(p_0, p; x, u)$. By a simple variational analysis on x^i, we have, necessarily,

$$-\left\langle C_i^*(C_i\hat{x}^i - z_i), y^i\right\rangle_{L_n^2} + \left\langle p_i, \dot{y}^i - Ay^i\right\rangle_{L_n^2} = 0, C^* = \text{ adjoint of } C, \tag{2.1}$$

for all $y^i \in H_{n0}^1, i = 1, \cdots, N$. The above has a solution $\hat{X}$ if and only if p satisfies

$$p \in [H_{0n}^1]^N. \tag{2.2}$$

Indeed, (2.2) is a necessary and sufficient condition for

$$\sup_{\substack{(X,v)\\ X(0)=X_0}} \mathbf{L}(p_0, p; x, u; X, v) = \mathbf{L}(p_0, p; x, u; X, v). \tag{2.3}$$

(2.1) and (2.2) yield

$$-\langle C_i^*(C_i\hat{x}^i - z_i) + \dot{p}_i + A * p_i, y^i\rangle = 0, \quad i = 1, \cdots, N.$$

Hence

$$\dot{p}_i = -A^* p_i - C_i^*(C_i\hat{x}^i - z). \tag{2.4}$$

Similar variational analysis on v_i gives

$$-\langle M_i\hat{v}_i, w_i\rangle - \langle p_i, B_i w_i\rangle = 0, \quad \forall w_i \in L_{m_i}^2,$$

or

$$\hat{v}_i = M_i^{-1} B_i^* p_i, \quad i = 1, \cdots, N. \tag{2.5}$$

Note that $(\hat{X}, \hat{v})$ is independent of (x, u).

Next, we consider $\inf_{\substack{(x,u)\\ x(0)=x_0}} \mathbf{L}(p_0, p; x, u; X, v)$. For given $p_0 \in L_n^2, p \in [H_{n0}^1]^N$, using the same reasoning as before, we can show that

$$\inf_{\substack{(x,u)\\ x(0)=x_0}} \mathbf{L}(p_0, p; x, u; \hat{X}, \hat{v}) = L(p_0, p; \hat{x}, \hat{u}; \hat{X}, \hat{v})$$

for some (x, u) if and only if

$$p_0 \in H_{0n}^1, \tag{2.6}$$

$$\dot{p}_0 = -A^* p_0 + \sum_{i=1}^{N} C_i^*(C_i\hat{x} - z_i), \tag{2.7}$$

$$\hat{u}_i = M_i^{-1}B_i^*\Big(p_0 + \sum_{j\neq i} p_j\Big) = M_i^{-1}B_i^*(p_0 + p_s - p_i), \quad p_s = \sum_{j=1}^{N} p_j. \tag{2.8}$$

Let $L(p_0,p)$ be as defined in §1. If the problem $\sup_{p_0}\inf_p L(p_0,p)$ attains its max-min at $(\hat{p}_0,\hat{p})$, $\hat{p}_0$ and $\hat{p}$ satisfy (2.6), (2.7), (2.3) and (2.4). Therefore, we obtain $\hat{X},\hat{v},\hat{x},\hat{u},\hat{p}_0,\hat{p}$ as the solution to the following two-point boundary problem:

Theorem 2.1. *Assume that* $\max_{p_0\in L_n^2}\min_{p\in[L_n^2]^N} L(p_0,p)$ *is attained by* $(\hat{p}_0,\hat{p})$. *Then* $(\hat{p}_0,\hat{p}) \in H_{0n}^1 \times [H_{0n}^1]^N$,

$$L(\hat{p}_0,\hat{p}) = \max_{p_0\in L_n^2}\ \min_{p\in[L_n^2]^N} L(p_0,p) = \max_{p_0}\min_{p} L(p_0,p;x,u;X,v)$$
$$= \max_{p_0}\min_{p}\ \min_{\substack{(x,u)\in H_{0n}^1\times U\\ x(0)=x_0}}\ \max_{\substack{(X,v)\in[H_{0n}^1]^N\times U\\ X(0)=X_0}} L(p_0,p;x,u;X,v)$$

and $x, X = (x^1,\cdots,x^N), p_0$ *and* $p=(p_1,\cdots,p_N)$ *are coupled through*

$$\frac{d}{dt}\begin{bmatrix}\hat{x}\\ \hat{x}^1\\ \vdots\\ \hat{x}^N\\ \hat{p}_0\\ \hat{p}_1\\ \vdots\\ \hat{p}_N\end{bmatrix} = \begin{bmatrix} A & 0 & & 0 & S & S_1 & \cdots & S_N\\ 0 & A & & 0 & S_1 & S_{11} & \cdots & S_{1N}\\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & & A & S_N & S_{N1} & \cdots & S_{NN}\\ \sum_{i=1}^N C_i^*C_i & 0 & & 0 & -A^* & 0 & \cdots & 0\\ 0 & -C_1^*C_1 & & 0 & 0 & -A^* & & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & & -C_N^*C_N & 0 & 0 & & -A^* \end{bmatrix}$$
$$\times\begin{bmatrix}\hat{x}\\ \hat{x}^1\\ \vdots\\ \hat{x}^N\\ \hat{p}_0\\ \hat{p}_1\\ \vdots\\ \hat{p}_N\end{bmatrix} + \begin{bmatrix} f\\ f\\ \vdots\\ f\\ \sum_{i=1}^N C_i^* z_i\\ C_1^* z_1\\ \vdots\\ C_N^* z_N\end{bmatrix}, \tag{2.9}$$

$$\hat{x}(0) = \hat{x}^1(0) = \cdots = \hat{x}^N(0) = x_0, \quad \hat{p}_0(T) = \hat{p}_1(T) = \cdots = \hat{p}_N(T) = 0,$$

and $\hat{u},\hat{v}$ *satisfy*

$$\hat{u}_i = M_i^{-1}B_i^*(p_0 + p_s - p_i), \quad \hat{v}_i = -M_i^{-1}B_i^* p_i,$$

with

$$S=\sum_{j=1}^{N}B_jM_j^{-1}B_j^*,\quad S_i=\sum_{j\neq i}B_jM_j^{-1}B_j^*,$$

$$S_{ik}=S-(1-\delta_{ik})B_iM_i^{-1}B_i^*-B_kM_k^{-1}B_k^*,\quad \delta_{ik}=\text{ Kronecker's }\delta. \tag{2.10}$$

We now study the dual problem. Henceforth, for simplicity, we denote the operators $C_i^*C_i$ and $\sum_{i=1}^N C_i^*C_i$ (induced by the matrices $C_i^*(t)C_i(t)$ and $\sum_{i=1}^N C_i^*(t)C_i(t)$) in L_n^2 as $C_i(1\le i\le N)$ and C_0, respectively.

Several assumptions are needed as we proceed. First, we assume

(A3) each operator $\mathbf{C}_i(1\le i\le N)$ is strictly positive definite in L^2.

From (2.4), we get

$$\hat{x}^i=-\mathbf{C}_i^{-1}(\dot{p}_i+A^*p_i-C_i^*z_i). \tag{2.11}$$

By (A3), $\mathbf{C}_0$ is also strictly positive definite. By (2.7), we get

$$\hat{x}=\mathbf{C}_0^{-1}\left(\dot{p}_0+A^*p_0+\sum_{i=1}^{N}C_i^*z_i\right). \tag{2.12}$$

We now substitute (2.11), (2.12), (2.5) and (2.8) into (1.5). Integrating by parts with respect to p_0 and $p_i(1\le i\le N)$ once, using the end conditions $p_i(T)=0,\ 0\le i\le N$, and simplifying, we get

$$\begin{aligned}L(p_0,p)&=\mathbf{L}(p_0,p;\hat{x},\hat{u};\hat{X},\hat{v})=-\frac{1}{2}\langle\dot{p}_0+A^*p_0,\mathbf{C}_0^{-1}(\dot{p}_0+A^*p_0)\rangle\\&+\frac{1}{2}\sum_{i=1}^{N}\left\langle\dot{p}_i+A^*p_i,\mathbf{C}_i^{-1}(\dot{p}_i+A^*p_i)\right\rangle-\frac{1}{2}\left\langle p_0+p_s,S(p_0+p_s)\right\rangle\\&+\left\langle p_0+p_s,\sum_{i=1}^{N}B_iM_i^{-1}B_i^*p_i\right\rangle-\left\langle\dot{p}_0+A^*p_0,\ \mathbf{C}_0^{-1}\sum_{i=1}^{N}C_i^*z_i\right\rangle\\&-\sum_{i=1}^{N}\left\langle\dot{p}_i+A^*p_i,\ \mathbf{C}_i^{-1}C_i^*z_i\right\rangle-\left\langle p_0+p_s,f\right\rangle-\left\langle p_0(0)+p_s(0),x_0\right\rangle\\&-\frac{1}{2}\left\langle\mathbf{C}_0^{-1}\Big(\sum_{j=1}^{N}C_j^*z_j\Big),\ \sum_{j=1}^{N}C_j^*z_j\right\rangle+\frac{1}{2}\|z\|^2\equiv\sum_{i=1}^{10}T_i,\end{aligned} \tag{2.13}$$

where $\|z\|^2=\sum_{i=1}^N\|z_i\|_{L^2}^2$, and p_s is defined as in (2.8).

It is easy to see that $L(p_0,p)$ is strictly concave in p_0 for any given p. However, for any given p_0, $L(p_0,p)$ is not necessarily convex in p because of the negative sign in front of T_3. This causes a severe handicap for the duality approach; see Remark 2.2 below. To circumvent this, we need the following important assumption:

(A4) The positive definite operators $\mathbf{C}_i^{-1}(1\le i\le N)$ in L_n^2 are large enough so that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{N}\left\langle\dot{p}_i+A^*p_i,\mathbf{C}_i^{-1}(\dot{p}_i+A^*p_i)\right\rangle-\frac{1}{2}\left\langle p_s,Sp_s\right\rangle\\&\quad+\left\langle p_s,\sum_{i=1}^{N}B_iM_i^{-1}B_i^*p_i\right\rangle\ge v\sum_{i=1}^{N}\|\dot{p}\|^2,\end{aligned} \tag{2.14}$$

for some $v > 0$, and for all $p \in [H^1_{0n}]^N$.

We remark that, even if $\mathbf{C}_i^{-1}, 1 \le i \le N$, are not large enough, the above assumption can still be valid provided that T is chosen sufficiently small, because in this case the first positive definite quadratic form in (2.14) will have a large coercivity coefficient to absorb L^2-norm, when the interval $[0,T]$ is small. This is consistent with the assumption that $t_1 - t_0$ is sufficiently small in [13].

Another special case where (A4) holds without requiring $\mathbf{C}_i^{-1}$, $1 \le i \le N$, be large is when

$$N = 2, \quad U_1 = U_2, \quad B_1 M_1^{-1} B_1^* = B_2 M_2^{-1} B_2^* \equiv B, \text{ for some } B \ge 0.$$

It is easily seen that now

$$\begin{aligned}(2.14) &= \frac{1}{2}\sum_{i=1}^{2}\left\langle \dot{p}_i + A^* p_i, \mathbf{C}_i^{-1}(\dot{p}_i + A^* p_i)\right\rangle - \frac{1}{2}2\left\langle p_s, Bp_s\right\rangle + \left\langle p_s, Bp_s\right\rangle \\ &= \frac{1}{2}\sum_{i=1}^{2}\left\langle \dot{p}_i + A^* p_i, \mathbf{C}_i^{-1}(\dot{p}_i + A^* p_i)\right\rangle, \end{aligned} \tag{2.15}$$

so (A4) holds.

Remark 2.2. The fact that an assumption like (A2) in [7] is indispensable for the tractability of the dual problem can be observed as follows: If $\mathbf{C}_i^{-1}, i = 1, \cdots, N$, are not large enough in comparison with $B_i M_i^{-1} B_i^*, i = 1, \cdots, N$, so as to cause the existence of some $\tilde{p} \in [H^1_{0n}]^N$ satisfying

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{N}\langle \dot{\tilde{p}}_i + A^*\tilde{p}_i,\ C_i^{-1}(\dot{\tilde{p}}_i + A^*\tilde{p}_i)\rangle - \frac{1}{2}\langle \tilde{p}_s, S\tilde{p}_s\rangle \\ &\qquad + \left\langle \tilde{p}_s,\ \sum_{i=1}^{N} B_i M_i^{-1} B_i^* \tilde{p}_i \right\rangle < 0, \end{aligned} \tag{2.16}$$

then for any given $p_0 \in H^1_{0n}$, we deduce from (2.15) that

$$\lim_{k\to\infty} L(p_0, k\tilde{p}) = -\infty \text{ and } \inf_{p\in[H^1_{0n}]^N} L(p_0, p) = -\infty$$

for any given $p_0 \in H^1_{0n}$. Therefore, the dual problem is rendered completely worthless. A situation like (2.16) should be avoided to ensure mathematical tractability. For the computational purpose we will need the uniqueness of p. Thus we take a step further to assume coercivity and strict convexity of p in $L(p_0, p)$ in hypothesis (A4) to achieve this goal.

Let us list the above and other usefull properties in the following, which is readily verifiable.

Lemma 2.3. *Assume* (A3) *and* (A4); *then*

(i) *For each given* $p_0 \in H^1_{0n}$, $L(p_0, p)$ *is strictly convex in* p *for all* $p \in [H^1_{0n}]^N$ *and, for each given* $p \in [H^1_{0n}]^N$, $L(p_0, p)$ *is strictly concave in* p_0 *for all* $p_0 \in H^1_{0n}$.

(ii) *The following coercivity conditions are satisfied:*

$$\lim_{\|p\|_{[H^1_{0n}]^N}\to\infty} L(p_0,p)=\infty,\quad \forall\ p_0\in H^1_{0n},$$

$$\lim_{\|p_0\|_{H^1_{0n}}\to\infty} L(p_0,p)=\infty,\qquad \forall\ p\in [H^1_{0n}]^N. \tag{2.17}$$

Using the above lemma and the minimax theorem, we conclude

Proposition 2.4. Under (A3) and (A4), the dual problem $\sup_{p_0}\inf_p L(p_0,p)$ has a unique solution $(\hat p_0,\hat p)$ satisfying

$$L(\hat p_0,\hat p)=\sup_{p_0\in H^1_{0n}}\ \inf_{p\in[H^1_{0n}]^N} L(p_0,p)=\max_{p_0\in H^1_{0n}}\ \min_{p\in[H^1_{0n}]^N} L(p_0,p)=\min_{p\in[H^1_{0n}]^N}\ \max_{p_0\in H^1_{0n}} L(p_0,p).$$

Theorem 2.5 (Primal-Dual Equivalence Theorem). *Let $C_i(t), z_i(t), i=1,\cdots,N$, $f(t)$ and $\mathbf{C}_0^{-1},\mathbf{C}_i^{-1}, i=1,\cdots,N$, be sufficiently smooth (as functions and operators, respectively). Let $F(x,u;X,v)$ be defined as in* (1.3). *Assume that there exists $(x,u)\in H^1_n\times U > (x,v)\in [H^1_n]^N\times U$ such that*

$$\inf_{\substack{(x,u)\\ \text{feasible}}}\ \sup_{\substack{(X,v)\\ \text{feasible}}} F(x,u;X,x)=\min_{\substack{(x,u)\\ \text{feasible}}}\ \max_{\substack{(X,v)\\ \text{feasible}}} F(x,u;X,v)=f(\hat x,\hat u;\hat X,\hat v)<\infty \tag{2.18}$$

and that (A2) *in* [7] *is also satisfied, i.e.,*

$$\psi(x,u)=\sup_{\substack{(X,v)\\ \text{feasible}}} F(x,u;X,v) \tag{2.19}$$

is convex in (x,u) for all $(x,u)\in H^1_n\times U$, $x(0)=x_0$. Assume that (A3) *and* (A4) *hold and let $(\hat p_0,\hat p)$ be the solution in Proposition* 2.4. *Then*

(i)
$$\begin{aligned} L(\hat p_0,\hat p) &= \max_{p_0\in H^1_{0n}}\ \min_{p\in[H^1_{0n}]^N} L(p_0,p)\\ &= \max_{p_0\in H^1_{0n}}\ \min_{p\in[H^1_{0n}]^N}\ \min_{\substack{(x,u)\in H^1_n\times U\\ x(0)=x_0}}\ \max_{\substack{(X,v)\in[H^1_{0n}]^N\\ X(0)=X_0}} \mathbf{L}(p_0,p;x,u;x,v)\\ &= \min_{\substack{(x,u)\\ \text{feasible}}}\ \max_{\substack{(X,v)\\ \text{feasible}}} F(x,u;X,v)=f(\hat x,\hat u;\hat X,\hat v); \end{aligned} \tag{2.20}$$

(ii) *$(\hat x,\hat u;\hat X,\hat v)$ is related to $(\hat p_0,\hat p)$ through*

$$\hat x=\mathbf{C}_0^{-1}\Big(\dot{\hat p}_0+A^*\hat p_0+\sum_{i=1}^N C_i^* z_i\Big), \tag{2.21}$$

$$\hat u_i=M_i^{-1}B_i^*(\hat p_0+\hat p_s-\hat p_i),\quad i=1,\cdots,N, \tag{2.22}$$

$$\hat x^i=-\mathbf{C}_i^{-1}(\dot{\hat p}_i+A^*\hat p_i-C_i^* z_i),\quad i=1,\cdots,N, \tag{2.23}$$

$$\hat v_i=-M_i^{-1}B_i^*\hat p_i,\quad i=1,\cdots,N \tag{2.24}$$

and $(\hat x,\hat u;\hat X,\hat v)$ satisfies differential equations (1.1) *and* (1.4).

Proof. Because all the assumptions are satisfied, we can apply Theorem 2.1 of [7] (particularly (2.17) in the proof) to conclude (i). Note that all the sup's and inf's can be replaced by max's and min's due to the quadratic nature of the problem.

(2.21)–(2.24) are verified in a straightforward way as in (2.8), (2.5) and as in (2.11) and (2.12), but now every procedure is justified.

To show that $(\hat{x}, \hat{u}; \hat{X}, \hat{v})$ satisfies differential equations (1.1) and (1.4), we can make a variational analysis on $L(p_0, p)$. Because

$$L(p_0, \hat{p}) \le L(\hat{p}_0, \hat{p}) \le L(\hat{p}_0, p), \quad \forall (p_0, p) \in H^1_{0n} \times [H^1_{0n}]^N,$$

we get

$$\frac{\partial}{\partial p} L(p_0, p)\Big|_{p=\hat{p}} = 0. \tag{2.25}$$

This yields the Euler-Lagrange equations

$$\begin{cases} \frac{d}{dt}\mathbf{C}_i^{-1}(\dot{\hat{p}}_i + A^*\hat{p}_i) - A\mathbf{C}_i^{-1}(\dot{\hat{p}}_i + A^*\hat{p}_i) + S(\hat{p}_0 + \hat{p}_s) - \sum_{j=1}^{N} B_j M_j^{-1} B_j^* \hat{p}_j \\ \qquad - B_i M_i^{-1} B_i^*(\hat{p}_0 + \hat{p}_s) + A\mathbf{C}_i^{-1} C_i z_i - \frac{d}{dt}(\mathbf{C}_i^{-1} C_i^* z_i) + f = 0, \\ p_i(T) = 0, \\ \mathbf{C}_i^{-1}(0)[\dot{\hat{p}}(0) + A^*(0)\hat{p}_i(0)] = -x_0 + \mathbf{C}_i^{-1}(0) C_i^*(0) z_i(0), \qquad \text{for } i = 1, \cdots, N. \end{cases} \tag{2.26}$$

From the assumption that $\mathbf{C}_i, C_i, z_i, f$ are sufficiently smooth, and that (2.20)–(2.24) hold, we see that the above equations agree with (1.4). Similarly, from

$$\frac{\partial}{\partial p_0} L(p_0, p)|_{p_0 = \hat{p}_0} = 0, \tag{2.27}$$

we can also show that (1.1) is satisfied by (2.21)–(2.22).

Note that for a linear-quadratic differential game, $\psi(x, u)$ in (2.19) can be calculated expticitly and is equal to

$$\begin{aligned} \psi(x,u) = \max_{\substack{(X,v) \\ \text{feasible}}} F(x,u;X,v) = \sum_{i=1}^{N} \frac{1}{2}\Big\{ &\|C_i x - z_i\|^2 + \langle M_i u_i, u_i \rangle \\ &- \|C_i(\mathbf{L}_0 x_0 + \sum_{j \ne i} \mathbf{L}_j u_j + \mathbf{L}_{N+1} f) - z_i\|^2 \\ &+ \Big\langle \mathbf{L}_i^* C_i^* \Big[C_i\Big(\mathbf{L}_0 x_0 + \sum_{j \ne i} \mathbf{L}_j u_j + \mathbf{L}_{N+1} f\Big) - z_i \Big], \\ &(M_i + \mathbf{L}_i^* C_i^* C_i \mathbf{L}_i)^{-1} \mathbf{L}_i^* C_i^* \Big[C_i\Big(\mathbf{L}_0 x_0 + \sum_{j \ne i} \mathbf{L}_j u_j + \mathbf{L}_{N+1} f\Big) - z_i \Big] \Big\rangle \Big\}. \end{aligned} \tag{2.28}$$

The reader should refer to Part I, §2, for the notations and derivation of the above.

Corollary 2.6. Consider the linear-quadratic differential game (1.1), (1.6). Assume

(A0)′ $\min\limits_{\substack{(x,u) \\ \text{feasible}}} \max\limits_{\substack{(X,v) \\ \text{feasible}}} F(x,u;X,v) = 0$ holds, so the differential game has a solution (x, u);

(A2)′ $\psi(x, u)$ given in (2.28) is convex in (x, u) for $(x, u) \in H^1_{0n} \times U$, $x(0) = x_0$; and assume also (A3) and (A4). Then there exists a unique $(p_0, p) \in H^1_{0n} \times [H^1_{0n}]^N$ such that (2.20) holds and the solution (x,u) of the differential game can be obtained from (p_0,p) via

$$x = \mathbb{C}_0^{-1}(\dot{\hat{p}}_0 + A^*\hat{p}_0 + \sum_{i=1}^{N} C_i^* z_i),$$

$$\hat{u}_i = M_i^{-1}B_i^*(\hat{p}_0 + \hat{p}_s - \hat{p}_i), \quad i = 1, 2, \cdots, N,$$

as given in (2.12) and (2.8).

Remark 2.7. Many evidences seem to suggest that assumption (A2)′ in Corollary 2.6 is redundant because of (A4). Nevertheless, we are still unable to prove or disprove this.

§3. The Dual Variational Problem and Finite Element Approximations

In this section we devote ourselves to the study of the finite element numerical method for differential games. It is fair to say that the methods of solution for N-person differential games are still very incomplete. More efforts are needed to develop good analytic and approximation methods to solve them. The finite etement method is a rigorously established, highly accurate numerical method which is becoming increasing popularly in recent years. Due to the special minimax structure of differential games, we are able to apply and generalize the existing theory of finite element analysis to our own problem to establish rigorous error bounds and to obtain numerical solutions.

The unique solution (p_0,p) of the max-min problem satisfies (2.25) and (2.27). From (2.13), by a simple calculation, we obtain

$$\partial_{p0}L(\hat{p}_0,\hat{p}).r = -\langle \dot{\hat{p}}_0 + A^*\hat{p}_0, \mathbb{C}_0^{-1}(\dot{r} + A^*r)\rangle - \langle \hat{p}_0 + \hat{p}_s, Sr\rangle + \left\langle r, \sum_i^N B_i M_i^{-1} B^* p_i \right\rangle$$

$$- \left\langle \dot{r} + A^*r, \mathbb{C}_0^{-1}\sum_1^N C_i^* z_i \right\rangle - \langle r, f\rangle - \langle r(0), x_0\rangle = 0, \quad \forall r \in H_{0n}^1, \quad (3.1)$$

$$\partial_p L(\hat{p}_0,\hat{p}).s = \sum_1^N \left\langle \dot{\hat{p}}_i + A^*\hat{p}_i, \mathbb{C}_i^{-1}(\dot{s} + A^* s_i)\right\rangle - \left\langle \hat{p}_0 + \hat{p}_s, S\sum_1^N s_i\right\rangle$$

$$+ \left\langle \hat{p}_0 + \hat{p}_s, \sum_1^N B_i M_i^{-1} B_i^* S_i\right\rangle + \left\langle \sum_1^N s_i, \sum_1^N B_i M_i^{-1} B_i^* \hat{p}_i\right\rangle$$

$$- \sum_1^N \left\langle \dot{s}_i + A^* s_i, \mathbb{C}_i^{-1} C_i^* z_i\right\rangle - \left\langle \sum_1^N s_i, f\right\rangle - \left\langle \sum_1^N s_i(0),\ x_0\right\rangle = 0,$$

$$\forall\ s = (s_1, \cdots, s_N) \in \left[H_{0n}^1\right]^N. \quad (3.2)$$

The above two relations induce a bilinear form on $H_{0n}^1 \times [H_{0n}^1]^N$: for $r^1, r^2 \in H_{0n}^1$ and $s^1 = (s_1^1, s_2^1 \cdots, s_N^1)$, $\quad s^2 = (s_1^2, \cdots, s_N^2) \in [H_{0n}^1]^N$,

$$a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right) \equiv -\left\langle \dot{r}^1 + A^* r^1,\ \mathbf{C}_0^{-1}(r^2 + A^* r^2)\right\rangle - \left\langle r^1 + \sum_{j=1}^N s_j^1,\ S r^2\right\rangle$$
$$+ \left\langle r^2, \sum_1^N B_i M_i^{-1} B_i^* s_i^1\right\rangle + \sum_i^N \left\langle \dot{s}_i^1 + A^* s_i^1, \mathbf{C}_i^{-1}(\dot{s}_i^2 + A^* s_i^2)\right\rangle - \left\langle r^1 + \sum_{j=1}^N s_j^1,\ S \sum_{j=1}^N s_j^2\right\rangle$$
$$+ \left\langle r^1 + \sum_1^N s_i^1,\quad \sum_1^N B_i M_i^{-1} B_i^* s_i^2\right\rangle + \left\langle \sum_1^N s_i^2, \sum_1^N B_i M_i^{-1} B_i^* s_i^1\right\rangle, \tag{3.3}$$

and a linear form θ: for $r \in H_{0n}^1$ and $s = (s_1, \cdots, s_N) \in [H_{0n}^1]^N$,

$$\theta\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) = \left\langle r + \sum_1^N s_j,\ f\right\rangle + \left\langle r(0) + \sum_1^N s_j(0), x_0\right\rangle + \left\langle \dot{r} + A^* r, \mathbf{C}_0^{-1} \sum_1^N C_i^* z_i\right\rangle$$
$$= \sum_1^N \langle \dot{s}_i + A^* s_i, \mathbf{C}_i^{-1} C_i^* z_i\rangle. \tag{3.4}$$

Thus, (3.1) and (3.2) are equivalent to

$$a\left(\begin{bmatrix} p_0 \\ p \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix}\right) = \theta\left(\begin{bmatrix} r \\ s \end{bmatrix}\right), \quad \forall\, (r,s) \in H_{0n}^1 \times [H_{0n}^1]^N. \tag{3.5}$$

We are now in a position to compute $(\hat{p}_0, \hat{p})$ be the finite element method. As in [1], we say that $S_h^2 \subset H_r^{t_2}(0,T)$ is a (t_1,t_2)-system (t_1, t_2 are nonnegative integers) if, for all $v \in H_r^{k_0}(0,T)$, there exists $v_h \in S_h$ such that

$$\|v - v_h\|_{H_r^k} \le K h^m \|v\|_{H_r^{m+k}}, \quad \forall\, 0 \le k \le \min(k_0,\ t_2), \quad k \in N, \tag{3.6}$$

where $m = \min(t-k,\ k_0 - k)$ and $K > 0$ is independent of h and v.

Let $S_h \subset H_{0n}^1$ be a $(t,1)$-system. We consider

$$\max_{p_0 \in S_h} \min_{p \in [S_h]^N} L(p_0, p). \tag{3.7}$$

It is easy to see that under (A4), there exists a unique saddle point $(\hat{p}_{0h}, \hat{p}_h) \in S_h \times [S_h]^N$ such that

$$L(\hat{p}_{0h}, \hat{p}_h) = \max_{p_0 \in S_h} \min_{p \in [S_h]^N} L(p_0, p).$$

This point (p_{0h}, p_h) is characterized as the solution to the variational equation

$$a\left(\begin{bmatrix} p_{0h} \\ p_h \end{bmatrix}, \begin{bmatrix} r_h \\ s_h \end{bmatrix}\right) = \theta\left(\begin{bmatrix} r_h \\ s_h \end{bmatrix}\right), \quad \forall\, (r_h, s_h) \in S_h \times [S_h]^N. \tag{3.8}$$

If $\{\phi^i\}_{i=1}^J, \{\psi\}_{i=1}^{N,J}$ are basis for $S_h, [S_h]^N$, respectively, then (3.8) is a matrix equation $\bar{M}_h \bar{\gamma}_h = \bar{\theta}_h$, where the entries of $\bar{M}_h$ and $\bar{\theta}_h$ are

$$[\bar{M}_h]_{ij} = a\left(\begin{bmatrix} \psi^i \\ \phi^i \end{bmatrix}, \begin{bmatrix} \psi^j \\ \phi^j \end{bmatrix}\right), \quad 1 \le i,j \le (N+1)J,$$

$$(\theta_h)_j = \theta\left(\begin{bmatrix} \psi^j \\ \phi^j \end{bmatrix}\right), \quad 1 \le j \le (N+1)J.$$

Proposition 3.1. Under (A4), the bilinear form $a(.,.)$ satisfies

$$\inf_{\left\|\begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right\|=1} \sup_{\left\|\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}\right\|=1} \left|a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right)\right| > 0, \tag{3.9}$$

and the space $\{S_h\}_h$ satisfies

$$\inf_{\left\|\begin{bmatrix} r_h^2 \\ s_h^2 \end{bmatrix}\right\|=1} \sup_{\left\|\begin{bmatrix} r_h^1 \\ s_h^1 \end{bmatrix}\right\|=1} \left|a\left(\begin{bmatrix} r_h^1 \\ s_h^1 \end{bmatrix}, \begin{bmatrix} r_h^2 \\ s_h^2 \end{bmatrix}\right)\right| = \gamma_h > \gamma > 0, \tag{3.10}$$

for some $\gamma > 0, \forall\, h > 0$.

Proof. In (3.3), for any given $(r^2, s^2) \in H_{0n}^1 \times [H_{0n}^1]^N$, with norm 1, let

$$r^1 = -r^2, \quad s^1 = s^2.$$

Then the norm of (r^1, s^1) in $H_{0n}^1 \times [H_{0n}^1]^N$ is also equal to 1, and

$$\begin{aligned}
&a\left(\begin{bmatrix} r^1 \\ s^1 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right) = a\left(\begin{bmatrix} -r^2 \\ s^2 \end{bmatrix}, \begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right) \\
&= \langle r^{2\prime} + A^* r^2, C_0^{-1}(r^2 + A^* r^2)\rangle + \langle r^2, S r^2\rangle \\
&+ \left\langle \sum_{i=1}^N s_i^2, \sum_{i=1}^N B_i M_i^{-1} B_i^* s_i^2 \right\rangle + \sum_{i=1}^N \langle s_i^2 + A^* s_i^2,\ C_0^{-1}(s_i^2 + A^* s_i^2)\rangle \\
&- \left\langle \sum_{i=1}^N s_i^2, S \sum_{i=1}^N s_i^2 \right\rangle + 2\left\langle \sum_{i=1}^N s_i^2,\ \sum_{i=1}^N B_i M_i^{-1} B_i^* s_i^2 \right\rangle \\
&\geq \langle \dot r_2 + A^* r^2, C_0^{-1}(\dot r^2 + A^* r^2)\rangle + \langle r^2, S r^2\rangle + 2\mu \sum_{i=1}^N \|s_i^2\|_{L^2} \qquad (\text{by}(A4) \\
&\geq \mu' \left\|\begin{bmatrix} r^2 \\ s^2 \end{bmatrix}\right\|_{H_{0n}^1 \times [H_{0n}^1]^N} = \mu'
\end{aligned}$$

for some $\mu' > 0$. So (3.9) also follows in exactly the same way.

Theorem 3.2. *Let $(\hat p_{0h}, \hat p_h)$ be the solution of (3.7) and let S_h be a $(t,1)$-system. Assume that $C_i(t), z_i(t)$, $i = 1, \cdots, N$, are sufficiently smooth. Under (A3), (A4), we have*

$$\|\hat p_0 - \hat p_{0h}\|_{H_{0n}^1} + \|\hat p - \hat p_h\|_{[H_{0n}^1]^N} \leq K h^m \left(\|\hat p_0\|_{H_n^r} + \|\hat p\|_{[H_n^r]^N}\right), \tag{3.11}$$

$$\|\hat p_0 - \hat p_{0h}\|_{L^2} + \|\hat p - \hat p_h\|_{[L^2]^N} \leq K h^{m+1} \left(\|\hat p_0\|_{H_n^r} + \|\hat p\|_{[H_n^r]^N}\right) \tag{3.12}$$

provided $(\hat p_0, \hat p) \in [H_{0n}^1 \cap H_n^r] \times [H_{0n}^1 \cap H_n^r]^N$, where $m = \min(t - i, r - 1)$ and $K > 0$ is a constant independent of $(\hat p_0, \hat p)$. Consequently,

$$|L(\hat p_0, \hat p) - L(\hat p_{0n}, \hat p_h)| \leq K_2 h^{2m} (\|\hat p_0\|_{H_n^r}^2 + \|\hat p\|_{[H_n^r]^N}^2) \tag{3.13}$$

holds for some $K_2 > 0$ independent of $(\hat p_0, \hat p)$.

Proof. Because (p_{0h}, p_h) satisfies (3.8) and $(\hat{p}_0, \hat{p})$ satisfies (3.5), we get

$$a\left(\begin{bmatrix} \hat{p}_0 - \hat{p}_{0h} \\ \hat{p} - \hat{p}_h \end{bmatrix}, \begin{bmatrix} r_h \\ s_h \end{bmatrix}\right) = 0, \quad \forall\, (r_h, s_h) \in S_h \times [S_h]^N.$$

Therefore[1], by Proposition 3.1, we get

$$\begin{aligned} &\|(\hat{p}_0 - \hat{p}_{0h}, \hat{p} - \hat{p}_h)\|_{H^1_{0n} \times [H^1_{0n}]^N} \\ &\quad \le \left(1 + \frac{c}{\gamma}\right) \inf_{(r_h, s_h) \in S_h \times [S_h]^N} \left(\|\hat{p}_0 - r_h\|_{H^1_{0n}} + \|\hat{p} - s_h\|_{[H^1_{0n}]^N}\right) \end{aligned}$$

for some $c > 0$ independent of h. Using (3.6), we obtain (3.11).

To prove (3.12) we use Nitsch's trick ([8], [15]). By Proposition 3.1 and [1], for any $g \in L^2 \times [L^2]^N$, we have a unique $w(g) \in H^1_{0n} \times [H^1_{0n}]^N$ such that

$$a(w(g), y) = \langle g, y \rangle_{L^2 \times [L^2]^N}, \quad \forall\, y \in H^1_{0n} \times [H^1_{0n}]^N.$$

Furthermore, we have $w(g) \in [H^1_{0n} \cap H^2_n] \times [H^1_{0n} \cap H^2_n]^N$, provided that $C_i(t)$ and $z_i(t), i = 1, 2, \cdots, N$, are sufficiently smooth (this $w(g)$ can be obtained explicitly from integration by parts). It is not difficult to verify that

$$\|w(g)\|_{H^2_n \times [H^2_n]^N} \le K' \|g\|_{L^2 \times [L^2]^N},$$

where K' is independent of g. By the very same proof of the Aubin-Nitsche lemma[7], which remains valid under Proposition 3.1, we get

$$\begin{aligned} \|\hat{p}_0 - \hat{p}_{0h}\|_{L^2} + \|\hat{p} - \hat{p}_h\|_{[L^2]^N} &\le C h^m \left(\|\hat{p}_0\|_{H^r_n} + \|\hat{p}\|_{[H^r_n]^N}\right) \\ &\quad \cdot \sup_{g \in L^2 \times [L^2]^N} \left(1/\|g\| \inf_{\zeta_h \in S_h \times [S_h]^N} \|w(g) - \zeta_h\|\right). \end{aligned} \tag{3.14}$$

But, by (3.6),

$$\frac{1}{\|g\|} \inf_{\zeta_h \in S_h \times [S_h]^N} \|w(g) - \zeta_h\| \le \frac{1}{\|g\|} K'' h \|w(g)\|_{H^2_n} \le \frac{1}{\|g\|} K'' h K' \|g\| = K' K'' h,$$

for some $K'' > 0$ independent of g and $w(g)$. Using the above in (3.14), we get (3.12).

To show (3.13), we note that

$$\begin{aligned} L(\hat{p}_{0h}, \hat{p}_h) - L(\hat{p}_0, \hat{p}) = 2\Bigg(a\left(\begin{bmatrix} \hat{p}_0 \\ \hat{p} \end{bmatrix}, \begin{bmatrix} \hat{p}_{0h} - \hat{p}_0 \\ \hat{p}_h - \hat{p} \end{bmatrix}\right) - \theta\left(\begin{bmatrix} \hat{p}_{0h} - \hat{p}_0 \\ \hat{p}_h - \hat{p} \end{bmatrix}\right) \\ + a\left(\begin{bmatrix} \hat{p}_{0h} - \hat{p}_0 \\ \hat{p}_h - \hat{p} \end{bmatrix}, \begin{bmatrix} \hat{p}_{0h} - \hat{p}_0 \\ \hat{p}_h - \hat{p} \end{bmatrix}\right). \end{aligned}$$

The first term on the right above is zero because of (3.5). The second term on the right can be estimated by using (3.11). Hence we get (3.13).

Corollary 3.3. Let

$$\hat{x}_h = \mathbf{C}_0^{-1}\left(\dot{\hat{p}}_{0h} + A^* \hat{p}_{0h} + \sum_{i=1}^{N} C_I^* z_i\right), \tag{3.15}$$

$$\hat{u}_{h,i} = M_i^{-1}B_i^*\Big(\hat{p}_{0h} + \sum_{j=1}^{N}\hat{p}_{h,j} - \hat{p}_{h,i}\Big), \quad i = 1, \cdots, N, \tag{3.16}$$

$$\hat{x}_h^i = -\mathbf{C}_0^{-1}(\dot{\hat{p}}_{h,i} + A^*\hat{p}_{h,i} - C_i^* z_i), \quad i = 1, \cdots, N, \tag{3.17}$$

$$\hat{v}_{h,i} = -M_i^{-1}B_i^*\hat{p}_{h,i}, \quad i = 1, \cdots, N \tag{3.18}$$

and

$$\hat{X}_h = (\hat{x}_h^1, \cdots, \hat{x}_h^N), \quad \hat{v}_h = (\hat{v}_{h,1}, \cdots, \hat{v}_{h,N}), \quad \hat{u}_h = (\hat{u}_{h,1}, \cdots, \hat{u}_{h,N}).$$

Then

$$\left\|\hat{u} - \hat{u}_h\right\|_{L^2} + \left\|\hat{v} - \hat{v}_h\right\|_{[L^2]^N} \le K_3 h^{m+i}\left(\left\|\hat{p}_0\right\|_{H_n^r} + \left\|\hat{p}\right\|_{[H_n^r]^N}\right), \tag{3.19}$$

$$\left\|\hat{x} - \hat{x}_h\right\|_{L^2} + \left\|\hat{X} - \hat{X}_h\right\|_{[L^2]^N} \le K_3 h^{m}\left(\left\|\hat{p}_0\right\|_{H_n^r} + \left\|\hat{p}\right\|_{[H_n^r]^N}\right), \tag{3.20}$$

for some $K_3 > 0$ independent of $\hat{x}, \hat{u}, \hat{X}, \hat{v}, p_0$ and p.

The convergence rate (3.19) is the sharpest possible[14],[2]. The rate (3.20) is not optimal. To obtain a faster rate of convergence for x and X, we can use $\hat{u}_h$ and $\hat{v}_h$ in $(DE) = 0$ and $(DE)_i = 0$, $i = 1, \cdots, N$, and integrate to solve for more accurate x and X.

§4. Examples and Computation Results

In this section, we apply the finite element method and the penalty method to some examples and present our numerical results.

Example 1. We consider the following two-person non-zero-sum game:

$$\begin{aligned}
&\dot{x}(t) = x(t) + u_1(t) + 2u_2(t) + 1, \quad t \in [0, T], \quad T = \pi/4, \\
&x(0) = 0, \\
&J_1(x, u) = \int_0^T [|x(t) + (\cos t + 1/2)|^2 + 1/2|u_1(t)|^2]dt, \\
&J_2(x, u) = \int_0^T [|x(t) - \sin t|^2 + 2|u_2(t)|^2]dt.
\end{aligned} \tag{4.1}$$

The Lagrangian $\mathbf{L}$ in (2.13) corresponding to this problem is

$$\begin{aligned}
L(p_0, p_1, p_2) = &-1/2\langle \dot{p}_0 + p_0,\ 1/2(\dot{p}_0 + p_0)\rangle + 1/2[\langle \dot{p}_1 + p_1, \dot{p}_1 + p_1\rangle \\
&+ \langle \dot{p}_2 + p_2,\ \dot{p}_2 + p_2\rangle] - 1/2\langle p_0 + p_1 + p_2,\ 4(p_0 + p_1 + p_2)\rangle \\
&+ \langle p_0 + p_1 + p_2,\ 2p_1 + 2p_2\rangle - \langle \dot{p}_0 + p_0,\ 1/2[(\cos t + 1/2) + \sin t]\rangle \\
&- [\langle \dot{p}_1 + p_1,\ \cos t + 1/2\rangle + \langle \dot{p}_2 + p_2,\ \sin t\rangle] - \langle p_0 + p_1 + p_2,\ 1\rangle \\
&- 1/2\langle 1/2[-(\cos t + 1/2) + \sin t], -(\cos t + 1/2) + \sin t\rangle \\
&+ 1/2[\langle \cos t + 1/2,\ \cos t + 1/2\rangle + \langle \sin t, \sin t\rangle].
\end{aligned} \tag{4.2}$$

In order to apply the theory and analysis in §3 to this example, we need to verify that assumptions (A2), (A3) and (A4) are satisfied, and

(A0)′ $\min_{\substack{(x,u)\\ \text{feasible}}} \max_{\substack{(X,v)\\ \text{feasible}}} F(x,u;X,v)=0$ holds so that the differential game has a solution $(\hat{x},\hat{u})$.

Instead of checking (A0)′ directly, we show that the "decision operator" $\mathbf{D}$ as defined in (2.6) in Part I[6] is invertible so that the differential game has a unique solution, so (A0)′ is satisfied. But here

$$\mathbf{D}=\begin{bmatrix} M_1+\mathbf{L}_1^*C_1^*C_1\mathbf{L}_1 & \mathbf{L}_1^*C_1^*C_1\mathbf{L}_2 \\ \mathbf{L}_2^*C_2^*C_2\mathbf{L}_1 & M_2+\mathbf{L}_2^*C_2^*C_2\mathbf{L}_2 \end{bmatrix}=\begin{bmatrix} 1/2I+\mathbf{L}_1^*L_1 & 2\mathbf{L}_1^*\mathbf{L}_1 \\ 2\mathbf{L}_1^*\mathbf{L}_1 & 2I+4\mathbf{L}_1^*\mathbf{L}_1 \end{bmatrix} \tag{4.3}$$

because $\mathbf{L}_2=2\mathbf{L}_1$ and $\mathbf{L}_2^*=2\mathbf{L}_1^*$, and $C_1=C_2=1$, $C_1^*=C_2^*=1$, where

$$\mathbf{L}_1: U\to H_n^1(0,t),\quad \mathbf{L}_1u=\int_0^t e^{t-s}u(s)ds.$$

We easily see that $\mathbf{D}$ in (4.3) above is symmetric and strictly positive definite, so $\mathbf{D}$ is invertible. Hence (A0)′ is satisfied.

To check (A2), we write out $\psi(x,u)$ explicitly:

$$\begin{aligned}\psi(x,u)=&\,1/2\{\|x(t)+(\cos t+1/2)\|^2+1/2\|u_1(t)\|^2+\|x(t)-\sin t\|^2+2\|u_2(t)\|^2\\ &-\|\mathbf{L}_0x_0+\mathbf{L}_1u_1+\mathbf{L}_3f-\sin t\|^2-\|\mathbf{L}_0x_0+2\mathbf{L}_1u_2+\mathbf{L}_3f+(\cos t+1/2)\|^2\\ &+\langle \mathbf{L}_1^*(\mathbf{L}_0x_0+2\mathbf{L}_1u_2+\mathbf{L}_3f+(\cos t+1/2)),(1/2I+\mathbf{L}_1^*\mathbf{L}_1)^{-1}\mathbf{L}_1^*(\mathbf{L}_0x_0\\ &+2\mathbf{L}_1u_2+\mathbf{L}_3f+(\cos t+1/2))\rangle+\langle 2\mathbf{L}_1^*(\mathbf{L}_0x_0+\mathbf{L}_1u_1+\mathbf{L}_3f-\sin t),\\ &(2I+4\mathbf{L}_1^*\mathbf{L}_1)^{-1}2\mathbf{L}_1^*(\mathbf{L}_0x_0+\mathbf{L}_1u_1+IL_3f-\sin t)\rangle,\end{aligned} \tag{4.4}$$

where we have used $\mathbf{L}_2=2\mathbf{L}_1, \mathbf{L}_2^*=2\mathbf{L}_1^*, C_1=C_2=I$, $C_1^*=C_2^*=I$ and

$$\mathbf{L}_0x_0=e^tx_0,\qquad \mathbf{L}_3=\int_0^t et^{-s}f(s)ds.$$

In (4.4), it is easy to see that $\psi(x,u)$ is convex with respect to x because $\psi(x,u)$ has $\|x\|^2$ as the only quadratic term involving x. The quadratic terms involving u_1 and u_2 are

$$\begin{aligned}&1/2\{\langle[1/2I-\mathbf{L}_1^*\mathbf{L}_1+4\mathbf{L}_1^*\mathbf{L}_1(2I+4\mathbf{L}_1^*\mathbf{L}_1)^{-1}\mathbf{L}_1^*\mathbf{L}_1]u_1,u_1\rangle\\ &\quad+\langle[2I-4\mathbf{L}_1^*\mathbf{L}_1+4\mathbf{L}_1^*\mathbf{L}_1(1/2I+\mathbf{L}_1^*\mathbf{L}_1)^{-1}\mathbf{L}_1^*\mathbf{L}_1]u_2,u_2\rangle\}\\ &\quad=1/2\{\langle(2I+4\mathbf{L}_1^*\mathbf{L}_1)^{-1}[1/2I+4\mathbf{L}_1^*\mathbf{L}_1)-(2I+4\mathbf{L}_1^*\mathbf{L}_1)\mathbf{L}_1^*\mathbf{L}_1\\ &\quad+4(\mathbf{L}_1^*\mathbf{L}_1)(\mathbf{L}_1^*\mathbf{L}_1)]u_1,u_1\rangle+\langle(2I+4\mathbf{L}_1^*\mathbf{L}_1^*)^{-1}[2(2I+4\mathbf{L}_1^*\mathbf{L}_1)\\ &\quad-4(2I+4\mathbf{L}_1^*\mathbf{L}_1)\mathbf{L}_1^*\mathbf{L}_1+16(\mathbf{L}_1^*\mathbf{L}_1)(\mathbf{L}_1^*\mathbf{L}_1)]u_2,u_2\rangle\}\\ &\quad=1/2\{\langle(2I+4\mathbf{L}_1^*\mathbf{L}_1)^{-1}u_1,u_1\rangle+4\langle(2I+4\mathbf{L}_1^*\mathbf{L}_1)^{-1}u_2,u_2\rangle\}.\end{aligned}$$

The above is a strictly positive definite quadratic form in u_1 and u_2. Therefore, $\psi(x,u)$ is also convex with respect to $u=(u_1,u_2)$. In fact, in this example, $\psi(x,u)$ is strictly convex with respect to x and u.

It is easy to see that (A3) is satisfied, so only (A4) remains. This can be done straight forwardly from (4.2) with little work. Hence all assumptions have been verified and by Theorem 2.5 $(\hat{x},\hat{u})$ is the solution.

We choose a (4.1)-system of Hermite cubic splines as in [15]. The interval $[0,T]$ is divided into N equal subintervals, each with mesh length $h=T/N$. The matrix M_h is a $(6N+3)\times(6N+3)$ matrix. We use the IMSL high accuracy subroutine LEQ2S to solve the matrix equation $\bar{M}_h\bar{\gamma}_h=\bar{\theta}_h$ with double precision on an IBM370/model 3033 at Pennsylvania State University.

Numerical results are plotted in Figures 1–4:

(i) Figure 1. Strategy u_1 is plotted, using $h=(\pi/4)/4$, $(\pi/4)/8$, $(\pi/4)/16$, $(\pi/4)/32$, $(\pi/4)/64$, respectively. These five curves show no visible difference in gragh. Numerical results for v_1 are found to be identical with u_1, as indicated in Corollary 2.7[6].

(ii) Figure 2. Strategy u_2 is plotted, using $h=(\pi/4)/4$, $(\pi/4)/8$, $(\pi/4)/16$, $(\pi/4)/32$, $(\pi/4)/64$, respectively. Numerical results for v_2 are identical with u_2.

(iii) Figure 3. State x is plotted, using $h=(\pi/4)/4$, $(\pi/4)/8$, $(\pi/4)/16$, $(\pi/4)/32$, $(\pi/4)/64$.

(iv) Figure 4. x, x^1 and x^2 are plotted, with $h=(\pi/4)/16$. Again, we see that the three curves show no visible difference in the graph. The values of $L(p_0,p_1,p_2)$ and $F(x,u;X,v)$ are found to be

$$L=F=6.529054\times10^{-10},\ h=\pi/4;\quad L=F=1.22479\times10^{-11},\quad h=(\pi/4)/8;$$
$$L=F=2.127\times10^{-13},\ h=(\pi/4)/16;\quad L=F=1.6\times10^{-15},\ h=(\pi/4)/64.$$

In Table 1, we list some values of $\hat{u}_1,\hat{u}_2,\hat{x},\hat{x}^1,\hat{x}^2,\hat{p}_0,\hat{p}_1$ and p_2 at certain selected nodal points. For this example, there is no known closed form exact solutions to compare. Therefore, the only way to show that our numerical scheme works is to check the rate of convergence (3.13) by a different method; see Example 3. Using the data, we have plotted the logarithmic error graph in Figure 5. The asymptotic rate of convergence, which is indicated by the slope of line segment is $O(h^{6.2})$ which is extremely close to the predicted rate $O(h^6)$ in (3.13). Note here that $m=4-1=3$, so $2m=6$ in (3.13), provided that $(\hat{p}_0,\hat{p})$ is at least $H_1^3\times[H_1^3]^2$ regular.

Example 2. We consider the following 2-person non-zero-sum game:

$$\begin{aligned}
&\dot{x}(t)=x(t)+\cos t u_1(t)+\sin t u_2(t)+1,\ 0\le t\le 2\pi,\\
&x(0)=0,\\
&J_1(x,u)=\int_0^T\left[|x(t)+(\cos t+1/2)|^2+1/3u_1^2(t)\right]dt,\\
&J_2(x,u)=\int_0^T\left[|x(t)-0.9\sin t|^2+u_2(t)\right]dt.
\end{aligned}$$

It is not clear to us as to whether conditions (A0)′, (A2), (A3) and (A4) are satisfied. The numerical evidence below suggests that the rate of convergence of L to 0 is not close to $O(h^6)$, thus it is likely that Corollaries 2.6 and 3.3 do not hold for this example. Thus we believe at least one of the conditions (A0)′, (A2), (A3), and (A4) is violated.

Using the computational scheme in §3, we obtain

$$L = 1.493 \times 10^{-1}, \quad h = 2\pi/4; \qquad L = 4.646 \times 10^{-2}, \quad h = 2\pi/8;$$

$$L = 1.267 \times 10^{-2}, \quad h = 2\pi/16; L = 1.771 \times 10^{-2}, \quad h = 2\pi/32;$$

$$L = 1.1755 \times 10^{-2}, \quad h = 2\pi/64.$$

The logarithmic error is also plotted on Fig. 5. Here we find that the rate of convergence is $O(h^{1.93})$ at best, which is way off the predicted rate $O(h^6)$.

Table 1. Numerical Values of u_1, u_2, x, x^1, x^2, P_0, P_1 and P_2 at $t = \frac{1}{4}\cdot\frac{\pi}{4}$, $\frac{1}{2}\cdot\frac{\pi}{4}$, $\frac{3}{4}\cdot\frac{\pi}{4}$, and $\frac{\pi}{4}$ for Example 1.

	$t = \frac{1}{4}\cdot\frac{\pi}{4}$			$t = \frac{1}{2}\cdot\frac{\pi}{4}$		
	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$
u_1	-2.020419	-2.078747	-2.078747	-1.239453	-1.239453	-1.239453
u_2	0.441072	0.441072	0.441072	0.285282	0.285282	0.285282
x	-0.125895	-0.125896	-0.125896	-0.136733	-0.136733	-0.136733
x^1	-0.125895	-0.125896	-0.125896	-0.136733	-0.136733	-0.136733
x^2	-0.125895	-0.125896	-0.125896	-0.136733	-0.136733	-0.136733
P_0	-0.598302	-0.598302	-0.598302	-0.334444	-0.334444	-0.334444
P_1	1.039374	1.039374	1.039374	0.619726	0.619726	0.619726
P_2	-0.441072	-0.441072	-0.441072	-0.285282	-0.285282	-0.285282

	$t = \frac{3}{4}\cdot\frac{\pi}{4}$			$t = \frac{\pi}{4} = T$		
	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$	$h = \frac{\pi}{4}/16$	$h = \frac{\pi}{4}/32$	$h = \frac{\pi}{4}/64$
u_1	-0.562914	-0.562913	-0.562913	0.0	0.0	0.0
u_2	0.131964	0.131964	0.131964	0.0	0.0	0.0
x	-0.053732	-0.053732	-0.053732	0.118645	0.118645	0.118646
x^1	-0.053731	-0.053732	-0.053732	0.118645	0.118646	0.118646
x^2	-0.053732	-0.053732	-0.053732	0.118644	0.118645	0.118646
P_0	-0.149492	-0.149492	-0.149492	0.0	0.0	0.0
P_1	0.281457	0.281457	0.281457	0.0	0.0	0.0
P_2	-0.131964	-0.131964	-0.131964	0.0	-0.0	-0.0

Remark. The numerical values of v_1, v_2 are identical, respectively, with u_1, u_2. All entries above are rounded off figures with six decimal place accuracy. " ″ " entries have the same values as the one immediately above.

The penalty method developed in §3 can also be combined with finite elements to do numerical calculations. Analysis of error can be found in [5]. The penalty-finite element scheme seems to be less stable than the duality-finite element as given in §3, and its error estimates are hard to verify experimentally. We have successfully computed Example 1 by the penalty-finite element scheme, as shown below.

Example 3. We consider the very same example as in (4.1). $F_\epsilon(x,u;X,v)$ is given as in [7]. We choose for x, x^1 and x^2 approximation spaces S_h^0 which are a (3,1)-system of quadratic splines, and use a (2,0)-system of piecewise linear elements as approximation spaces S_h^1 for u_1, u_2, v_1 and v_2.

Numerical data for $\hat{u}_1, \hat{u}_2$ and $\hat{x}$ at selected points are given in Table 2 below, with

$$h = (\pi/4)/32, \qquad \text{uniform meshes for } S_h^0 \text{ and } S_h^1, \quad e_0 = e_1 = e_2.$$

They compare very well with the duality-finite element solutions, which use (4,1)-cubics and $h = (\pi/4)/32$.

Note that numerical solutions of $\hat{x}^1, \hat{x}^2, \hat{v}_1$ and $\hat{v}_2$ also satisfy

$$\hat{x}^1 = \hat{x}^2 = \hat{x}, \quad \hat{v}_1 = \hat{u}_1, \quad \hat{v}_2 = \hat{u}_2.$$

For more numerical examples and detailed discussions, see[7].

Table 2. P_1: penalty solution with $e_0 = e_1 = e_2 = 10^{-3}$.
P_2: penalty solution with $e_0 = e_1 = e_2 = 10^{-5}$.
D: duality solution.

$t =$		$(\pi/4)/4$	$(\pi/4)/2$	$(\pi/4)(3/4)$	$\pi/4 = T$
u_1	P_1	-2.077473	-1.238577	-0.562432	0.000086
	P_2	-2.078433	-1.239262	-0.562789	-0.004539
	D	-2.078747	-1.239453	-0.562913	0.0
u_2	P_1	0.440848	0.285103	0.131847	-0.000053
	P_2	0.441103	0.285264	0.131923	-0.002366
	D	0.441072	0.285282	0.131964	0.0
u_x	P_1	-0.125946	-0.136808	-0.053823	0.118535
	P_2	-0.125870	-0.136707	-0.053713	0.118661
	D	-0.125896	-0.136733	-0.053732	0.118645

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