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ON THE NUMERICAL METHOD OF FOLLOWING HOMOTOPY PATHS*[1)]

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Abstract

In this paper, we develop one kind of method, called self-adaptive method (SAM), to trace a continuous curve of a homotopy system for the solution of a nonlinear system of equations in finite steps. The existence of the continuous solution, the determination of safe initial points, and the test of regularity and stop criterion corresponding to this method are discussed. As a result the method can follow the curve efficiently. The numerical results show that our method is satisfactory.

§1. Introduction

The homotopy extension method, as a kind of algorithm for finding the solution of nonlinear systems

$$F(x) = 0, \quad F : D \subset R^n \to R^n, \tag{1}$$

is very important. Like to the simplicial algorithm, this method will attract more attention because both of them are continuation techniques for finding the fixed points or zeros , and are related with Newton's method.

The principal idea of the homotopy extension method is to transform (1) into the following form (2) by homotopy mapping:

$$H(x(t)) = 0 \tag{2}$$

for arbitrary $t \in [0, 1]$, and to follow the continuous curves of (2). In this respect, many results have been presented, of which one inmportant result is the local convergence theorem on "Newton following" given by Oterga and Rheinboldt [5]. But a series of problems on the existence of the solution, the partition on "Newton following", the computer implementability and the regularity of $\partial_x H(x(t), t)$ have not been solved yet, and a lot of difficulties are yet to be overcome for the numerical procedure of homotopy extension.

In this paper, we develop a kind of method to follow a continuous curve of the solutions in finite steps of the homotopy systems by the self-adaptive method. The numerical result shous that the algorithm can be implemented on computer.

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§2. Existence of the Solutions

Consider the following nonlinear systems:

$$F(x) = 0, F : D \subset R^n \to R^n. \tag{3}$$

The famous Newton-Kantorovich theorem shows the relations between local pionts and the solution. Our idea is to trace the homotopy path by using the Newton-Kantorovich theorem, so we need the following facts.

Definition 1. *Let $f : D \subset R^n \to R^1$ be a mapping, and $\rho \in R^1$ be positive real. Define the set*

$$\Gamma_{D,f}(\rho) = \{x | f(x) \leq \rho, \forall x \in D\}. \tag{4}$$

We say that $\Gamma_{D,f}$ is a level set of f on D and ρ.

Definition 2. *We say that D_r is an r-interior set of $D_0 \subset D$, if D_r satisfies*

$$D_r = \{x | \min \|y - x\| \geq r, x \in D_0, Y \in \partial D_0\} \tag{5}$$

where ∂D_0 is the boundary of D_0, and r is positive.

Theorem 1. *Let $F : D \subset R^n \to R^n$ be C_1 smooth and F' satisfy*

$$\|F'(x) - F'(y)\| \leq L\|x - y\|, x, y \in D_0. \tag{6}$$

For $x \in D_0, F'(x)$ is invertible and satisfies

$$\|F'(x)^{-1}\| \leq \beta. \tag{7}$$

Then there exist solutions if and only if there is a positive ε_0 such that

$$\text{int}\,(\Gamma_{D_0}) = \text{int}\,\left(\Gamma_{D_0, \|F'(x)^{-1}F(x)\|}(\eta)\right) \neq \phi \tag{8}$$

where η satisfy

$$\eta \leq \min\left\{\frac{1}{2\beta L}, \varepsilon_0 - \frac{1}{2}\beta L \varepsilon_0^2\right\} \tag{9}$$

and ϕ is empty.

Proof. If ε_0 and η satisfy (8) and (9), then the interior of Γ_{D_0} is nonempty. Let $x^0 \in \text{int}\,(\Gamma_{D_0})$. By Definition 2, we have $x^0 \in D_{\varepsilon_0} \subset D_0$ and x^0 satisfies

$$\|F'(x^0)^{-1}F(x^0)\| \leq \eta.$$

By (9),

$$\alpha = \beta L \eta \leq \frac{1}{2} \quad \text{and} \quad \overline{S}(x^0, \varepsilon_0) \subset D_0.$$

Let

$$t^* = (1 - \sqrt{1 - 2\alpha})\eta/\alpha.$$

Then we have

$$t^* - \frac{1}{2}\beta L t^* = \eta \leq \varepsilon_0 - \frac{1}{2}\beta L \varepsilon_0^2.$$

Thus

$$(\varepsilon_0 - t^*)(1 + \frac{1}{2}\beta L(t^* - \varepsilon_0)) \geq 0.$$

So we get $\varepsilon_0 \geq t^*$, and thus $\overline{S}(x^0, t^*) \subset D_0$. Then x^0 satisfies all the conditions of the Newton-Kantorovich theorem, which gurantees the existence of the solution.

Let $x^* \in \text{int}(D_0)$ be a solution of the nonlinear systems. Then there is an $\varepsilon_1 > 0$ such that $S(x^*, \varepsilon_1) \subset D_0$.

We now consider the following estimate:

$$\|F'(x)^{-1}F(x)\| \leq \beta\|F(x)\| \leq \beta\|F(x) - F(x^*) - F'(x^*)(x - x^*)\|$$
$$+ \beta\|F'(x^*)\|\|x - x^*\| \leq \frac{1}{2}\beta L\|x - x^*\|^2 + \beta\|F'(x^*)\|\|x - x^*\|$$

for $\forall x \in D_0$. Let $\varepsilon_0 < \frac{1}{2}\varepsilon_1$ and $\varepsilon_0 - \frac{1}{2}\beta L\varepsilon_0^2 > 0$. Then from (9), we have $\eta > 0$.

Considering the equation

$$\frac{1}{2}\beta Lt^2 + \beta\|F'(x^*)\|t \leq \eta.$$

We have a positive solution $\tilde{t}$, and if $0 < t < \tilde{t}$, we always get

$$\frac{1}{2}\beta Lt^2 + \beta\|F'(x^*)\|t \leq \eta.$$

Thus we have

$$S(x^*, \tilde{t}) \subset \Gamma_{D_{\varepsilon_0}} = \Gamma_{D_{\varepsilon_0}, \|F'(x)^{-1}F(x)\|}(\eta).$$

This shows that the interior of $\Gamma_{D_{\varepsilon_0}}$ is nonempty.

Remark. It is not necessary that D_0 be closed. For $x^0 \in int(\Gamma_{D_{\varepsilon_0}})$, we always have

$$S(x^0, \varepsilon_0) \subset D_0.$$

By using the Mysovskii theorem (9) can be rewritten as

$$\eta \leq \min\{2/\beta L, \varepsilon_0 - \frac{1}{2}\beta L\varepsilon_0^2\}.$$

For the homotopy system

$$H(x,t) = 0, \quad H : D \times [0,1] \subset R^n \times R^1 \to R^n, \tag{10}$$

we have a new exsitence theorem for its solution.

Definition 3. *Let $D \subset R^m$ be open and $f : D \to R^n$ be smooth. We say that $y \in R^n$ is a regular value for f if*

$$\text{Rannge}\, Df(x) = R^n \quad \text{for all} \quad x \in f^{-1}(y)$$

where Df denotes the matrix of partial derivatives of $f(x)$.

The following two principal lemmas can be found in [4].

Lemma 1. *Let $f : D \subset R^{n+1} \to R^n$ be a $C_l (l \geq 1)$ smooth map, and $y \in R^n$ be a regular value of f. Then $f^{-1}(y)$ is a one-dimensional C_1 smooth manifold.*

Lemma 2. *A one-dimensional compact C_l smooth manifold contains only a finite number of connected arcs and circles.*

Now we prove an existence theorem for the continuous curve.

Theorem 2 (Continuous curve existence theorem). *Let $H : D \times [0,1] \subset R^n \times R \to R^n$ be C_l smooth and D be bounded and closed, $H^{-1}(0) \subset \text{int}(D) \times [0,1]$, and for all $(x,t) \in H^{-1}(0), \partial_x H(x,t)$ exists. Then there is a smooth curve $x(t)$, for $t \in [0,1]$, satisfying $H(x,t) = 0$ and $x(t) \in int(D)$. If $x(0) = x_0$, then the curve is unique.*

Proof. By the assumptions, 0 is of cours a regular value. By Lemmas 1 and 2, $H^{-1}(0)$ is a one-dimensional manifold. For D closed and bounded, it is easy to prove that $H^{-1}(0)$ is bounded and closed, so $H^{-1}(0)$ is compact. By Lemma 2,

$$L_i: \quad (x,t) : [0,s_i] \times R^1 \to R^{n+1} \quad i = 1,2,\cdots l.$$

Since $\partial_x H^{-1}$ exists on L_i, $t(s)$ is monotonic on s. So we can rewrite L_i in the following form by the inverse function theorem:

$$L_i: \quad x : [a_i, b_i] \subset [0,1] \to R^n \tag{11}$$

where the interval on t is closed, for $H^{-1}(0)$ is closed. If $a_i \neq 0$,then the point $(x(a_i), a_i) \in \text{int}(D) \times (0,1)$. By the condition, $\partial_x H(x(a_i), a_i)$ exists. By the implicit function theorem, there will exist $\varepsilon > 0$, for $t \in (a_i - \varepsilon, a_i + \varepsilon)$, x is uniquely expressed by t. But L_i as a component cannot be extended, This is a contradiction. So $a_i = 0$.

By the same argument, we can obtain $b_i = 1$. This shows that there are no circle and that arbifrary on the arcs leading from $D \times 0$ to $D \times 1$ are of C_l. So if a point and that arbifrary the arc is given, the arc can be determined uniquely.

This theorem shows that if, the conditions are satisfied, then the solution curve of the problem

$$H(x,t) = 0, \quad x(0) = x_0 \tag{12}$$

is unique and smooth.

On the basis of Theorems 1 and 2, we can obtain another theorem from which the algorithm in our paper comes.

Theorem 3 (Chain level set existence theorem). *Let* $H : D \times [0,1] \subset R^n \times R^1 \to R^n$ *satisfy the conditions of Theorem 2. Then there exists a chain contructed by a finite number of level sets* $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$ *which cover the projection of the solutoin curve on* D, *and*

(i) $\Gamma_i \cap \Gamma_{i+1} \neq \phi, \qquad i = 1,2,\cdots,l-1;$

(ii) $\Gamma_i \subset int(D), \qquad i = 1,2,\cdots,l;$

(iii) $\text{int}(\Gamma_i) \neq \phi, \qquad i = 1,2,\cdots,l;$

(iv) *there is a* t_i *with respect to* Γ_i*; for all* $x_i^0 \in int(\Gamma_i)$, x_i^0 *is a safe initial point on the systems* $H(x, t_i) = 0$.

Proof. By Theorem 2, there is a unique curve $x : [0,1] \subset R^1 \to R^n$ satisfyiny (12); we denote it as L. Define the projection map $P : L \to D$ as follows:

$$P(x,t) = x, \qquad \text{for all } (x,t) \in L.$$

By the continuity and compactness of L, the projection PL is compact for $L \subset \text{int}(D) \times [0,1]$; thus $PL \subset \text{int}(D)$.

Since $\partial_x H$ exists by the condition, using the continuity of $\partial_x H$ and the compactness of L, we have $\|\partial_x H^{-1}\| \leq \beta$. By the Banach lemma there exists a $\delta > 0$, such that $\|\partial_x H\| \leq \overline{\beta}$ 0n $\overline{S}_\delta(L, \delta)$, where

$$(\overline{S})_\delta = \overline{S}_\delta(L,\delta) = \{(x,t) \mid |(x,t) - (\overline{x},\overline{t})| \leq \delta, \forall (\overline{x},\overline{t}) \in L, (x,t) \in D \times [0,1]\}.$$

Using Theorem 1 on $P\overline{S}_\delta$ (the projection of $\overline{S}_\delta$ on D), for each $t \in [0,1]$, the system $H(x,t) = 0$ has a unique solution. Then there exists a level set Γ_t whoses interior is nonempty. For arbitrary $x \in PL$, there exists at least one t, such that $(x,t) \in L$, and $x \in \operatorname{int}(\Gamma_t)$. This shows that $\operatorname{int}(\Gamma_t)$ must cocer PL. By the compactness of PL, there exists a finite number of level sets which can be ordered as $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$, satisfying (i), and we can obtain (ii), (iii), (iv) by using Theorem 1.

§3. Step Selection

By Theorem 3(iv), for each Γ_i, there exists a t_i with respect to Γ_i, and if $x_i^0 \in \operatorname{int}(\Gamma_i)$, x_i^0 is a safe initial point on the nonlinear system

$$H(x, t_i) = 0.$$

This will offer us an algorithm for following the fixed points or zeros, so we will discuss how to obtain the step t_i in the sequel.

Theorem 4 (Estimation of the initial step). *For problem* (12), *let H be F-differentiable, and let $\partial_x H^{-1}(x^0, 0)$ exist and satisfy the following Lipschitz condition:*

$$\|\partial_x H(\tilde{x}, t) - \partial_x H(\tilde{y}, t)\| \le \gamma \|\tilde{x} - \tilde{y}\| \quad \text{for all } \tilde{x}, \tilde{y} \in D, t \in [0,1], \tag{13}$$

$$\|\partial_x H(\tilde{x}, t) - \partial_x H(\tilde{y}, s)\| \le æ |t - s| \quad \text{for all } \tilde{x} \in D, s, t \in [0,1], \tag{14}$$

$$\|H(x, s) - H(x, t)\| \le \omega |s - t| \quad \text{for all } x \in D, s, t \in [0,1]. \tag{15}$$

Let $\delta^0 = \operatorname{Dist}(x^0, \partial D)$. Then we obtain an expression of t_1:

$$\begin{cases} t_1 = \min\{\max\limits_{0<\delta<\delta^0} (2\delta - \beta\gamma\delta^2)/2\beta\omega, 1/2\beta^2\omega\gamma\} & \text{for } æ = 0, \\ t_1 = \min\{\max\limits_{0<\delta<\delta^0} (2\delta - \beta\gamma\delta^2)/(2\delta\beta\, æ + 2\beta\omega), & \\ \qquad [\beta\, æ + \beta^2\omega\gamma - \sqrt{\beta^4\gamma^2\omega^2 + 2\beta^2\, æ\,\omega\gamma}]/\beta^2\, æ^2\} & \text{for } æ \ne 0. \end{cases}$$

If $0 < t < t_1$ and x^0 is an initial point, then the following Newton iteration

$$x(k+1) = x(k) - \partial_x H^{-1}(x(k), t) H(x(k), t), \quad k = 0, 1, \cdots$$

will converge, where $\beta = \|\partial_x H^{-1}(x^0, 0)\|$, $\operatorname{Dist}(A, B) = \min\{\|x - y\| \mid x \in A, y \in B\}$.

Proof. We first have the following estimate:

$$\|I - \partial_x H^{-1}(x^0, 0)\partial_x H(x^0, t_1)\| \le \beta \|\partial_x H(x^0, 0) - \partial_x H(x^0, t_1)\| \le \beta\, æ\, t_1.$$

If $\beta\, æ\, t_1 < 1$, using the Banach lemma, we have

$$\|\partial_x H^{-1}(x^0, t_1)\| \le \beta(1 - \beta\, æ\, t_1)^{-1} = \tilde{\beta};$$

thus

$$\|\partial_x H^{-1}(x^0, t_1) H(x^0, t_1)\| \le \tilde{\beta} \|H(x^0, t_1) - H(x^0, 0)\| \le \tilde{\beta}\omega t_1 = \eta.$$

If x^0 is a safe initial point, there must hold

$$\alpha = \tilde{\beta}\gamma\eta \le \frac{1}{2};$$

thus

$$\beta^2\omega\gamma t_1 (1 - \beta\, æ\, t_1)^{-2} \le \frac{1}{2}.$$

Hence we have

$$\begin{cases} t_1 = 1/2\beta^2\omega\gamma & \text{for } æ = 0\,, \\ t_1 = (\beta æ + \beta^2\omega\gamma - \sqrt{\beta^4\gamma^2\omega^2 + 2\beta^3 æ\omega\gamma})/\beta^2 æ^2 & \text{for } æ \neq 0\,. \end{cases}$$

So we obtain

$$t^* = \frac{1}{\alpha}(1 - \sqrt{1-2\alpha})\eta.$$

If $\overline{S}(x^0, t^*) \subset D, t^*$ should be less than δ^0. For $0 < \delta < \delta^0$, we have

$$\eta \leq \delta - \frac{1}{2}\tilde{\beta}\gamma\delta^2;$$

thus

$$\beta\omega t_1/(1 - \beta æ t_1) \leq \delta - \frac{1}{2}\beta\gamma\delta^2/(1 - \beta æ t_1).$$

Hence we have

$$t_1 \leq (2\delta - \beta\gamma\delta^2)/2(\beta æ \delta + \beta\omega).$$

So we can choose

$$t_1 \leq \max_{0<\delta<\delta^0} \Big(\frac{2\delta - \beta\gamma\delta^2}{2(\beta\gamma\delta + \beta\omega)}\Big).$$

Then the statment above shows that, if $0 < t < t_1, x^0$ is a safe initial point on the nonlinear system $H(x,t) = 0$.

Theorem 4 says that if we obtain the message near $(x^0, 0)$, we can construct the first level set which contains x^0. In fact, the level set covers $x(t)$ for $t \in [0,1]$, and we can easily find that the set constructed shouled belong to the level set Γ_1, if Γ_1 is constructed by x^0. Next, we will construct in a similar way all the level sets.

Theorem 5. *Let* $\|H(\bar{x}(\tilde{t}_i), \tilde{t}_i)\| \leq \varepsilon_i$, *where* $\tilde{t}_i \in [0,1]$, *and* ε_i *is a small control number. Let* $\delta^i =$ Dist $((\tilde{x}(\tilde{t}_i), \tilde{t}_i), \partial D), \partial_x H^{-1}(\bar{x}(\tilde{t}_i), \tilde{t}_i)$ *exist, and* (13)–(15) *be satisfied. Then,*

$$\begin{cases} t_i = \min\Big\{\dfrac{(1 - 2\beta_i^2\varepsilon_i\gamma)}{2\beta_i^2\omega\gamma}, \max\limits_{0<\delta\leq\delta^i}(2\delta - \beta_i\gamma\delta^2 - 2\varepsilon_i)/2\beta_i\omega\Big\} & \text{if } æ = 0, \\ t_i = \min\Big\{((\beta_i^2\gamma\omega + \beta_i æ) - \sqrt{(\beta_i^2\gamma\omega + \beta_i æ)^2 - \beta_i^2 æ^2(1 - 2\beta_i^2\gamma\varepsilon_i)})/\beta_i^2 æ^2, & \\ \qquad \max\limits_{0<\delta\leq\delta^i}(2\delta - \beta_i\gamma\delta^2 - 2\beta_i\varepsilon_i)/2(\beta_i\omega + \delta\beta_i æ)\Big\} & \text{if } æ \neq 0 \end{cases}$$

where

$$\|\partial_x H^{-1}(\bar{x}(\tilde{t}_i), \tilde{t}_i)\| \leq \beta_i.$$

If ε_i *is small enough such that* $t_i > 0$, *then for all* $t \in [0, t_i], \bar{x}(\tilde{t}_i)$ *is a safe initial point on the nonlinear system* $H(x, \tilde{t}_i + t) = 0$.

Proof. The proof of Theorem 5 is similar to that of Theorem 4, so we need only to estimate

$$\begin{aligned} &\|\partial_x H^{-1}(\bar{x}(\tilde{t}_i), \tilde{t}_i + t_i)H(\bar{x}(\tilde{t}_i), \tilde{t}_i + t_i)\| \\ &\leq \Big[\frac{\beta_i}{(1 - \beta_i æ t_i)}\Big][\|H(\bar{x}(\tilde{t}_i), \tilde{t}_i + t_i) - H(\bar{x}(\tilde{t}_i), \tilde{t}_i\| + \|H(\bar{x}(\tilde{t}_i), \tilde{t}_i)\| \\ &\leq \Big[\frac{\beta_i}{(1 - \beta_i æ t_i)}\Big](\omega t_i + \varepsilon_i) = \eta. \end{aligned}$$

By the same argument as in Theorem 4 we can obtain the expressions of t_i.

Theorem 5 says that, if we obtain the message near the approximate solution $\tilde{x}(\tilde{t}_i)$ of the nonlinear system $H(x, \tilde{t}_i) = 0$, we can construct the set which must belong to the level set $\Gamma_{\tilde{t}_i}$. In fact, this set covers $x(t)$ for all $t \in [\tilde{t}_i, \tilde{t}_i + t]$.

§4. SAM Algorithm

Theorem 4 and 5 guarantee that our algorithm below is practical.

Algorithm. 1^0 if $\partial_x H(x^0, 0)$ is invertible, compute $\beta = \|\partial_x H^{-1}(x^0, 0)\|, \delta^0 =$ Dist $(x^0, \partial D)$.

2^0 Calculate t_1 by Theorem 4. If $t_1 > 1$, goto 7^0; otherwise goto 3^0.

3^0 Choose an initial x^0, and run the following iterative procedure.

4^0 Let $k := 0$

$$x(k+1) = x(k) - \partial_x H^{-1}(x(k), t_1) H(x(k), t_1).$$

5^0 If $\|H(x(k), t_1)\| \leq \varepsilon$, goto 6^0; otherwise let $k := k + 1$, goto 4^0.

6^0 If $t_1 \geq 1$, goto 7, otherwise if $\partial_x H(x(k+1), t_1)$ is invertable, calculate $\beta_i = \|\partial_x H^{-1}(x(k+1), t_1)\|, \delta^i =$ Dist $(x(k+1), \partial D)$, and obtain t_i by Theorem 5. Let $x^0 = x(k+1), t_1 = t_1 + t_i, k := 0$, goto 3^0, otherwise GOTO $8°$.

7^0 Let $x^0 := x(k+1)$, and do the following iterartion:

$$x(k+1) = x(k) - \partial_x H^{-1}(x(k), 1) H(x(k), 1), \quad k = 0, 1, \cdots, M.$$

8^0 Stop.

It is easy to find that the algorithm above is a calculation and test procedure, so we can not only trace the continuous curve, but also test the regular value.Meanwhile, we can obtain the following theorem.

Theorem 6. *If the algorithm can proceed to $\tilde{t}$, then there exists C_1 smooth curve $x : [0, \tilde{t}] \to R^n$, satisfying $x(t) \in$ int (D).*

Proof. By the algorithm we can obtain $\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_l$ of the approximate chain $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \cdots, \tilde{\Gamma}_l$ of Theorem 3. Again by the procedore and the smoothness assumption, there exists a unique solution of the nonlinear system $H(x, y) = 0$ in $\tilde{\Gamma}_i$ (for $t \in [\tilde{t}_i, \tilde{t}_i + t_i]$), so x can be uniquely expressed as a function of t in $\tilde{\Gamma}_i (i = 1, \cdots, l)$. For all the sets connected with one another, x can be uniquely extended from $\tilde{\Gamma}_1$ to $\tilde{\Gamma}_l$. Using the implicit function theorem, we have $x(t) \in C_1$.

Since we have built the algorithm, we should see to it that the conditions can guarantee finite extention, such that we can obtain an approximate solution of the nonlinear system $H(x, 1) = 0$.

Theorem 7. *Let the conditions of Theorem 2 be satisfied, and suppose that* (13)–(15) *hold. If ε_i is small enough, the extension is finite.*

Proof. If there exists $t_i > 0$ for all $\delta_i > \delta t$, by the compactness of [0, 1], the extension can be finished by a finite number of steps. If this is not true, then $\sum t_i \leq 1$.

Thus, there exists a positive integer M, such that, for $i > M, t_i$ is sufficiently small.By Theorem 2, we have

$$\|\partial_x H^{-1}(x(t),t)\| \le \beta.$$

Since $x(t) \in int(D)$ and $x(t)$ is continuous, we obtain

$$0 < \delta \le \text{Dist}\,(x(t), \partial D).$$

For $t \in [\tilde{t}_i, \tilde{t}_i + t_i]$, the Newton-Kantorovich theorem goarantees that there exist a unique solution curve $x(t) \in \overline{S}(x^{m_i}, t_i^*)$ satisfying $H(x,t) = 0$.where

$$t_i^* = \left[1 - \sqrt{1-2\alpha}\right]/\beta_i^* r; \qquad \alpha \le \beta_i^* \gamma\eta \le \frac{1}{2};$$

$$\beta_i^* = \frac{\beta_i}{1 - \beta_i æ t_i^*}, \qquad \eta = \beta_i(\omega t_i + \varepsilon_i)/(1 - \beta_i æ t_i).$$

By the Banach lemma, we have

$$\beta_i \le \beta/(1 - \beta æ t_i).$$

For $i > M, t_i$ is small enough. Let $t_i < 1/8\beta æ$. Then $\beta_i < 2\beta$ and $\beta_i^* < 2\beta$, and

$$\delta^i = \text{Dist}\,(x^{m_i}, \partial D) > \delta - t_i^*.$$

If ε_i can be kept small enough, then x^{m_i} is fairly near to $x(\tilde{t}_i+1)$. Let $\|x^{m_i} - x(\tilde{t}_i+1)\| < \frac{1}{2}\delta$. Then $\delta^i > \frac{1}{2}\delta$. Using the value above, we can obtain a common $\delta t > 0$ which has no relation with i but $\beta, \delta, æ, \omega$. This gives a contradiction.

§5. A Numerical Exmple

We know that for some concrete problems, there are various ways to contruct the homotopy mapping. And we often get the regolarity of the mapping by using Sard's theorem and the transversality theorem. If $D = R^n$, in general we get the local Lipschitz continuity of the homotopy mapping instead of the Lipschitz condition. The following problem is an example of using the local Lipschitz condition on an IBM PC-XT.

Consider the following system:

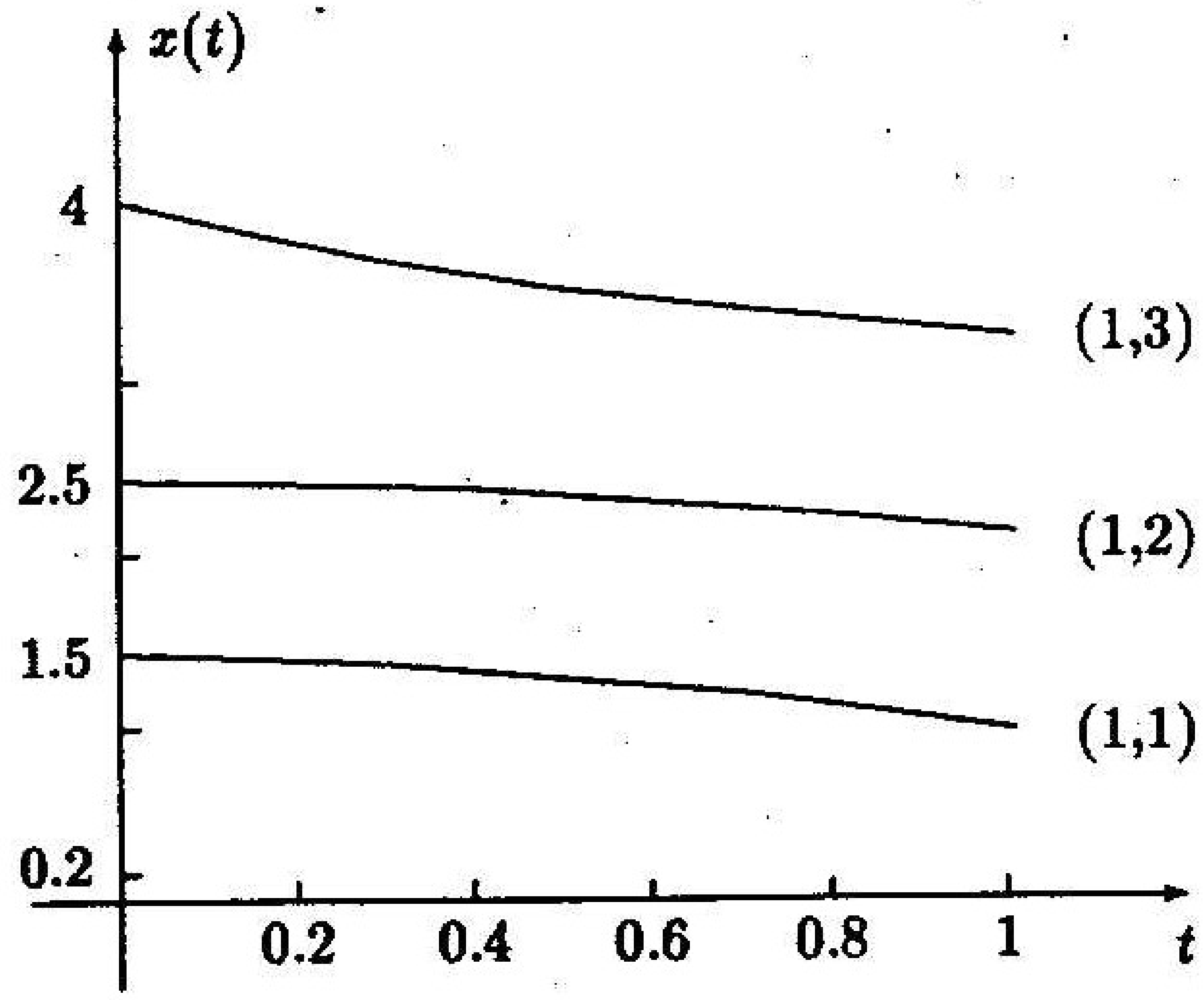
$$\begin{cases} x_1 + x_2 + x_3 = 6, \\ x_1x_2 + x_1x_3 + x_2x_3 = 11, \\ x_1x_2x_3 = 6. \end{cases}$$

We know a solution of the system above is (1, 2, 3), and we can construct the homotopy mapping as follows:

$$H(x,t) = F(x) - (1-t)F(x^0)$$

where $x^0 = (1.5, 2.5, 4)$.

The following graph shows that the procedure of following the path is very satisfactory.



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