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EXTRAPOLATION OF NYSTRÖM SOLUTIONS OF BOUNDARY INTEGRAL EQUATIONS ON NON-SMOOTH DOMAINS*

I.G. Graham
(*School of Mathematical Sciences, University of Bath, United Kingdom*)
Lin Qun Xie Rui-feng
(*Institute of Systems Science, Academia Sinica, Beijing, China*)

Abstract

The interior Dirichlet problem for Laplace's equation on a plane polygonal region Ω with boundary Γ may be reformulated as a second kind integral equation on Γ. This equation may be solved by the Nyström method using the composite trapezoidal rule. It is known that if the mesh has $O(n)$ points and is graded appropriately, then $O(1/n^2)$ convergence is obtained for the solution of the integral equation and the associated solution to the Dirichlet problem at any $\underset{\sim}{x} \in \Omega$. We present a simple extrapolation scheme which increases these rates of convergence to $O(1/n^4)$.

§1. Extrapolation on Non-Uniform Meshes

In this paper we examine a technique for extrapolating numerical solutions (obtained by the Nyström method) to an integral equation defined on the boundary of a polygonal planar domain. At the corners of the boundary the integral operator and the solution suffer from loss of regularity, and the mesh should be graded to compensate for this.

We shall show that, even for such non-uniform meshes, extrapolation of the numerical solutions is still possible. For simplicity we shall restrict ourselves to the case that the Nyström scheme is based on the trapezoidal rule (since this has some computational advantages), but the results obtained should generalise easily to other commonly used quadrature rules.

To motivate the extrapolation procedure, consider first the simple quadrature problem for a function v over $[0,1]$. Let Π_n be an arbitrary mesh $0 = x_0 < x_1 < \cdots < x_n = 1$, and let $h_i = x_i - x_{i-1}$. Let $T_n v$ denote the composite trapezoidal rule with respect to Π_n applied to v. Assume v has sufficient derivatives. Then the Euler-Maclaurin series gives

$$T_n v - \int_0^1 v = \frac{B_2}{2}\sum_{i=1}^{n} h_i^2 \int_{x_{i-1}}^{x_i} D^2 v + R_n v, \tag{1}$$

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where B_2 is the second Bernoulli number, and where

$$|R_n v| \le C \sum_{i=1}^{n} h_i^4 \int_{x_{i-1}}^{x_i} |D^4 v| \tag{2}$$

for some C independent of n and i. Now let Π_{2n} be the mesh obtained by dividing each of the intervals of Π_n exactly in half, and let $T_{2n}v$ be the corresponding composite trapezoidal rule applied to v. It is clear from (1) that

$$(4T_{2n}v - T_n v)/3 - \int_0^1 v = (4R_{2n}v - R_n v)/3.$$

Hence, under the conditions that $D^4 v$ is integrable and $h_i \le C(1/n)$ for each i, we have, from (2),

$$\left|(4T_{2n}v - T_n v)/3 - \int_0^1 v\right| = O(1/n^4). \tag{3}$$

However, under the same conditions, direct use of (1) yields only

$$\left|T_n v - \int_0^1 v\right| = O(1/n^2).$$

Thus, one step of Richardson extrapolation has doubled the rate of convergence of the trapezoidal approximations to the integral of v. The conditions on v and Π_n under which this result holds may be weakened: It is easy to see that (3) remains true provided

$$\sum_{i=1}^{n} h_i^4 \int_{x_{i-1}}^{x_i} |D^4 v| = O(1/n^4),$$

a criterion which naturally leads to the selection of a graded mesh with smaller subintervals where v varies most rapidly.

In §3 we shall show that the same principle holds for Nyström-trapezoidal solutions of boundary integral equations on polygonal domains. It turns out that, provided the mesh reflects accurately enough the (known) qualitative behaviour of appropriate higher derivatives of the unknown solution of the integral equation, extrapolation, analogous to that described above, can be performed.

There has been a long history of interest in extrapolation as a means of accelerating the convergence of numerical solutions of differential and integral equations. In recent years, considerable progress has been made on the extrapolation of finite element solutions of partial differential equations (see e.g., [2], [5], [6] and the references therein).

The use of extrapolation in the solution of integral equations was initiated by Baker[1], who obtained asymptotic error expansions for trapezoidal-Nyström solutions of second kind equations with smooth or Green's function kernels. Baker then used his expansions to prove convergence of a deferred correction technique. The results of Baker have recently been considerably extended by McLean, who obtained in [8] asymptotic error expansions for a wide range of Nyström, collocation, and Galerkin solutions to smooth equations.

The approaches of [5], [7], [9] are different from the above. There the authors sought circumstances under which one step of Richardson extrapolation will lead to acceleration of convergence. They found that they needed only much weaker assumptions on the integral equation. In particular, the need for asymptotic error expansions could be avoided. Thus Lin and Liu were able to show in [5], that one step of extrapolation increases the accuracy of piecewise linear iterated collocation solutions of integral equations with Green's function kernels from $O(n^{-2})$ to $O(n^{-4})$. In [7], [9] extrapolation of solution of boundary integral equations was considered for the first time. There difficulties arose when the boundary had corners which led to singularities in the integral operator and in the solution. Nevertheless, Lin and Xie were able to show in [7] that for the piecewise constant iterated Galerkin method with a graded mesh on a polygonal boundary, one step of extrapolation increases the convergence rate from $O(n^{-2})$ to $O(n^{-4})$. These results are described in a greater detail in the recent Ph.D. thesis of Xie[9], where they are also extended to a broader range of collocation and Galerkin methods for integral equations arising from interior and exterior Dirichlet problems on smooth and non-smooth domains. In This paper, the techniques of [7], [9] are modified and used to prove the convergence of extrapolation of the more practical Nyström methods which were recently analysed in [4].

§2. A Boundary Integral Equation

Let Ω be a bounded domain in $\mathbb{R}^2$, with polygonal boundary Γ. For a distribution $u : \Gamma \to \mathbb{R}$, define the double layer potential:

$$Wu(\underset{\sim}{x}) = \int_\Gamma G'(\underset{\sim}{x}, \underset{\sim}{\xi})u(\underset{\sim}{\xi})d\Gamma(\underset{\sim}{\xi}), \quad \underset{\sim}{x} \in \mathbb{R}^2. \tag{4}$$

Here $G'(\underset{\sim}{x}, \underset{\sim}{\xi}) = -(2\pi)^{-1}(\partial/\partial n(\underset{\sim}{\xi}))(\log|\underset{\sim}{x} - \underset{\sim}{\xi}|), \underset{\sim}{x} \in \mathbb{R}^2, \underset{\sim}{\xi} \in \Gamma$, and $\partial/\partial n(\xi)$ denotes the outward normal derivative from Γ. Consider the integral equation

$$u(\underset{\sim}{x}) - 2Wu(\underset{\sim}{x}) + \chi(\underset{\sim}{x})u(\underset{\sim}{x}) = -2g(\underset{\sim}{x}), \quad \underset{\sim}{x} \in \Gamma \tag{5}$$

where $\chi(\underset{\sim}{x}) \in (-1, 1)$ is such that $(1+\chi(\underset{\sim}{x}))\pi$ is the exterior angle between the tangents to Γ at $\underset{\sim}{y}$, as $\underset{\sim}{y} \to \underset{\sim}{x}\pm$. Equation (5) arises as a reformulation of various harmonic boundary value problems on Ω and on $\Omega' := \mathbb{R}^2 \backslash \bar{\Omega}$. A particular example is discussed in §4.

Following the notation of [4], suppose Γ is a polygon parameterised by $\underset{\sim}{x} = \underset{\sim}{x}(s)$ (s = arc length), which corners at $\underset{\sim}{x}_{2\ell} = \underset{\sim}{x}(s_{2\ell}), \ell = 0, \cdots, L$, where $\underset{\sim}{x}_{2L} = \underset{\sim}{x}_0$. For each ℓ, let $s_{2\ell-1}$ be the mid-point of $[s_{2\ell-2}, s_{2\ell}]$, define $\Gamma_\ell = \{\underset{\sim}{x}(s) : s \in [s_{2\ell-1}, s_{2\ell+1}]\}$, and for any $\phi : \Gamma \to \mathbb{R}$, let ϕ_ℓ denote the restriction of ϕ to Γ_ℓ.

It is known that the solution of (5) is not smooth at corners of Γ. To compensate for this we use a mesh which is graded near each corner. Specifically, each Γ_ℓ will be subdivided by a mesh which in a neighbourhood of $\underset{\sim}{x}_{2\ell}$ will be required to coincide with points of Γ with arc length

$$\left.\begin{array}{ll} s = s_{2\ell} + (i/n)^{q_\ell} a_\ell, & i = 0,1,2,\cdots \\ \text{or} \quad s = s_{2\ell} - (i/n)^{q_\ell} b_\ell, & i = 0,1,2,\cdots \end{array}\right\} \tag{6}$$

where $a_\ell, b_\ell > 0$ are constants usually chosen by the user to reflect the lengths of $(s_{2\ell+1} - s_{2\ell})$ and $(s_{2\ell} - s_{2\ell-1})$ respectively, and $q_\ell \geq 1$ will be specified later. Away from $\underset{\sim}{x}_{2\ell}$, we require only that the width of each subinterval of the mesh should be uniformly $O(1/n)$, as $n \to \infty$.

For any $v : \Gamma \to I\!R$, let $T_{\ell,n}(v)$ denote the composite trapezoidal approximation to $\int_{\Gamma_\ell} v$, obtained using the mesh on Γ_ℓ described above. The Nyström approximation W_n to W (given by (4)) is then defined by

$$W_n\phi(\underset{\sim}{x}) = \sum_{\ell=1}^{L} T_{\ell,n}(G'(\underset{\sim}{x},\cdot)\phi(\cdot)), \quad \underset{\sim}{x} \in I\!R^2. \tag{7}$$

It is well known that $G'(\underset{\sim}{x},\underset{\sim}{\xi}) \to \infty$ when $\underset{\sim}{x},\underset{\sim}{\xi}$ both approach the same corner from different sides, but that $G'(\underset{\sim}{x},\underset{\sim}{\xi})$ is smooth otherwise. To offset this bad behaviour we first rearrange (5) slightly before approximating W by W_n. For each ℓ, set $\tilde{u}_\ell = (u - u(\underset{\sim}{x}_{2\ell}))_\ell$. Then (5) may be rearranged as

$$u(\underset{\sim}{x}) - 2\sum_{\ell=1}^{L}[W\tilde{u}_\ell + u(\underset{\sim}{x}_{2\ell})W1_\ell(\underset{\sim}{x}))] + \chi(\underset{\sim}{x})u(\underset{\sim}{x}) = -2g(\underset{\sim}{x}), \tag{8}$$

with 1 denoting the function with constant value 1. The integrals $W1_\ell(\underset{\sim}{x})$ are known analytically, and $\tilde{u}_\ell(\underset{\sim}{x}_{2\ell}) = 0$, for each ℓ. Our numerical approximation to (8), then, is to seek $u_n : \Gamma \to I\!R$ satisfying

$$u_n(\underset{\sim}{x}) - 2\sum_{\ell=1}^{L}[W_n\tilde{u}_{n,\ell}(\underset{\sim}{x}) + u_n(\underset{\sim}{x}_{2\ell})W1_\ell(\underset{\sim}{x})] + \chi(\underset{\sim}{x})u_n(\underset{\sim}{x}) = -2g(\underset{\sim}{x}), \tag{9}$$

where $\tilde{u}_{n,\ell}$ is derived from u_n exactly as $\tilde{u}_\ell$ is derived from u.

(9) was analysed in [4], where it was shown that the stability result ($\|u_n\|_\infty/\|g\|_\infty$ bounded in n) implies

$$\|u - u_n\|_\infty \leq Cn^{-2}, \tag{10}$$

provided $q_\ell > 2(1 + |\chi(\underset{\sim}{x}_{2\ell})|)$, for each ℓ. Stability, however, has only been established, in general, provided the method is modified slightly. Modification involves the introduction of an integer parameter $i* \geq 0$, and the removal from (7) of all contributions from $i*$ subintervals on each side of each corner. Then the results of [4] show that there exists fixed $i* \geq 1$ independent of n which ensures stability, and preserves the rate of convergence (10). In particular, however, $i* = 0$ is usually sufficient to ensure stability. The results on extrapolation given in this paper assume that the method is stable with $i* = 0$. Identical results for any fixed $i* > 0$ follow by direct analogy.

To prove our results, we first rewrite (8) and (9) as systems. Let B denote the vector space of all $\underset{\sim}{v} = (v_1, \cdots, v_L)^T$, where, for each ℓ, v_ℓ is continuous on Γ_ℓ and $v(\underset{\sim}{x}_{2\ell}) = 0$.

Then define $\mathcal{B}$ to be the Banach space $B \times I\!R^L$ with norm

$$\|(\underset{\sim}{v},\underset{\sim}{\nu})^T\| := \max_{\ell=1,\cdots,L} \|v_\ell(\cdot) + \nu_\ell\|_\infty,$$

where $\underset{\sim}{\nu} = (\nu_1, \cdots, \nu_L)^T$.

Equation (8) may now be recast as a 2×2 system of equations in the space $\mathcal{B}$. The second equation in this system is (8) evaluated at each corner point. The first equation is then obtained by subtracting the second equation from (8). This yields a system

$$\mathcal{T}\begin{pmatrix}\tilde{\underset{\sim}{u}}\\ \underset{\sim}{\mu}\end{pmatrix} := (\mathcal{I} - \mathcal{S} - \mathcal{M})\begin{pmatrix}\tilde{\underset{\sim}{u}}\\ \underset{\sim}{\mu}\end{pmatrix} = \begin{pmatrix}-2\tilde{\underset{\sim}{g}}\\ -2\underset{\sim}{\gamma}\end{pmatrix}$$

where $(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T$ is derived from u by

$$\tilde{\underset{\sim}{u}} = (\tilde{u}_1, \cdots, \tilde{u}_\ell)^T, \quad \underset{\sim}{\mu} = (u(\underset{\sim}{x}_2), \cdots, u(\underset{\sim}{x}_{2L}))^T,$$

and where $\tilde{\underset{\sim}{g}}$ and $\underset{\sim}{\gamma}$ are obtained from g analogously. $\mathcal{S}$ is a diagonal matrix of operators containing the non-smooth part of $\mathcal{T}$. In fact, for $(\underset{\sim}{v}, \underset{\sim}{\nu})^T \in \mathcal{B}$,

$$\mathcal{S}\begin{pmatrix}\underset{\sim}{v}\\ \underset{\sim}{\nu}\end{pmatrix} = \begin{pmatrix}\underset{\sim}{z}\\ \underset{\sim}{0}\end{pmatrix}, \tag{11a}$$

where

$$z_\ell = (2Wv_\ell)_\ell, \quad \ell = 1, \cdots, L. \tag{11b}$$

$\mathcal{M}$ is a smooth (compact) remainder. Similarly (9) may be written

$$\mathcal{T}_n\begin{pmatrix}\tilde{\underset{\sim}{u}}_n\\ \underset{\sim}{\mu}_n\end{pmatrix} := (\mathcal{I} - \mathcal{S}_n - \mathcal{M}_n)\begin{pmatrix}\tilde{\underset{\sim}{u}}_n\\ \underset{\sim}{\mu}_n\end{pmatrix} = \begin{pmatrix}-2\underset{\sim}{g}\\ -2\underset{\sim}{\gamma}\end{pmatrix},$$

where $\tilde{\underset{\sim}{u}}_n = (\tilde{u}_{n,1}, \cdots, \tilde{u}_{n,L})^T, \underset{\sim}{u}_n = (u_n(\underset{\sim}{x}_2), \cdots, u_n(\underset{\sim}{x}_{2L}))^T$, and where $\mathcal{S}_n$ and $\mathcal{M}_n$ are obtained from $\mathcal{S}$ and $\mathcal{M}$ by replacing all the integral operators which operate on $\tilde{\underset{\sim}{u}}$ by their corresponding Nyström approximations using (7). (See [4] for fuller details).

Then $\mathcal{T}^{-1}$ is bounded on $\mathcal{B}$ and subject to the possibility of $i* \geq 1$ (see [4]), $\mathcal{T}_n^{-1}$ is uniformly bounded on $\mathcal{B}$. Thus, say,

$$\begin{aligned}(\tilde{\underset{\sim}{u}}_n, \underset{\sim}{\mu}_n)^T - (\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T &= \mathcal{T}_n^{-1}(\mathcal{T} - \mathcal{T}_n)(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T = \mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}_n)(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T\\ &+ \mathcal{T}_n^{-1}(\mathcal{T} - \mathcal{T}_n)\mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}_n)(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T := \mathcal{A}_n + \mathcal{A}'_n.\end{aligned} \tag{12}$$

Equation (12) provides the basis for the extrapolation argument. Let u_{2n} be the solution on the mesh obtained by precisely halving each subinterval of the original mesh. Then we shall show in the next section that, provided g, the right hand side of (5), is sufficiently smooth, an appropriate choice of grading exponents $\{q_\ell\}$ yields

$$\|\mathcal{A}'_n\| = O(1/n^4),$$

and

$$\|4\mathcal{A}_{2n} - \mathcal{A}_n\| = O(1/n^4).$$

These show

$$\|(\tilde{\underset{\sim}{u}},\underset{\sim}{\mu})^T - (4(\tilde{\underset{\sim}{u}}_{2n},\underset{\sim}{\mu}_{2n})^T - (\tilde{\underset{\sim}{u}}_n,\underset{\sim}{\mu}_n)^T)/3\| = O(1/n^4),$$

and hence

$$\|u - (4u_{2n} - u_n)/3\|_\infty = O(1/n^4).$$

We now give fuller details of this argument.

§3. Extrapolation of the Integral Equation Solution

Let D_s denote differentiation with respect to s. Abbreviate this by D when there is no ambiguity. We shall work in a subspace of $\mathcal{B}$, defined as follows. Suppose $(\underset{\sim}{v},\underset{\sim}{\nu})^T \in \mathcal{B}$. For $\ell = 1,\cdots,L$, let $1 > \beta_\ell > 0$, and for any integer $m \geq 1$, set

$$|v_\ell|_{m,\beta_\ell} = \sup\{|s_{2\ell} - s|^{m-\beta_\ell}|D_s^m v_\ell(\underset{\sim}{x}(s))| : s \in [s_{2\ell-1}, s_{2\ell+1}]\backslash\{s_{2\ell}\}\}.$$

Let $\underset{\sim}{\beta} = (\beta_1,\cdots,\beta_L)$. Then for any integer $k \geq 1$, we define $\mathcal{B}^k_{\underset{\sim}{\beta}}$ to be the completion of the space:

$$\{(\underset{\sim}{v},\underset{\sim}{\nu})^T \in \mathcal{B} : v_\ell \text{ infinitely continuously differentiable on } \Gamma_\ell,\ l = 1,\cdots,L\},$$

under the norm:

$$\|(\underset{\sim}{v},\underset{\sim}{\nu})^T\|_{k,\underset{\sim}{\beta}} := \max\{\|(\underset{\sim}{v},\underset{\sim}{\nu})^T\|,\quad |v_\ell|_{m,\beta_\ell} : m = 1,\cdots,k, \ell = 1,\cdots,L\}.$$

Define

$$\alpha^*_\ell = (1 + |\chi(\underset{\sim}{x}_{2\ell})|)^{-1}, \quad \ell = 1,\cdots,L. \tag{13}$$

It is known [3] that if $\alpha_\ell < \alpha^*_\ell$ for each ℓ, then $\mathcal{T}^{-1}$ is bounded on $\mathcal{B}^k_{\underset{\sim}{\alpha}}$ for all $k \geq 1$.

From now on, C will denote a generic constant independent of n. The principal step of our analysis of (12) involves a detailed study of

$$(\mathcal{T}_n - \mathcal{T})(\underset{\sim}{v},\underset{\sim}{\nu})^T = ((\mathcal{S} - \mathcal{S}_n) + (\mathcal{M} - \mathcal{M}_n))(\underset{\sim}{v},\underset{\sim}{\nu})^T. \tag{14}$$

By (11),

$$(\mathcal{S} - \mathcal{S}_n)(\underset{\sim}{v},\underset{\sim}{\nu})^T = (\underset{\sim}{z} - \underset{\sim}{z}_n, \underset{\sim}{0})^T,$$

with $\underset{\sim}{z}$ given by (11b), and $\underset{\sim}{z}_n$ its Nyström approximant. We shall examine the behaviour of a typical component of, say, $\underset{\sim}{z} - \underset{\sim}{z}_n$, $z_\ell - z_{n\ell}$. Without loss of generality we can assume $s_{2\ell-1} = -1$, $s_{2\ell} = 0$, $s_{2\ell+1} = 1$. Then (see [3] or [4]),

$$z_\ell(\underset{\sim}{x}(s)) = \begin{cases} \displaystyle\int_0^1 K\Big(\frac{s}{\sigma}\Big)\frac{1}{\sigma}v_\ell(\underset{\sim}{x}(\sigma))\,d\sigma, & s \in [-1,0], \quad (15a)\\ \displaystyle\int_0^1 K\Big(-\frac{s}{\sigma}\Big)\frac{1}{\sigma}v_\ell(\underset{\sim}{x}(-\sigma))\,d\sigma, & s \in [0,1], \quad (15b)\end{cases}$$

where

$$K(\sigma) = \frac{\sin(\chi(\underset{\sim}{x}_{2\ell})\pi)}{\pi}\,\frac{\sigma}{\sigma^2 + 1 - 2\sigma\cos(\chi(\underset{\sim}{x}_{2\ell})\pi)}.$$

We shall discuss (15a) and its Nyström approximant. (15b) may be dealt with by an identical argument. Recall that the mesh on Γ is defined by (6) and the subsequent discussion. Without loss of generality we can assume the mesh on $[0,1] \subset \Gamma_\ell$ is given by

$$s_i = (i/n)^{q_\ell}, \quad i = 0, \cdots, n. \tag{16}$$

Applying the composite trapezoidal rule with respect to this mesh to (15a), and recalling that $v_\ell(\underset{\sim}{x}(0)) := 0$, we have

$$z_{n,\ell}(\underset{\sim}{x}(s)) = \sum_{i=1}^{n} w_i K\Big(\frac{s}{s_i}\Big)\frac{1}{s_i} v_\ell(\underset{\sim}{x}(s_i)), \quad s \in [-1,0] \tag{17}$$

with $w_i = (h_{i+1} + h_i)/2, i = 1, \cdots, n-1$, and $w_n = h_n/2$, where $h_i = s_i - s_{i-1}$. Denote the integral in (15a) by $\mathcal{K}v_\ell(s)$, and the sum in (17) by $\mathcal{K}_n v_\ell(s)$. Now, without loss of generality, bounds on $(\mathcal{K} - \mathcal{K}_n)v_\ell$ yield analogous bounds on $(\mathcal{S} - \mathcal{S}_n)(\underset{\sim}{v}, \underset{\sim}{\nu})^T$.

To bound $(\mathcal{K} - \mathcal{K}_n)v_\ell$, we shall need some elementary properties of K. First, by direct calculations,

$$\int_0^\infty \sigma^{m-\rho}|D^m K(\sigma)|\frac{d\sigma}{\sigma} < \infty, \tag{18}$$

for all $0 \leq \rho < 1$, and all integers $m \geq 0$. Next, a simple inductive calculation shows that, for all $k \geq 0, m \geq 0$,

$$(sD_s)^k(\sigma D_\sigma)^m\Big(K\Big(\frac{s}{\sigma}\Big)\frac{1}{\sigma}\Big)$$

is a linear combination of

$$\Big\{\Big(\frac{s}{\sigma}\Big)^j (D^j K)\Big(\frac{s}{\sigma}\Big)\frac{1}{\sigma}, \quad j = k, \cdots, k+m\Big\},$$

and hence, by (18), we have, for all $0 \leq \rho < 1$,

$$\int_0^1 \Big|(s^{k-\rho}D_s^k \sigma^{m+\rho} D_\sigma^m)\Big(K\Big(\frac{s}{\sigma}\Big)\frac{1}{\sigma}\Big)\Big| d\sigma \leq C, \quad s \in [-1,0], \tag{19}$$

with C independent of s.

Lemma 1. *For each ℓ, let $0 < \alpha_\ell < \alpha_\ell^*$, where α_ℓ^* given by (13). Choose $q_\ell > 1/\alpha_\ell$ and set $\beta_\ell = \alpha_\ell - 2/q_\ell$. Then*

$$\|(\mathcal{T} - \mathcal{T}_n)(\underset{\sim}{v}, \underset{\sim}{\nu})^T\|_{2,\underset{\sim}{\beta}} \leq C\frac{1}{n^2}\|(\underset{\sim}{v}, \underset{\sim}{\nu})^T\|_{2,\underset{\sim}{\alpha}},$$

for all $(\underset{\sim}{v}, \underset{\sim}{\nu}))^T \in \mathcal{B}^2_{\underset{\sim}{\alpha}}$.

Proof. We work with the expression (14) for $\mathcal{T} - \mathcal{T}_n$. Without loss of generality, we estimate the first term of (14) dealing with a typical component using explicit expressions (15) and (17). For $k \geq 0$, introduce the operator

$$D^k_{\beta_\ell}\phi(s) = \begin{cases} \phi(s), & k = 0, \\ s^{k-\beta_\ell}D^k\phi(s), & k \geq 1. \end{cases}$$

Then we shall show that for all $(\underset{\sim}{v}, \underset{\sim}{\nu})^T \in \mathcal{B}^2_{\underset{\sim}{\alpha}}$, for all $k = 0, 1, 2$, and for all $s \in [-1, 0]$,

$$|D^k_{\beta_\ell}(\mathcal{K} - \mathcal{K}_n)v_\ell(s)| \leq C\frac{1}{n^2}\|(\underset{\sim}{v}, \underset{\sim}{\nu})^T\|_{2,\underset{\sim}{\alpha}}. \tag{20}$$

By the above remarks, once this is proved, the appropriate bound for the first term of (14) follows.

So, let $(\underset{\sim}{v},\underset{\sim}{\nu})^T \in \mathcal{B}^2_{\underset{\sim}{\alpha}}$. Observe that by (15a), (17) and the fact that $h_1 = s_1$,

$$\mathrm{D}^k_{\beta_\ell}\Big[(\mathcal{K}-\mathcal{K}_n)v_\ell(s) - \Big(\int_0^{s_1} K(\frac{s}{\sigma})\frac{1}{\sigma}v_\ell(\underset{\sim}{x}(\sigma))d\sigma - \frac{1}{2}K(\frac{s}{s_1})v_\ell(\underset{\sim}{x}(s_1))\Big)\Big]$$

is the error in the trapezoidal rule (with respect to σ) applied to

$$D^k_{\beta_\ell}\Big(K\Big(\frac{s}{\sigma}\Big)\frac{1}{\sigma}v_\ell(\underset{\sim}{x}(\sigma))\Big)$$

over $[s_1, 1]$. Standard error estimates for quadrature (e.g. [4, Lemma 2]), together with (18) then yield

$$\begin{aligned} |D^k_{\beta_\ell}(\mathcal{K}-\mathcal{K}_n)v_\ell(s)| &\leq C_1 \sup\{|v_\ell(\underset{\sim}{x}(\sigma))| : \sigma \in [0, s_1]\} \\ &\quad + C_2 \sum_{i\geq 2} h_i^2 \int_{s_{i-1}}^{s_i} \Big|D^2_\sigma D^k_{\beta_\ell}\Big(K\Big(\frac{s}{\sigma}\Big)\frac{1}{\sigma}v_\ell(\underset{\sim}{x}(\sigma))\Big)\Big| d\sigma. \end{aligned} \tag{21}$$

Now, for $0 \leq \sigma \leq s_1$,

$$\begin{aligned} |v_\ell(\underset{\sim}{x}(\sigma))| = |v_\ell(\underset{\sim}{x}(\sigma)) - v_\ell(\underset{\sim}{x}(0))| &\leq s_1 \sup\{|D_\sigma v_\ell(\underset{\sim}{x}(\sigma))| : \sigma \in [0, s_1]\} \\ &\leq s_1^{\alpha_\ell}|v_\ell|_{1,\alpha_\ell} \leq (1/n^2)|v_\ell|_{1,\alpha_\ell}, \end{aligned} \tag{22}$$

by (16) and choice of q_ℓ. This bounds the first term of (21) in the appropriate form.

Using essentially the same argument as in [4, Theorem 3], the second term of (21) may be bounded by

$$\begin{aligned} C_2 \sum_{i\geq 2} h_i^2 s_{i-1}^{-2+\alpha_\ell-\beta_\ell} \sum_{p=0}^{2} \int_{s_{i-1}}^{s_i} \Big|\sigma^{p+\beta_\ell} D^p_\sigma D^k_{\beta_\ell}\Big(K\Big(\frac{s}{\sigma}\Big)\frac{1}{\sigma}\Big)\Big| d\sigma \|(\underset{\sim}{v},\underset{\sim}{\nu})^T\|_{2,\underset{\sim}{\alpha}} \\ \leq C \max\{h_i^2 s_{i-1}^{-2+\alpha_\ell-\beta_\ell} : 2 \leq i \leq n\} \|(\underset{\sim}{v},\underset{\sim}{\nu})^T\|_{2,\underset{\sim}{\alpha}}, \end{aligned} \tag{23}$$

where we have used (19) and the fact that $\beta_\ell < \alpha_\ell < \alpha_\ell^* < 1$. Now for $i \geq 2$, (16) gives

$$\begin{aligned} h_i^2 s_{i-1}^{-2+\alpha_\ell-\beta_\ell} &\leq C\frac{1}{n^2}\Big(\frac{i}{n^2}\Big)^{2q_\ell-2}\Big(\frac{i-1}{n}\Big)^{(-2+\alpha_\ell-\beta_\ell)q_\ell} \\ &\leq C\frac{1}{n^2}\Big(\frac{i}{n}\Big)^{-2+q_\ell(\alpha_\ell-\beta_\ell)} = C\frac{1}{n^2}, \end{aligned} \tag{24}$$

by choice of β_ℓ. Substitution of (22), (23), (24) into (21) now gives the required estimate (20).

The second term of (14) is easier to estimate, since each of the operators in $\mathcal{M}$ has a smooth kernel, and we omit the details.

Now let us consider an extrapolation procedure. For a given n, let $\mathcal{T}_n$ be the composite trapezoidal approximation to $\mathcal{T}$ with respect to the mesh introduced in §2. Let $\mathcal{T}_{2n}$ be the corresponding approximation with respect to the mesh obtained by dividing each of the subintervals of the original mesh into two equal pieces. Let u_n, u_{2n} be the corresponding approximations to u defined using (9).

Lemma 2. *For each ℓ, let $0 < \alpha_\ell < \alpha_\ell^*$, and choose $q_\ell > 4/\alpha_\ell$. Then*

$$\|(\mathcal{T} - (4\mathcal{T}_{2n} - \mathcal{T}_n)/3)(\underset{\sim}{v}, \underset{\sim}{\nu})^T\| \leq C\frac{1}{n^4}\|(\underset{\sim}{v}, \underset{\sim}{\nu})\|_{4,\underset{\sim}{\alpha}},$$

for all $(\underset{\sim}{v}, \underset{\sim}{\nu})^T \in \mathcal{B}^4_{\underset{\sim}{\alpha}}$.

Proof. Again, we consider (14), and we obtain the required estimate for the most difficult part

$$(\mathcal{S} - (4\mathcal{S}_{2n} - \mathcal{S}_n)/3)(\underset{\sim}{v}, \underset{\sim}{\nu})^T. \tag{25}$$

The rest of the estimates required for the lemma are analogous, but easier.

Without loss of generality consider the ℓth component of (25). For $s \in [-1, 0]$ this may be bounded, using (1), (2) by

$$|\mathcal{K}v_\ell(s) - (4\mathcal{K}_{2n}v_\ell(s) - \mathcal{K}_n v_\ell(s))/3| \leq C_1 \sup\{|v_\ell(\underset{\sim}{x}(\sigma))| : \sigma \in [0, s_1]\}$$
$$+ C_2 \sum_{i \geq 2} h_i^4 \int_{s_{i-1}}^{s_i} \left| D_\sigma^4 \left(K\left(\frac{s}{\sigma}\right) \frac{1}{\sigma} v_\ell(\underset{\sim}{x}(\sigma)) \right) \right| d\sigma.$$

Using an almost identical argument to that in Lemma 1, and the fact that $q_\ell > 4/\alpha_\ell$ for each ℓ, it is straightforward to show that

$$|\mathcal{K}v_\ell(s) - (4\mathcal{K}_{2n}v_\ell(s) - K_n v_\ell(s))/3| \leq C\frac{1}{n^4}\|(\underset{\sim}{v}, \underset{\sim}{\nu})^T\|_{4,\underset{\sim}{\alpha}}.$$

An identical argument estimates the ℓth component of (25) for $s \in [0, 1]$, and the lemma follows.

Theorem 3. *Let $g \in C^4(\Gamma)$, and choose $q_\ell > 4/\alpha_\ell^*$ for all ℓ. Then*

$$\|u - (4u_{2n} - u_n)/3\|_\infty \leq C\frac{1}{n^4}\|(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{4,\underset{\sim}{\alpha}^*},$$

where $(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T$ is derived from u, as in §2.

Proof. A sketch of the proof is given at the end of §2. Here we give more details. Recall that if $0 < \alpha_\ell < \alpha_\ell^*$ for all ℓ, then $\mathcal{T}^{-1}$ is bounded on $\mathcal{B}^k_{\underset{\sim}{\alpha}}$ for all $k \geq 1$, and hence $(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T \in \mathcal{B}^4_{\underset{\sim}{\alpha}}$. Since $\mathcal{T}^{-1}$ is also bounded on $\mathcal{B}$, Lemma 2 then ensures

$$\|(4\mathcal{A}_{2n} - \mathcal{A}_n)\| \leq C\frac{1}{n^4}\|(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{4,\underset{\sim}{\alpha}} \leq C\frac{1}{n^4}\|(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{4,\underset{\sim}{\alpha}^*}. \tag{26}$$

Now choose $0 < \alpha_\ell < \alpha_\ell^*$ such that $q_\ell > 4/\alpha_\ell$, for each ℓ, and set $\beta_\ell = \alpha_\ell - 2/q_\ell$. Then, since $q_\ell > 4/\alpha_\ell > 2/\beta_\ell$, we apply Lemma 1 twice to obtain

$$\|(\mathcal{T} - \mathcal{T}_n)\mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}_n)(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\| \leq C\frac{1}{n^2}\|\mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}_n)(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{2,\underset{\sim}{\beta}}$$
$$\leq C\frac{1}{n^2}\|(\mathcal{T} - \mathcal{T}_n)(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{2,\underset{\sim}{\beta}} \leq C\frac{1}{n^4}\|(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{2,\underset{\sim}{\alpha}}.$$

Since $\mathcal{T}_n^{-1}$ is uniformly bounded on $\mathcal{B}$, we have

$$\|\mathcal{A}_n'\| \leq C\frac{1}{n^4}\|(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{2,\underset{\sim}{\alpha}} \leq C\frac{1}{n^4}\|(\underset{\sim}{\tilde{u}}, \underset{\sim}{\mu})^T\|_{2,\underset{\sim}{\alpha}^*}. \tag{27}$$

Then (26), (27) and the discussion at the end of §2 suffices to prove the theorem.

§4. Extrapolation of the Solution of the Interior Dirichlet Problem

The problem $\Delta U = 0$ in Ω, subject to $U = g$ on Γ, may be solved representing U as the double layer potential $U = Wu$, provided u solves the boundary integral equation (5). Suppose we have solved (5) numerically as described in §2, to get u_n and u_{2n}. Define

$$U_n = \sum_{\ell=1}^{L}[W_n\tilde{u}_{n,\ell} + u_n(\underset{\sim}{x}_{2\ell})W1_\ell],$$

$$U_{2n} = \sum_{\ell=1}^{L}[W_{2n}\underset{\sim}{u}_{2n,\ell} + u_{2n}(\underset{\sim}{x}_{2\ell})W1_\ell].$$

We shall show that extrapolation may be performed on U_n, U_{2n} analogously to the extrapolation of u_n, u_{2n} in §3.

Theorem 4. *Under the conditions of Theorem 3,*

$$|U(\underset{\sim}{x}) - (4U_{2n}(\underset{\sim}{x}) - U_n(\underset{\sim}{x}))/3| \le C(\underset{\sim}{x})\frac{1}{n^4}\|(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T\|_{4,\underset{\sim}{\alpha}*}$$

for $\underset{\sim}{x} \in \Omega$, where $C(\underset{\sim}{x})$ depends on $\underset{\sim}{x}$.

Proof. Since $U = Wu$,

$$U - U_n = \sum_{\ell=1}^{L}\{[W\tilde{u}_\ell - W_n\tilde{u}_{n,\ell}] + [u(\underset{\sim}{x}_{2\ell}) - u_n(\underset{\sim}{x}_{2\ell})]W1_\ell\},$$

so that, by Theorem 3,

$$|U(\underset{\sim}{x}) - (4U_{2n}(\underset{\sim}{x}) - U_n(\underset{\sim}{x}))/3| = \sum_{\ell=1}^{L}[W\tilde{u}_\ell - (4W_{2n}\tilde{u}_{2n,\ell} - W_n\tilde{u}_{n,\ell})/3] + O\left(\frac{1}{n^4}\right). \quad (28)$$

(From now on we use $O\left(\frac{1}{n^4}\right)$ to denote a quantity which can be bounded by $C(\underset{\sim}{x})\frac{1}{n^4}$ $\|(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T\|_{4,\underset{\sim}{\alpha}*}$). Now write

$$\begin{aligned} W\tilde{u}_\ell(\underset{\sim}{x}) - W_n\tilde{u}_{n,\ell}(\underset{\sim}{x}) &= W(\tilde{u}_\ell - \tilde{u}_{n,\ell})(\underset{\sim}{x}) + (W - W_n)\tilde{u}_\ell(\underset{\sim}{x}) \\ &\quad - (W - W_n)(\tilde{u}_\ell - \tilde{u}_{n,\ell})(\underset{\sim}{x}) \end{aligned} \quad (29)$$

and observe that, for $\underset{\sim}{x} \in \Omega, W$ is an integral operator with a smooth kernel. Hence, by Theorem 3, the first term of (29) yields $O(1/n^4)$ after extrapolation. Since $W_n\tilde{u}_\ell$ is the trapezoidal approximation of $W\tilde{u}_\ell$, (1), (2) and the same arguments as in Lemma 2 show that the second term may be extrapolated to give $O(1/n^4)$. To deal with the third term in (29) we proceed as in Theorem 3. Let $0 < \alpha_\ell < \alpha_\ell^*$ be such that $q_\ell > 4/\alpha_\ell$, and set $\beta_\ell = \alpha_\ell - 2/q_\ell$. Then,

$$\begin{aligned} |(W - W_n)(\tilde{u}_\ell - \tilde{u}_{n,\ell})(\underset{\sim}{x})| &\le C(\underset{\sim}{x})\frac{1}{n^2}\|(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T - (\tilde{\underset{\sim}{u}}_n, \underset{\sim}{\mu}_n)^T\|_{2,\underset{\sim}{\beta}} \\ &\le C(\underset{\sim}{x})\frac{1}{n^2}\|\mathcal{T}_n^{-1}(\mathcal{T} - \mathcal{T}_n)(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T\|_{2,\underset{\sim}{\beta}} \le C(\underset{\sim}{x})\frac{1}{n^4}\|(\tilde{\underset{\sim}{u}}, \underset{\sim}{\mu})^T\|_{2,\underset{\sim}{\alpha}}. \end{aligned}$$

Substituting these observations into (28) gives the theorem.

§5. A Numerical Experiment

The above theory has been developed only for a polygonal boundary, but it is anticipated that it can be extended without difficulty to a piecewise smooth boundary. This conjecture is tested by the numerical experiments in this section on a boundary consisting of a simple smooth curve with a single corner.

We solve (5) when Γ is given by

$$\underset{\sim}{x}(s) = (\sin s\pi \cos s\pi/4, \sin s\pi \sin s\pi/4), \quad s \in [0,1]. \tag{30}$$

This is the boundary of a "teardrop-shaped" region, symmetric about $x_2 = \tan(\pi/8)x_1$, having s single corner at $\underset{\sim}{0}$ with external angle $7\pi/4$ (i.e. $\chi(\underset{\sim}{0}) = 3/4$). Although s is not arc length here, $\underset{\sim}{x}(s)$ is smooth on $[0,1]$, $|\dot{\underset{\sim}{x}}(s)| > 0$ for $s \in [0,1]$, and the theory of the previous sections is perfectly valid for such a parameterisation of Γ.

For a given even n, we subdivide Γ into n pieces with break points given (cf. (16)) by

$$\left.\begin{aligned} s_i &= (2i/n)^q/2 \\ s_{n-i} &= 1 - s_i \end{aligned}\right\} i = 0, \cdots, n/2 \tag{31}$$

where $q \geq 1$ will be chosen below. Whilst u_n denotes the approximate solution of (5) using this mesh, u_{2n} denotes the approximate solution of (5) using the mesh obtained by subdividing each interval of this mesh into two equal subintervals.

We solve (5) for the particular case when g is the restriction of the harmonic function $U(\underset{\sim}{x}) = e^{x_1}\cos x_2$ to Γ. Then Wu is the unique solution to the interior Dirichlet problem with boundary data g, i.e. $Wu = U$ on Ω. The exact solution of an arbitrary interior Dirichlet problem on Ω typically has a (very) weak singularity near $\underset{\sim}{0}$, but in this particular test example we have forced U to be smooth. However, this is not an unrealistic example, since the (unknown) solution u of (5) will in general still have the typical singularity assumed by the theory of §3. Thus, referring to §2, we have $L = 1$ and $\alpha_1* = (1 + |\chi(\underset{\sim}{0})|)^{-1} = 4/7$, and so by Theorem 3 we would expect that the extrapolated approximation will converge to u with $O(1/n^4)$, provided $q > 4/\alpha_1* = 7$. The choices $q = 3.5, q = 7$ were compared experimentally. It was found that the results for $q = 3.5$ were usually as good and sometimes better than those with $q = 7$. From now on we shall discuss only the results with $q = 3.5$.

Table 1 gives the errors at $\underset{\sim}{0}$ for u_n, u_{2n} and $(4u_{2n} - u_n)/3$. The (unknown) exact value $u(\underset{\sim}{0})$ used in calculating the error was estimated as -0.88506188, correct to 8 decimal places using the extrapolated solution for $n = 256$. Table 2 gives the analogous errors at $\underset{\sim}{x}(1/2) = (\cos(\pi/8), \sin(\pi/8))$. In this case we took the exact value $u(\underset{\sim}{x}(1/2)) = -2.572980$, which is provided, correct to 6 decimal places by the extrapolated solution with $n = 256$. In both tables u_n and u_{2n} converge with $O(1/n^2)$ as expected. The extrapolated solution is much better than either u_n or u_{2n}, although the rate of convergence has not settled down yet to the $O(1/n^4)$ expected by Theorem 3.

In all tables $*\times 10^{-N}$ means that the computed solution agreed with the exact solution up to N decimal places of accuracy.

The proof that the extrapolation procedure works depends strongly on the technique of subdividing each subinterval of the original mesh into two equal subintervals before computing u_{2n} (see discussion in §1). However, in practice the user is more likely to compute simply the sequence u_n for, say, $n = 2, 4, 8, 16, \cdots$, where u_n is the solution to (5) obtained using the mesh (31). It is interesting to ask whether extrapolation will work on this sequence. In Table 3 we compute the error $|(u - u_n^*)(\underset{\sim}{0})|$, where $u_n^* = (4u_{2n} - u_n)/3$, and in this case u_{2n} is obtained using (31) with $2n$ points. We took $u(\underset{\sim}{0})$ to have the same value as in Table 1. Interestingly the errors in this procedure are slightly smaller than those given in Table 1 for the extrapolation procedure analysed in this paper. We have no proof of the performance of this procedure, but the fact that it seems to work well should be of interest in practice.

To provide some results relevant to Theorem 4, we choose 3 points in Ω :

$$\underset{\sim}{x}_i = \gamma_i(\cos(\pi/8), \sin(\pi/8)), \quad i = 1, 2, 3,$$

where $\gamma_1 = 0.1, \gamma_2 = 0.5, \gamma_3 = 0.9$. We give in Table 4 the errors in $U_n(\underset{\sim}{x}_i), U_{2n}(\underset{\sim}{x}_i)$ and $(4U_{2n} - U_n)(\underset{\sim}{x}_i)/3$. Note that the true values $U(\underset{\sim}{x}_i)$ are now known since $U(\underset{\sim}{x}) = e^{x_1}\cos x_2$. Although the observed rates of convergence are irregular, they are often much faster than those predicted in Theorem 4. More remarkable however is the dramatic increase in accuracy obtained by this simple extrapolation procedure.

Table 1. Error at $\underset{\sim}{0}$ assuming $u(\underset{\sim}{0}) = -0.88506188$

n	$\|(u-u_n)(\underset{\sim}{0})\|$	$\|(u-u_{2n})(\underset{\sim}{0})\|$	$\|(u-(4u_{2n}-u_n)/3)(\underset{\sim}{0})\|$
8	1.37×10^{-2}	1.98×10^{-3}	1.92×10^{-3}
16	2.55×10^{-3}	4.47×10^{-4}	2.19×10^{-4}
32	5.03×10^{-4}	1.24×10^{-4}	3.0×10^{-6}
64	1.26×10^{-4}	3.13×10^{-5}	1.1×10^{-7}
128	3.14×10^{-5}	7.80×10^{-6}	1.0×10^{-8}
256	7.80×10^{-5}	1.90×10^{-6}	$*\times 10^{-8}$

Table 2. Error at $\underset{\sim}{x}(\frac{1}{2})$, assuming $u(\underset{\sim}{x}(\frac{1}{2})) = -2.572980$

n	$\|(u-u_n)(\underset{\sim}{x}(\frac{1}{2}))\|$	$\|(u-u_{2n})(\underset{\sim}{x}(\frac{1}{2}))\|$	$\|(u-(4u_{2n}-u_n)/3)(\underset{\sim}{x}(\frac{1}{2}))\|$
8	6.44×10^{-2}	3.67×10^{-3}	2.64×10^{-2}
16	1.41×10^{-2}	5.85×10^{-3}	3.11×10^{-3}
32	1.03×10^{-2}	2.54×10^{-3}	3.6×10^{-5}
64	2.86×10^{-3}	7.07×10^{-4}	9.0×10^{-6}
128	7.24×10^{-4}	1.81×10^{-4}	1.0×10^{-6}
256	1.81×10^{-4}	4.60×10^{-5}	$*\times 10^{-6}$

Table 3. Error in $u_n^* = (4u_{2n} - u_n)/3$, where u_n, u_{2n} are both defined using (29)

n	$\lvert(u - u_n^*)(\underset{\sim}{0})\rvert$
8	1.16×10^{-3}
16	1.81×10^{-4}
32	2.9×10^{-7}
64	6.0×10^{-8}
128	$* \times 10^{-8}$

Table 4

	n	$\lvert(U - U_n)(\underset{\sim}{x}_i)\rvert$	$\lvert(U - U_{2n})(\underset{\sim}{x}_i)\rvert$	$\lvert(U - (4U_{2n} - U_n)/3)(\underset{\sim}{x}_i)\rvert$
$i = 1$	32	1.59×10^{-4}	4.42×10^{-5}	5.87×10^{-6}
	64	1.00×10^{-4}	2.71×10^{-5}	2.70×10^{-6}
	128	2.69×10^{-5}	6.72×10^{-6}	2.00×10^{-10}
	256	6.73×10^{-6}	1.68×10^{-6}	2.10×10^{-11}
$i = 2$	32	2.41×10^{-3}	5.18×10^{-4}	1.12×10^{-4}
	64	3.18×10^{-4}	8.06×10^{-5}	1.59×10^{-6}
	128	8.07×10^{-5}	2.02×10^{-5}	6.00×10^{-9}
	256	2.02×10^{-5}	5.05×10^{-6}	3.74×10^{-10}
$i = 3$	32	2.75×10^{-3}	3.64×10^{-4}	1.40×10^{-3}
	64	9.15×10^{-4}	2.25×10^{-4}	4.55×10^{-6}
	128	2.06×10^{-4}	5.19×10^{-5}	7.00×10^{-7}
	256	5.06×10^{-5}	1.27×10^{-5}	4.36×10^{-8}

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