Journal of Computational Mathematics, Vol.10, No.3, 1992, 224–230.

THE DIMENSIONS OF SPLINE SPACES AND THEIR SINGULARITY*

Shi Xi-quan
(*Jilin University, Changchun China*)

Abstract

In this paper, the dimensions of spaces $S_k^\mu(\Delta_n)(k \geq 2^n\mu + 1)$ are obtained, where Δ_n is a general simplicial partition of a bounded region with piecewise linear boundary. It is also pointed that the singularity of spaces $S_k^\mu(\Delta_n)$ can not disappear when $n \geq 3$ no matter how large k is. At the same time, a necessary and sufficient condition that Morgen and Scott's structure is singular is obtained.

§1. Dimension

Let $D_n \in I\!R^n$ be a bounded region with piecewise linear boundary, Δ_n a simplicial partition of D_n, and $S_j^{(i)}(i = 0, 1, \cdots, n; j = 1, 2, \cdots, T_i)$ be all i-simplices of Δ_n. $R(S^{(i)}) = \cup\{S^{(n)} \in \Delta_n : S^{(i)} \subset S^{(n)}\}$ is called an i-incident region of $S^{(i)}$.

Definition 1. *Let $S^{(i)} \in \Delta_n$ and P_0 be an inner point of $S^{(i)}$. Then*

$$T(S^{(i)}) = M \cap R(S^{(i)})$$

is called a transversal surface of $S^{(i)}$, where $M = \{P - P_0 \in I\!R^n : (P - P_0, V - P_0) = 0, V \in S^{(i)}\}$.

When $n = 2$ and $k \geq 4\mu + 1$,

$$\dim S_k^\mu(\Delta_2) = \frac{1}{2}(k - 3\mu - 1)(k - 3\mu - 2)T_2 + \frac{1}{2}(\mu + 1)(2k - 7\mu - 2)T_1 + \sum_{i=1}^{T_0} \dim S_{2\mu}^\mu(R(S_i^{(0)})), \tag{1}$$

(see [1] and [2]) and when $n = 3$ and $k \geq 8\mu + 1$,

$$\dim S_k^\mu(\Delta_3) = (C_{k-4\mu-1}^3 - 4C_\mu^3)T_3 + \frac{1}{2}(\mu + 1)((k - 5\mu - 1)(k - 4\mu - 2) + 2\mu)T_2 + \sum_{i=1}^{T_1} \Big[(k - 6\mu - 1)\dim S_{2\mu}^\mu(T(S_i^{(1)})) - \sum_{j=0}^{2\mu-1} \dim S_j^\mu(T(S_i^{(1)}))\Big] + \sum_{i=1}^{T_0} \dim S_{4\mu}^\mu(R(S_i^{(0)})), \tag{2}$$

* Received November 30, 1988.

(see [3]) where $T(S_i^{(1)})$ is a transversal surface of $S_i^{(0)}$, and

$$C_m^n = \begin{cases} \frac{m!}{n!(m-n)!}, & \text{when } m \ge n, \\ 0, & \text{otherwise.} \end{cases}$$

For a general case, we have

Theorem 1. *When $k \ge 2^n\mu + 1$,*

$$\dim S_k^\mu(\Delta_n) = \sum_{i=1}^{T_0} \dim S^\mu_{2^{n-1}\mu}(R(S_i^{(0)}))$$

$$+ \sum_{i=1}^{T_1}[(k - 3\cdot 2^{n-2}\mu - 1)\dim S^\mu_{2^{n-2}\mu}(T(S_i^{(1)})) - \sum_{j=0}^{2^{n-2}\mu-1} \dim S_j^\mu(T(S_i^{(1)}))]$$

$$+ \sum_{d=2}^{n-1}\sum_{j=1}^{T_d}[M(2^{n-d-1}\mu, d)\dim S^\mu_{2^{n-d-1}\mu}(TS_i^{(d)})) - \sum_{j=0}^{2^{n-d-1}\mu-1} (M(j+1,d)$$

$$- M(j,d))\dim S_j^\mu(T(S_i^{(d)}))] + M(0,n)T_n,$$

where $T(S_i^{(d)})$ is a transversal surface of $S_i^{(d)}$, and

$$M(d,i) = C^{\phi}_{A(i,d)} - (d+1)N(0,d,i) - \sum_{j=1}^{d-2} C_{d+1}^{j+1}[M(j, 2^{n-j-1}\mu)N(j,d,2^{n-d-1}\mu)$$

$$- \sum_{l=1}^{2^{n-d-1}\mu} (M(j, 2^{n-j-1}\mu - l + 1) - M(j, 2^{n-j-1}\mu - l)) \cdot L(j,d,i,l)],$$

$$A(i,d) = k - (d+1)2^{n-d}\mu + d_i - 1, \quad M(1,i) = C^1_{k-2^n\mu-1+i},$$

$$M(2,i) = C^2_{k-3\cdot 2^{n-2}+2i-1} - 3N(0,2,i), \quad N(m,d,i) = C^{d-m}_{B(i,d,m)},$$

$$B(i,d,m) = 2^{n-m-1}\mu - (d-m)\mu\cdot 2^{n-d} + (d-m-1)i,$$

$$L(j,d,i,l) = C^{d-j}_{B(i,d,j)-l}.$$

To prove Theorem 1, we give the folowing interpolation conditions:

i) Let $S^{(0)} \in \Delta_n$, $\{S_1^{(1)}, \cdots, S_m^{(1)}\} \subset \Delta_n$ be all 1-simplices with $S^{(0)}$ as a common end point, and τ_i be the unit vector of $S_i^{(1)}$, if $S_{i_1}^{(1)}, \cdots, S_{i_n}^{(1)}$ are the edges of a n-simplex in Δ_n, then the given conditions are

$$\left\{\frac{\partial^{m_1}}{\partial\tau_{i_1}^{m_1}} \cdots \frac{\partial^{m_n}}{\partial\tau_{i_n}^{m_n}} f(S^{(0)}_{i_1,\cdots,i_n})\right\}, \quad 0 \le m_1 + \cdots + m_n \le 2^{n-1}\mu,$$

and if $V_1 = V[S_{b_1}^{(1)}, S_{a_2}^{(1)}, \cdots, S_{a_n}^{(1)}]$ and $V_2 = V[S_{b_2}^{(1)}, S_{a_2}^{(1)}, \cdots, S_{a_n}^{(1)}$ have a common $(n-1)$-simplex surface, then

$$\frac{\partial^{m_1}}{\partial\tau_{b_2}^{m_1}} \frac{\partial^{m_2}}{\partial a_{a_2}^{m_2}} \cdots \frac{\partial^{m_n}}{\partial\tau_{a_n}^{m_n}} f(S^{(0)}_{b_2,a_2,\cdots,a_n})$$

$$= \left(d_1\frac{\partial}{\partial\tau_{b_1}} + \sum_{j=2}^{n} d_j \frac{\partial}{\partial\tau_{a_j}}\right)^{m_1} \frac{\partial^{m_2}}{\partial\tau_{a_2}^{m_2}} \cdots \frac{\partial^{m_n}}{\partial\tau_{a_n}^{m_n}} f(S^{(0)}_{b_1,a_2,\cdots,a_n})$$

where $0 \le m_1 \le \mu, 0 \le m_1 + \cdots + m_n \le 2^{n-1}\mu, d_i (1 \le i \le n)$ are constants and $\tau_{b_2} = d_1 \cdot \tau_{b_1} + \sum_{i=2}^{n} d_i \tau_{a_i}$.

ii) Let $S^{(1)} \in \Delta_n, S_i^{(2)} (1 \le i \le m) \in \Delta_n$ be all 2-simplices with $S^{(1)}$ as a common edge, $P_i (1 \le i \le k - 3 \cdot 2^{n-2}\mu - 1)$ be different points in $S^{(1)}$, and $\tau_i \| S_i^{(2)}$ and $\tau_i \perp S^{(1)}$. If $S_{a_i}^{(2)} (1 \le i \le n-1)$ are surfaces of a n-simplex in Δ_n, then the conditions are

$$\Big\{ \frac{\partial^{m_1}}{\partial \tau_{a_1}^{m_1}} \cdots \frac{\partial^{m_{m-1}}}{\partial \tau_{a_{n-1}}^{m-1}} f(S^{(1)}_{l, a_1, \cdots, a_{n-1}}) \Big\},$$

$$\max\{0, l + 2^n\mu + 1 - k\} \le h_1 + \cdots + h_{n-1} \le 2^{n-2}\mu,$$

and if two simplices $V_i = V[S_{b_i}^{(2)}, S_{a_2}^{(2)}, \cdots, S_{a_{n-1}}^{(2)}]$ $(i = 1, 2)$ have a common $(n-1)$-simplex surface, then

$$\frac{\partial^{m_1}}{\partial \tau_{b_2}^{m_1}} \frac{\partial^{m_2}}{\partial_{a_2}^{m_2}} \cdots \frac{\partial^{m_{n-1}}}{\partial \tau_{a_{n-1}}^{m_{n-1}}} f(S^{(1)}_{l, b_2, a_2, \cdots, a_{n-1}})$$
$$= \Big(d_1 \frac{\partial}{\partial \tau_{b_1}} + \sum_{i=2}^{n-1} d_i \frac{\partial}{\partial \tau_{a_i}} \Big)^{m_1} \frac{\partial^{m_2}}{\partial \tau_{a_2}^{m_2}} \cdots \frac{\partial^{m_{n-1}}}{\partial \tau_{a_{n-1}}^{m_{n-1}}} f(S^{(2)}_{l, b_1, a_2, \cdots, a_{n-1}})$$

where $\max\{0, l + 2^n\mu + 1 - k\} \le m_1 + \cdots m_{n-1} \le 2^{n-2}\mu$, $\max\{0, l + 2^n\mu + 1 - k\} \le h_1 + \cdots + h_{n-1} \le 2^{n-2}\mu$, $0 \le m_1 \le \mu, 1 \le l \le k - 3$. $d_i (1 \le i \le n-1)$ are constants and $\tau_{b_2} = d_1 \tau_{b_1} + \sum_{i=2}^{n-1} d_i \tau_{a_i}$.

iii) Let $S^{(d)} (2 \le d \le n-1) \in \Delta_n, S_i^{(d+1)} (1 \le i \le m)$ be all $(d+1)$-simplices with $S^{(d)}$ as a common surface, $P_j (1 \le j \le M(i, d))$ be suitable points in $S^{(d)}$, and $\tau_i \| S_i^{(d+1)}$ and $\tau_i \perp S^{(d)}$. If $S_{a_i}^{(d+1)} (1 \le i \le n-d)$ are surface of a n-simple of belonging to Δ_n, then the conditions are

$$\Big\{ \frac{\partial^{m_1}}{\partial \tau_{a_1}^{m_1}} \cdots \frac{\partial^{m_{n-d}}}{\partial \tau_{a_{n-d}}^{m_{n-d}}} f(P_{l, a_1, \cdots, a_{n-d}}) \Big\},$$

where $m_1 + m_2 + \cdots + m_{n-d} = i, 0 \le i \le 2^{n-d-1}\mu, 1 \le l \le M(i, d)$, and if two n-simplices $V_i = [S_{b_i}^{(d)}, S_{a_2}^{(d)} \cdots S_{a_{n-d}}^{(d)}] (i = 1, 2)$ have a common $(n-1)$-simplex surface, then

$$\frac{\partial^{m_1}}{\partial \tau_{b_2}^{m_1}} \frac{\partial^{m_2}}{\partial \tau_{a_2}^{m_2}} \cdots \frac{\partial^{m_{n-d}}}{\partial \tau_{a_{n-d}}^{m_{n-d}}} f(S^{(d)}_{l, b_2, a_2, \cdots, a_{n-d}})$$
$$= \Big(d_1 \frac{\partial}{\partial \tau_{b_1}} + \sum_{j=2}^{n-d} d_j \frac{\partial}{\partial \tau_{a_j}} \Big)^{m_1} \frac{\partial^{m_2}}{\partial \tau_{a_2}^{m_2}} \cdots \frac{\partial^{m_{n-d}}}{\partial \tau_{a_{n-d}}^{m_{n-d}}} f(S^{(d)}_{l, b_1, a_2, \cdots, a_{n-d}}),$$

where $0 \le m_1 \le \mu, d_i (1 \le i \le n-d)$ are constants and $\tau_{b_2} = d_1 \tau_{b_1} + \sum_{i=2}^{n-d} d_i \tau_{a_i}$.

iv) Let $S_n^{(n)} \in \Delta_n$, then the number of $M(n)$ interpolation conditions are given in the interior of $S^{(n)}$ such that they and boundary conditions of $S^{(n)}$ determine a unique polynomial of degree k.

It is not difficult to prove that the conditions i)–iv) determine a unique spline function beloging to $S_k^\mu(\Delta_n)$, and we can obtain Theorem 1 by computing the number of the independent interpolation conditions i)–iv) directly.

§2. Singularity

It is well known that dimensions of spline spaces are closely related to the geometric structure of Δ_n. We introduce the following definition for distinguishing singularity.

Definition 2. *The sigularity of spline space dimensions caused by the coplanarity of $(n-1)$-simplices in Δ_n is called* I*-singularity. The other singularities are called* II*-singularity.*

Obviously, $\dim S_k^\mu(\Delta_n)$ has no singularity when $n = 1$, and for $n = 2$, II-singularity vanishes when k are sufficiently large. But this result is not true when $n \geq 3$. For example, assume $n = 3$, and let D_3 and Δ_3 be shown in Fig. 1, $\Pi_{i,j}$ be the plane determined by points T_i, T_j and $T_0, \Pi_{i,6} = \alpha_{i,6}\Pi_{5,6} + \beta_{i,6}\Pi_{4,6} (i = 1,2), \Pi_{i,4} = \alpha_{i,4}\Pi_{4,6} + \beta_{i,4}\Pi_{4,5} (i = 2,3)$, and $\Pi_{i,5} = \alpha_{i,5}\Pi_{4,5} + \beta_{i,5}\Pi_{5,6} (i = 1,3)$. We also suppose that any two planes in them are not coplanar. Then by computing directly, we can obtain

$$\dim S_k^1(\Delta_3) = \begin{cases} M+1, & \text{when} \quad \mu_{2,4}\mu_{3,4}\mu_{1,5}\mu_{3,5}\mu_{1,6}\mu_{2,6} = 1, \\ M, & \text{otherwise}, \end{cases}$$

where $k \geq 2, \mu_{i,j} = \alpha_{i,j}/\beta_{i,j}, M = 7C_{k-5}^2 + 24C_{k-3}^2 + 12C_{k-5}^1 + 17C_{k-4}^1 + 6C_{k-2}^1 + 10C_{k-1}^1$. This implies that the singularity of space $S_k^1(\Delta_3)$ cannot disappear no matter how large the number k is.

§3. Morgan and Scott's Example

Let Δ_2 be a triangulation shown in Fig. 2. Morgan and Scott, Schumaker (cf.[4]) showed that, if the figure is symmetric, then

$$\dim S_2^1(\Delta_2) = 7. \tag{3}$$

Y.S. Chou, L.Y. Su and R.H. Wang (cf.[5]) have proved that, if

$$\begin{vmatrix} (A_5 - A_7)(A_5 - A_6) & (A_4 - A_2)(A_4 - A_1) \\ (A_5 - A_8)(A_5 - A_9) & (A_4 - A_8)(A_4 - A_9) \end{vmatrix} = 0,$$

then equation (3) is also true, where A_i is a slope of the corresponding line. Now we will prove.

Theorem 2. *Equation* (3) *holds if and only if the segments T_1T_4, T_2T_5 and T_3T_6 (or their prolongation) are concurrent lines or more precisely, if and only if one of the following cases occurs:*

1) *The common crossover point is on the edge of triangle $T_4T_5T_6$ (i.e. Δ_2 has a singular point, cf. Fig.* 3(a));

2) *The common crossover point is at the inner of the triangle $T_4T_5T_6$(cf. Fig.* 3 (b));

3) *The common crossover point is at the outside of triangle $T_4T_5T_6$ (cf. Fig 3 3(b)).*

To prove this theorem, obviously, we only need to prove the second case. We take the same symbols as in Fig. 1, and let A_{ij} be the smooth factor on Π_{ij}. Then we can obtain

$$A_{4,6}\Pi^2_{4,6} + A_{1,6}\Pi^2_{1,6} + A_{2,6}\Pi^2_{2,6} + A_{5,6}\Pi_2 5,6 = 0,$$

$$A_{4,5}\Pi^2_{4,5} + A_{2,4}\Pi^2_{2,4} + A_{3,4}\Pi^2_{3,4} - A_{4,6}\Pi_2 4,6 = 0,$$

$$A_{4,5}\Pi^2_{4,5} + A_{1,5}\Pi^2_{1,5} + A_{3,5}\Pi^2_{3,5} + A_{5,6}\Pi_2 5,6 = 0,$$

which are equal to the following equations:

$$A_{1,6}\alpha^2_{1,6} + A_{2,6}\alpha^2_{2,6} + A_{5,6} = 0, \quad A_{1,6}\alpha_{1,6}\beta_{1,6} + A_{2,6}\alpha_{1,5}\beta_{2,6} = 0,$$

$$A_{1,6}\beta^2_{1,6} + A_{2,6}\beta^2_{1,5} + A_{4,6} = 0, \quad A_{1,5}\alpha^2_{1,5} + A_{3,5}\alpha^2_{3,5} + A_{4,5} = 0,$$

$$A_{1,5}\alpha_{1,5}\beta_{1,5} + A_{3,5}\alpha_{3,5}\beta_{3,5} = 0, \quad A_{1,5}\beta^2_{1,5} + A_{3,5}\beta^2_{3,5} + A_{5,6} = 0,$$

$$A_{2,4}\alpha^2_{2,4} + A_{3,4}\alpha^2_{3,4} - A_{4,6} = 0, \quad A_{2,4}\alpha_{2,4}\beta_{2,4} + A_{3,4}\alpha_{3,4}\beta_{3,4} = 0,$$

$$A_{2,4}\beta^2_{2,4} + A_{3,4}\beta^2_{3,4} + A_{4,5} = 0,$$

denoted by

$$AX = 0,$$

where $X = (A_{1,5}, A_{1,6}, A_{5,6}, A_{2,6}, A_{2,4}, A_{4,6}, A_{3,4}, A_{3,5}, A_{4,5})$, and

$$A = \begin{pmatrix} 0 & \alpha^2_{1,6} & 1 & \alpha^2_{2,6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{1,6}\beta_{2,6} & 0 & \alpha_{2,6}\beta_{2,6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta^2_{1,6} & 0 & \beta^2_{2,6} & 0 & 1 & 0 & 0 & 0 \\ \alpha^2_{1,5} & 0 & 0 & 0 & 0 & 0 & 0 & \alpha^2_{3,5} & 1 \\ \alpha_{1,5}\beta_{1,5} & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{3,5}\beta_{3,5} & 0 \\ \beta^2_{1,5} & 0 & 1 & 0 & 0 & 0 & 0 & \beta^2_{3,5} & 0 \\ 0 & 0 & 0 & 0 & \alpha^2_{2,4} & -1 & \alpha^2_{3,4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{2,4}\beta_{2,4} & 0 & \alpha_{3,4}\beta_{3,4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta^2_{2,4} & 0 & \beta^2_{3,4} & 0 & 1 \end{pmatrix}$$

Hence

$$\det A = (\alpha_{1,6}\alpha_{2,6}\alpha_{2,4}\alpha_{3,4}\alpha_{1,5}\alpha_{3,5} - \beta_{1,6}\beta_{2,6}\beta_{2,4}\beta_{3,4}\beta_{1,5}\beta_{3,5})(\alpha_{2,6}\beta_{1,6} - \alpha_{1,6}\beta_{2,6})$$
$$\times(\alpha_{2,4}\beta_{3,4} - \alpha_{3,4}\beta_{2,4})(\alpha_{3,5}\beta_{1,5} - \alpha_{1,5}\beta_{3,5}).$$

It is not difficult to show that any two segments in Δ_2 are not collinear in the second case. Thus we have

$$\alpha_{i,j} \neq 0, \quad \beta_{i,j} \neq 0.$$

Therefore,

$$\det A = 0 \iff \mu_{2,4}\mu_{3,4}\mu_{1,5}\mu_{3,5}\mu_{1,6}\mu_{2,6} = 1,$$

where $\mu_{i,j} = \alpha_{i,j}/\beta_{i,j}$.

Computing directly, we obtain

$$\alpha_{1,6} = -\frac{\det(T_4 - T_6, T_1 - T_6)}{\det(T_5 - T_6, T_4 - T_6)}, \quad \beta_{1,6} = -\frac{\det(T_1 - T_6, T_5 - T_6)}{\det(T_5 - T_6, T_4 - T_6)},$$

$$\alpha_{1,5} = -\frac{\det(T_1 - T_5, T_6 - T_5)}{\det(T_6 - T_5, T_4 - T_5)}, \quad \beta_{1,5} = -\frac{\det(T_4 - T_5, T_1 - T_5)}{\det(T_6 - T_5, T_4 - T_5)}.$$

Thus,

$$\mu_{1,5}\mu_{1,6} = -\frac{\det(T_4 - T_6, T_1 - T_6)}{\det(T_4 - T_5, T_1 - T_5)}.$$

Let A be the intersection point of two lines T_1T_4 and T_5T_6, and $A - T_5 = \alpha_A(T_6 - T_5)$. Then

$$\mu_{1,5}\mu_{1,6} = \frac{1-\alpha_A}{\alpha_A}.$$

For the same reason,

$$\mu_{2,4}\mu_{2,6} = \frac{1-\alpha_B}{\alpha_B}, \quad \mu_{3,4}\mu_{3,5} = \frac{1-\alpha_C}{\alpha_C}.$$

Therefore,

$$\det A = 0 \iff \frac{(1-\alpha_A)(1-\alpha_B)(1-\alpha_C)}{\alpha_A\alpha_B\alpha_C} = 1.$$

According to Ceva's theorem, the conclusions of Theorem 2 holds.

For example, the triangulation as in Fig. 3(b) is denoted by Δ. Let O be the barycenter of triangle $T_1T_2T_3$, $|OT_i| = \frac{t_i}{2}|OT_{i-3}|$ $(0 < t_i < 1, 4 \le i \le 6)$. We can obtain that $S \in S_2^0(\Delta)$ belongs to spline space $S_2^1(\Delta)$, where S is determined by the following conditions:

$$S((T_i + T_j)/2) = 0 (i,j = 1,2,3), S(T_i) = (1 - t_i)(2 + t_i)$$

and

$$S((T_i + T_{i+1})/2) = \frac{1}{4}(S(T_i) + T_{i+1}) + (2 + t_i)(2 + t_{i+1})),$$

where $4 \le i \le 6$, $T_7 = T_4$,and

$$S((T_1 + T_6)/2) = S((T_2 + T_6)/2) = \frac{1}{4}S(T_6),$$

$$S((T_2 + T_4)/2) = S((T_3 + T_4)/2) = \frac{1}{4}S(T_4),$$

$$S((T_3 + T_5)/2) = S((T_1 + T_5)/2) = \frac{1}{4}S(T_5).$$

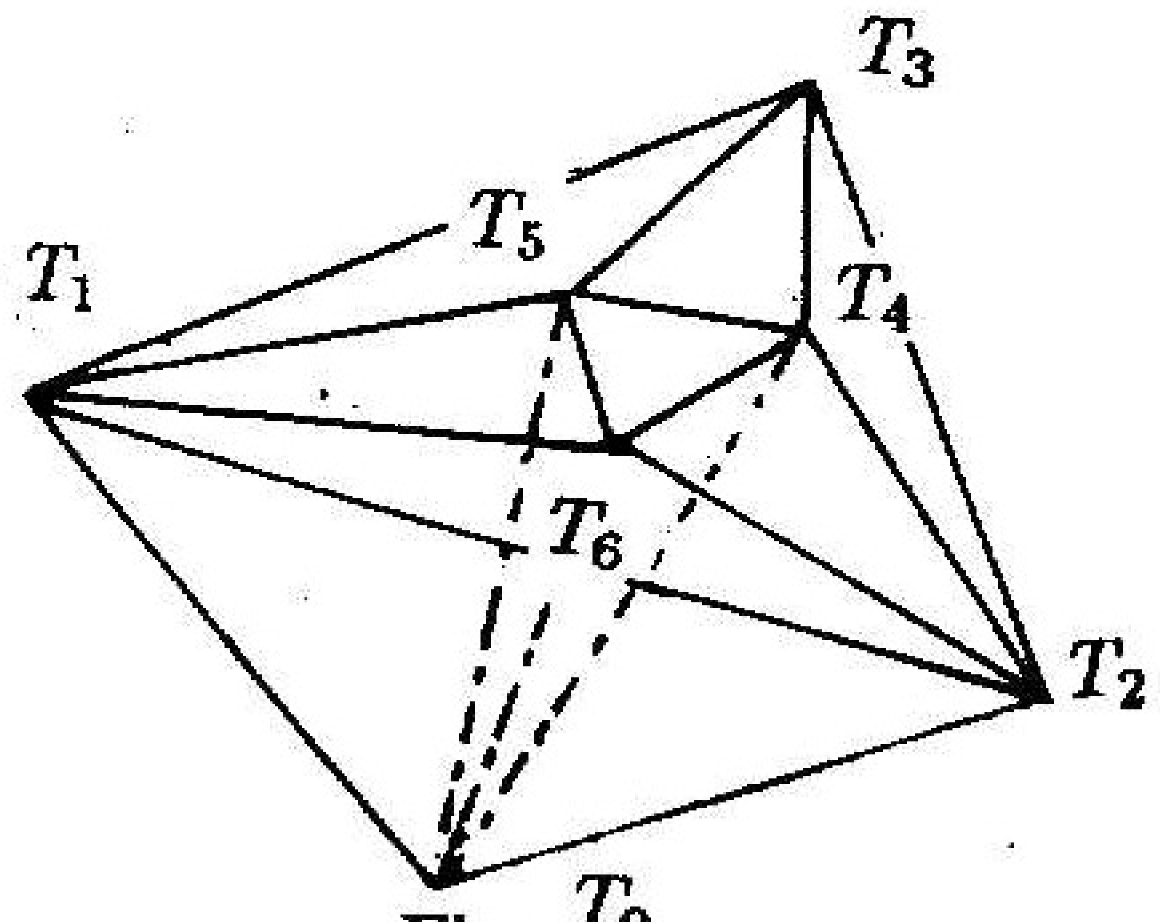


Fig.

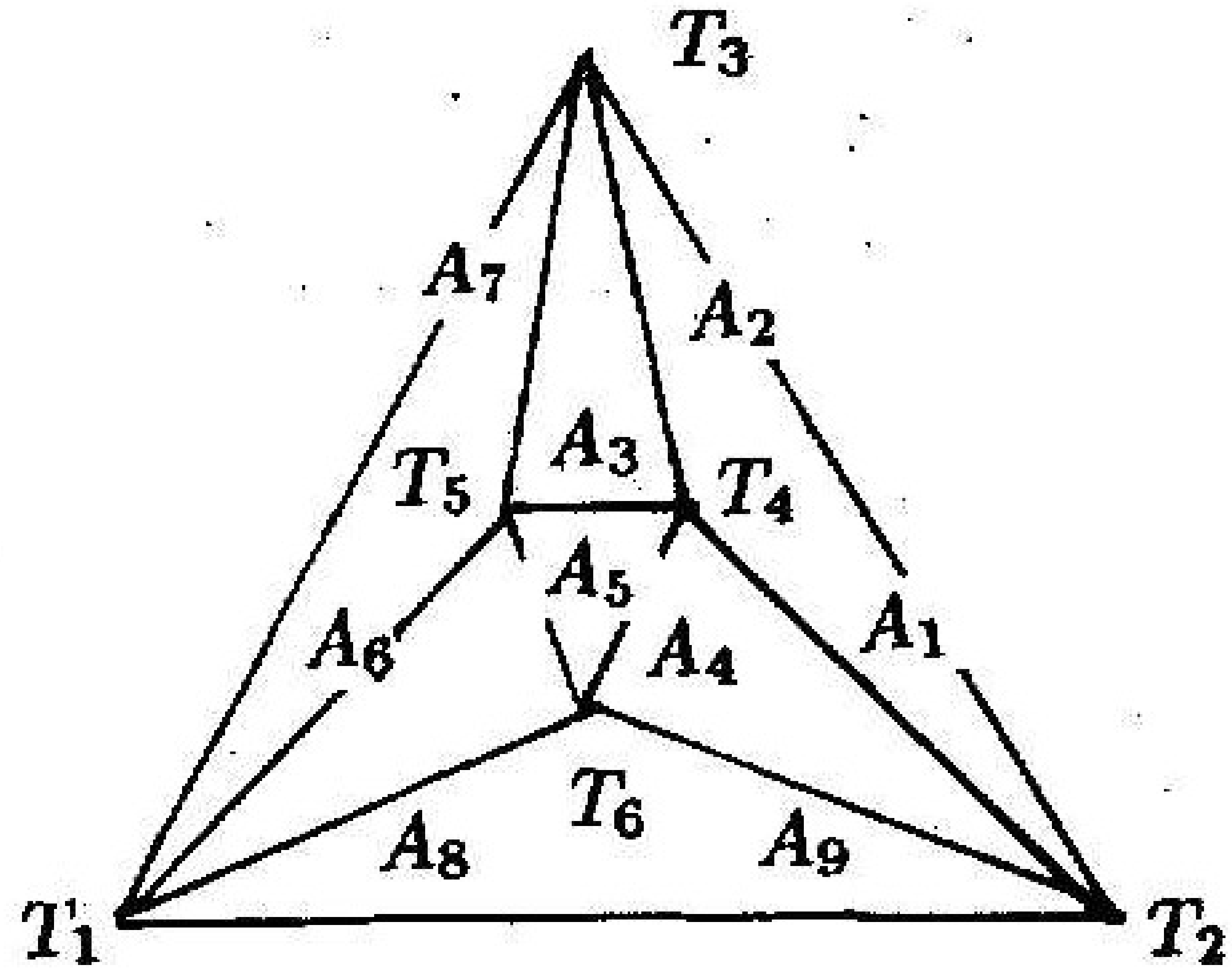
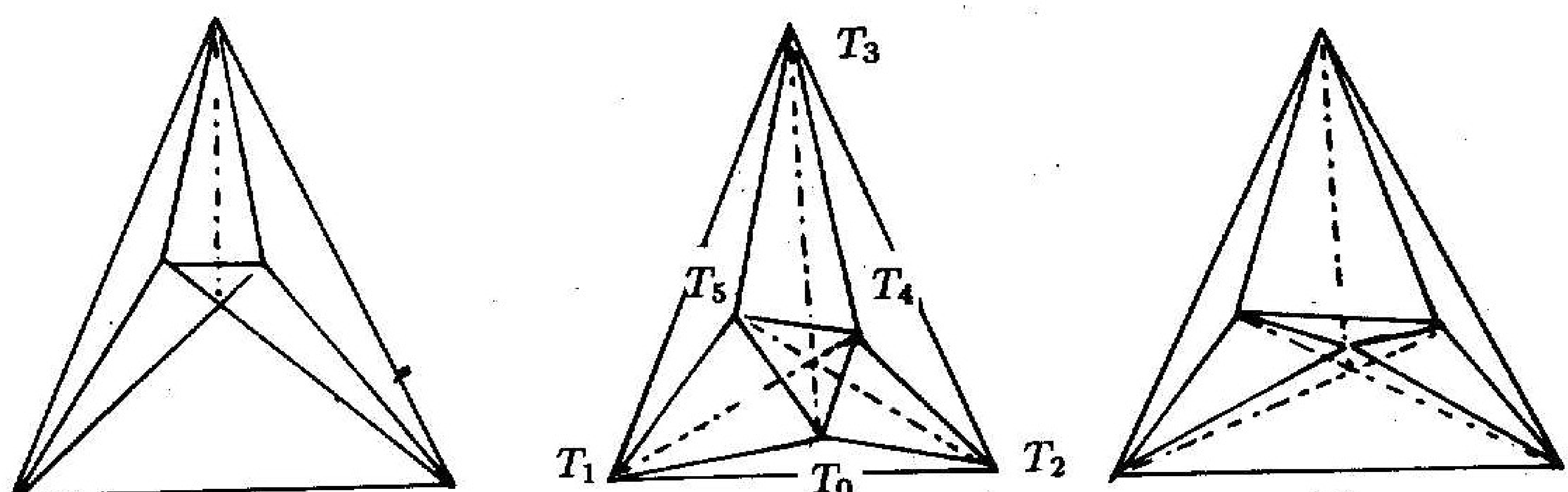


Fig. 2



(a) (b) (c)

Fig. 3

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