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A NEW APPROACH TO STABILITY OF FINITE ELEMENTS UNDER DIVERGENCE CONSTRAINTS*1)

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Abstract

A new stability inequality for velocity-pressure F.E. approximations of Stokes (or Navier-Stokes) problems is presented in this paper. It is proved that the inequality holds if the so-called patch test of rank non-deficiency is passed. As a use of the new criterion, the stability of various new and old combinations of velocity interpolations with pressure interpolations is discussed.

§1. Introduction

For finite element analysis of incompressible flow of viscous fluids, it is important that the stability inequality

$$\sup_{u\in U_h(\Omega)} \frac{\int_\Omega \operatorname{div} u\cdot p d\Omega}{\|u\|_{1,\Omega}} \geq C\|p\|_{L^2_p(\Omega)}, \quad p\in V_h(\Omega) \tag{1.1}$$

hold for F.E. velocity-pressure space $U_h(\Omega)\times V_h(\Omega)\subset (H_0^1(\Omega))^n\times L_0^2(\Omega)$.

Up to now, some efforts have been made for the construction of velocity-pressure finite element spaces and their stability analysis (see [3], [6–9], [11], [15]). In particular, the following macroelement condition was presented in [7, 9]:

H) (1.1) holds for a regular partition J_h under the condition that all of the macroelements M, i.e. the union of one or more neighboring elements, forms a new subdivision $\mathcal{M}_M$ of the domain Ω, and for each $h>0$ and $\Phi^h\in V_h(\Omega), \exists\ u_M^h\in U_h(\Omega)$ with $u_M^h|_{\Omega\backslash M}=0$ such that

$$\int_M \hat{\Phi}\operatorname{div} u_M^h d\Omega \geq \|\hat{\Phi}_M\|^2_{L^2(\Omega)}, \quad |u_M^h|_{1,M}\leq C\|\hat{\Phi}_M\|_{L^2(M)}$$

and

$$\sup_{v\in U_h(\Omega)} \frac{\int_\Omega \operatorname{div}\ v\bar{\Phi}}{\|v\|_{1,\Omega}} \geq C\|\bar{\Phi}\|_{L^2(\Omega)}$$

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where $\bar{\Phi}_M = \frac{1}{|M|}\int_M \Phi^h d\Omega, \forall M \in \mathcal{M}_M, \hat{\phi}_M = (\Phi^h - \bar{\Phi})|_M$, the constant C is independent of h, Φ^h and u^h_M.

This condition is local. Stenberg [9] pointed out that it can be used to determine the stability of various combinations. It is, however, not "primitive". It seems to have no more adaptation than the condition of patch type used widely in finite elements owing to the requirement that the partition be composed of macroelements.

As an improvement of the macroelement condition H), a new stability condition is presented in this paper, which is also local and has the same feature of rank non-deficiency condition used in finite element analysis of solid mechanics, and can reduce the judgment of the stability condition (1.1) on macroelements to the determination on element patches; therefore there is no restriction on the partition.

The main result in this paper is the following:

H)′ If for each $p \in V_h(\Omega)$ and each possible element patch M, $(\operatorname{div} v, p)_{(M)} = 0$ $\forall v \in U_h(M) \subset (H_0^1(M))^n$ implies $p|_M =$ constant (i.e. rank non-deficiency), then there exists a constant C independent of the mesh h of finite element such that

$$\sup_{u \in U_h(\Omega)} \frac{\int_\Omega \operatorname{div} u \cdot p d\Omega}{\|u\|_{1,h,\Omega}} \geq C\|p\|_{0,h,\Omega}, \quad p \in V_h \tag{1.2}$$

where norms $\|\cdot\|_{l,h,\Omega}$ and $\|\cdot\|_{0,h,\Omega}$, which will be defined in Section 2, are different from that in (1.1). Though (1.2) is not identical with (1.1), it can be proved that the inequality can play the same role as (1.1). In many cases, we can easily give the proof for (1.2), but not for (1.1).

On the basis of the new criterion, a new stable combination of piecewise linear velocity and piecewise constant pressure is constructed in Section 4. The same conclusions are extended to the three-dimensional case, and the stability of various combinations of velocity-pressure discussed in [6, 9, 15] will be checked one by one in a simple and unified manner by virtue of the new stability condition.

§2. New Stability Inequality

2.1. New stability inequality

Let Ω be a convex polygonal domain with the boundary Γ in $R^n (n = 2,3)$. The stationary Stokes equation is to find $u = (u_1, u_2, \cdots, u_n)$ and p such that

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma \end{cases} \tag{2.1.1}$$

where u is the velocity vector, p is the pressure, and f is the body force. The above problem is equivalent to the following variational problem:

Find $(u,p) \in (H_0^1(\Omega))^n \times L_0^2(\Omega)$ such that

$$\nu(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f, v), \quad \forall v \in (H_0^1(\Omega))^n, \tag{2.1.2}$$

$$(\operatorname{div} u, q) = 0, \quad \forall q \in L_0^2(\Omega). \tag{2.1.3}$$

Let $U_h(\Omega) \subset (H_0^1(\Omega))^n, V_h(\Omega) \subset L_0^2(\Omega) \cap H_p^1(\Omega)$. The following discrete problem is obtained:

Find $(u_h, p_h) \in U_h(\Omega) \times V_h(\Omega)$ such that

$$v(\nabla u_h, \nabla v) - (\operatorname{div} v, p_h) = (f, v), \quad \forall v \in U_h(\Omega), \tag{2.1.4}$$

$$(\operatorname{div} u_h, q) = 0, \quad \forall q \in V_h(\Omega) \tag{2.1.5}$$

where $H_p^1(\Omega) = \{q \in L_0^2(\Omega) : q|_{\Omega_e^*} \in H^1(\Omega_e^*), U\Omega_e^* = \Omega\}, \{\Omega_e^*\}$ is a subdivision of Ω.

Now let $\mathcal{T}_h = \{\Omega_i\}$ be another triangulation of Ω, where intersubdomain boundaries $\partial\Omega_i$ and $\partial\Omega_e^*$ do not generally overlap. $U \equiv (H_0^1(\Omega))^n$, $V \equiv L_0^2(\Omega) \cap H_0^1(\Omega)$. Then the following norm and semi-norm can be defined in U and V:

$$|p|^2_{V(\Omega_i)} \triangleq |p|^2_{0,h,\Omega_i} = \sum_e h_i^2 \|\nabla p\|^2_{0,\Omega_i\cap\Omega_e^*}, + h_i \oint_{\partial,(\Omega_i\cap\Omega_e^*)\backslash\partial\Omega_t} |\, p_+ - p_-|^2 ds, \quad \forall p \in V,$$

$$\|q\|^2_{0,h,\Omega} \triangleq \sum_t |q|^2_{0,h,\Omega_t}, \quad \forall q \in V,$$

$$\|v\|^2_{1,h,\Omega_t} \triangleq \sum_e h_i^{-2} \|v\|^2_{o,\Omega_t\cap\Omega_e^*} + |v|^2_{1,\Omega_t}, \quad \forall v \in U,$$

$$\|v\|^2_{1,h,\Omega} \triangleq \sum_t \|v\|^2_{1,h,\Omega_t}, \quad \forall v \in U.$$

We assert that if there is a constant C independent of h such that

$$\sup_{v\in U_h(\Omega)} \frac{(\operatorname{div} v, q))_{(\Omega)}}{\|v\|_{1,h,\Omega}} \geq C\|q\|_{0,h,\Omega}, \quad \forall q \in V_h(\Omega),$$

then the theory of existence and uniqueness and convergence of (2.1.4)–(2.1.5) can be established.

Lemma 2.1. *There is a constant C independent of h such that*

$$(\operatorname{div} v, p) \leq C\|v\|_{1,h,\Omega} \cdot \|p\|_{0,h,\Omega}, \quad \forall v \in U_h(\Omega), p \in V_h(\Omega).$$

Proof. See Lemma 3.2.

Remark 1. Throughout the paper, C or C_j denotes a positive constant, possibly different at different occurences, which is independent of the mesh parameter h, but may depend on Ω, ν, and other parameters introduced in the text.

Remark 2. All notations, if not otherwise specified, will be used in the usual meanings.

2.2. Existence, uniqueness and convergence of finite element solution

As the norm $\|\cdot\|_{1,h,\Omega}$ may not satisfy $(\nabla u, \nabla u) \geq c\|u\|^2_{1,h,\Omega}$, we can not straightforwardly use the framework of [2, 3] to obtain the error estimate, but we still have the following results.

Theorem 2.1. *If* $\sup\limits_{v\in U_h(\Omega)} \dfrac{(\operatorname{div} v, q)_{(\Omega)}}{\|v\|_{1,h,\Omega}} \geq C\|q\|_{0,h,\Omega}$, $\forall q \in V_h(\Omega)$ *holds, then problem (2.1.4)–(2.1.5) has a unique solution* $(u_h, p_h) \in U_h(\Omega) \times V_h(\Omega)$.

Furthermore, if $(u_f, p_f) \in (H^1_0(\Omega))^n \times H^1_p(\Omega)$ is the solution of problem (2.1.2)–(2.1.3), then the following abstract error estimates hold:

$$\|u_f - u_h\|_{1,\Omega} \leq C\Big[\inf_{v\in U_h(\Omega)} \|u_f - v\|_{1,h,\Omega} + \inf_{q\in V_h(\Omega)} \|p_f - q\|_{0,\Omega}\Big],$$

$$\|p_f - p_h\|_{0,h,\Omega} \leq C\Big[\inf_{v\in U_h(\Omega)} \|u_f - v\|_{1,h,\Omega} + \inf_{q\in V_h(\Omega)} (\|p_f - q\|_{0,h,\Omega} + \|p_f - q\|_{0,\Omega})\Big].$$

Proof. (I) The existence and uniqueness of the solution. We need only to prove that, for $f = 0$, the solution of (2.1.4)-(2.1.5) is only zero. And the proof is very simple.

(II) The estimates of $\|u_f - u_h\|_{1,\Omega}$ and $\|p_f - p_h\|_{0,h,\Omega}$. We refer to the proof of Theorem 1.1 in [6] (Chap. II, P59), and the estimates of $\|u_f - u_h\|_{1,\Omega}$ and $\|p_j - p_h\|_{0,h,\Omega}$ can be derived. See also [16].

Theorem 2.2. *Assume*

$$\sup_{v\in U_h(\Omega)} \frac{(\operatorname{div} v, q)_{(\Omega)}}{\|v\|_{1,h,\Omega}} \geq C\|q\|_{0,h,\Omega}, \quad \forall q \in V_h(\Omega).$$

Then the following error estimate holds:

$$\|p_f - p_h\|_{0,\Omega} \leq C\Big[\inf_{v\in U_h(\Omega)} \|u_f - v\|_{1,h,\Omega} + \inf_{q\in V_h(\Omega)} \|p_f - q\|_{0,\Omega}\Big]$$

where (u_f, p_f) *is the solution of* (2.1.2).

The proof of this theorem is very similar to Theorem 2.2 in [16]; See also Theorem 6.1 in [13].

§3. Stability Condition

In this section, let $U_h(\Omega)$ and $V_h(\Omega)$ be finite element spaces such that $U_h(\Omega) \subset U$, $V_h(\Omega) \subset V$. For $U_h(\Omega)$, $V_h(\Omega)$, we will prove the new stability inequality under some conditions. That is to say, there exists a constant C independent of h such that

$$\sup_{v\in U_h(\Omega)} \frac{(\operatorname{div} v, q)_{(\Omega)}}{\|v\|_{1,h,\Omega}} \geq C\|p\|_{0,h,\Omega}, \quad \forall p \in V_h(\Omega).$$

In order to prove this inequality, we first give some lemmas.

Definition 3.1. *A pair of element subspaces* (U_h, V_h) *is said to pass the element test of rank non-deficiency if for a regular triangulation* J_h, *for each* $P \in V_h(\Omega)$ *and* $\Omega_i \in J_h$,

$$(\operatorname{div} v, p)_{(\Omega_i)} = 0 \quad \forall v \in U_h(\Omega_i) \quad \text{implies} \quad |p|_{0,h,\Omega_i} = 0 \ \text{(i.e. } p = \text{constant in } \Omega_i)$$

where $U_h(\Omega_i) = U_h(\Omega)|_{\Omega_i} \cap (H_0^1(\Omega_i))^n$.

Remark 3. We may extend $U_h(\Omega_i)$ as zeros in $\Omega\backslash\Omega_i$, so we will consider $U_h(\Omega_i) \subset U_h(\Omega)$.

Lemma 3.1. *There is a constant C independent of h such that for each $\Omega_i \in \mathcal{J}_h$ and $v \in H^1(\Omega_i)$,*

$$\oint_{\partial\Omega_i} v^2 ds \le C h_i^{-1} \|v\|_{0,\Omega_i}(h_i|v|_{1,\Omega_i} + \|v\|_{0,\Omega_i}).$$

For the proof, see [13].

Lemma 3.2. *For any $v \in U_h(\Omega)$ and $p \in V_h(\Omega)$, we have*

$$(\operatorname{div} v, p)_{(\Omega_i)} - \int_{\partial\Omega_i} pv \cdot n ds \le C\|v\|_{1,h,\Omega_i}|p|_{0,h,\Omega_i} \quad \forall \Omega_i \in \mathcal{T}_h,$$

$$(\operatorname{div} v, p)_{(\Omega)} \le C\|v\|_{1,h,\Omega}\|p\|_{0,h,\Omega}.$$

Proof. $\forall v \in U_h(\Omega), p \in V_h(\Omega)$, we have

$$\begin{aligned}(\operatorname{div} v, p)_{(\Omega_i)} - \int_{\partial\Omega_i} pv \cdot n ds &= \sum_e \oint_{\partial(\Omega_i\cap\Omega_e^*)\backslash\partial\Omega_i} (p_+ - P_-) v \cdot n ds - (v, \nabla p)_{(\Omega_i\cap\Omega_e^*)} \\ &\le C\Big(\sum_e h_i^2\|\nabla P\|^2_{0,\Omega_i\cap\Omega_e^*} + h_i \oint_{\partial(\Omega_i\cap\Omega_e^*)\backslash\partial\Omega_i} (P_+ - P_-)^2 ds)^{1/2}(h_i^{-2}\|c\|^2_{0,\Omega_i} \\ &+ h_i^{-1}\oint_{\partial(\Omega_i\cap\Omega_e^*)\backslash\Omega_i)} (v\cdot n)^2 ds\Big)^{1/2} \\ &\le C\Big(\sum_e h_i^2\|\nabla P\|^2_{0,\Omega_i\cap\Omega_e^*} + h_i \oint_{\partial(\Omega_i\cap\Omega_e^*)\backslash\partial\Omega_i} (P_+ - P_-)^2)^{1/2}(h_i^{-2}\|v\|^2_{0,\Omega_i} \\ &+ |v|^2_{1,\Omega_i})\Big)^{1/2} = C\|v\|_{1,h,\Omega_i}|p|_{0,h,\Omega_i} \qquad \text{(by Lemma 3.1)}.\end{aligned}$$

As $v \in U_h(\Omega) \subset (H_0^1(\Omega))^n, P|_{\partial\Omega_i\backslash\partial\Omega}$ is continuous. Thus $\sum_i \oint_{\partial\Omega_i} pv \cdot n ds = 0$. Therefore,

$$\begin{aligned}(\operatorname{div} v, p)_{(\Omega)} &= \sum_i (\operatorname{div} v, p)_{(\Omega_i)} = \sum_i (\operatorname{div} v, p)_{(\Omega_i)} - \oint_{\partial\Omega_i} pv \cdot n ds \\ &\le C\Big(\sum_i \|v\|^2_{1,h,\Omega_i}\Big)^{1/2}\Big(\sum_i |p|^2_{0,h,\Omega_i}\Big)^{1/2} \\ &= C\|v\|_{1,h,\Omega}\cdot\|P\|_{0,h,\Omega}.\end{aligned}$$

Lemma 3.3. *For every $\Omega_i \in \mathcal{T}_h$, if $p \in V_h(\Omega)$ with $(\operatorname{div} v, p)_{(\Omega_i)} = 0$ $\forall v \in U_h(\Omega_i)$ implies $|p|_{0,h,\Omega_i} = 0$ (i.e. $p =$ const. in Ω_i), then there exists a constant $\beta(\Omega_i) > 0$ such that*

$$\sup_{v\in U_h(\Omega_i)} \frac{(\operatorname{div} v, p)_{(\Omega_i)}}{\|v\|_{1,h,\Omega_i}} \ge \beta(\Omega_i)\|p\|_{0,h,\Omega_i}, \quad \forall p \in V_h(\Omega).$$

Proof. If the above is not true, we could choose a sequence $\{p_m\}_{m=1}^{\infty} \in V_h(\Omega)$ such that

$$\|p_m\|_{0,h,\Omega_i} = 1 \quad \text{and} \quad \lim_{m\to\infty} (\operatorname{div} v, p_m)_{(\Omega_i)} = 0, \quad \forall v \in U_h(\Omega_i).$$

Since a bounded set in the finite dimensional space $V_h(\Omega)|_{\Omega_i}$ is compact, there exist a convergent subsequence $\{p_m\}$ and $p_0 \in V_h(\Omega)$ such that

$$\lim_{m\to\infty} |p_m - p_0|_{0,h,\Omega_i} = 0 \quad \text{and} \quad |p_0|_{0,h,\Omega_i} = 1.$$

Since $U_h(\Omega_i) \subset (H_0^1(\Omega))^n$, by Lemma 3.2, we can easily obtain

$$(\operatorname{div} v, p_m - p_0)_{(\Omega_i)} \le C\|v\|_{o,h,\Omega_i} |p_m - p_0|_{0,h,\Omega_i}.$$

Thus

$$(\operatorname{div} v, p_0)_{(\Omega_i)} = \lim_{m\to\infty} (\operatorname{div} v, p_m)_{(\Omega_i)} = 0 \quad \forall v \in U_h(\Omega_i)$$

This is a contradiction to the assumption of the lemma.

Lemma 3.4. *Define the set function:*

$$W(\Omega_i) = \inf_{\substack{p\in V_h(\Omega) \\ |p|_{0,h,\Omega_i}\neq 0}} \sup_{\substack{v\in U_h(\Omega) \\ \|v\|_{1,h,\Omega_i}\neq 0}} \frac{(\operatorname{div} v, p)_{(\Omega_i)}}{\|v\|_{1,h,\Omega_i} |p|_{0,h,\Omega_i}}, \quad \forall \Omega_i \in \mathcal{T}_h.$$

If $W(\Omega_i)$ is always positive for every $\Omega_i \in \mathcal{T}_h$, then there exists a constant β_0 independent of h such that $W(\Omega_i) \ge \beta_0$ $\forall \Omega_i \in \mathcal{T}_h$.

Proof. For each $\Omega_i \in \mathcal{T}_h$, let $h_i = \operatorname{diam}(\Omega_i)$, $h = \max_i(h_i)$. Since $\mathcal{T}_h = \{\Omega_i\}$ is a regular triangulation, there exists a family of affine invertible mappings $\{F_i(y)\}$ such that

(i) $F_i(y) = B_{F_i} y + b_{F_i}$, (ii) $x = F_i(y) : K \xrightarrow{\text{onto}} \Omega_i$, $y = F_i^{-1}(x) : \Omega_i \xrightarrow{\text{onto}} K$, where B_{F_i} is an $n \times n$ matrix, b_{F_i} is an $n \times 1$ matrix, and K is the usual unit reference triangle (or triangular pyramid). For convenience, we introduce some notations.

$$J_{F_i} = \det\left[\frac{\partial \chi_l}{\partial y_m}\right] = \det(B_{F_i}) \quad \text{i.e. the determinant of Jacobi matrix.}$$

$$v = (v_1, v_2) \quad \text{or} \quad v = (v_1, v_2, v_3); n = (n_1, n_2) \quad \text{or} \quad n = (n_1, n_2, n_3);$$

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^T \text{ the gradient operator;} \quad \bar{v} = v(F_i(y)), \bar{p} = p(F_i(y)).$$

Then $\forall v \in U_h(\Omega_i), p \in V_h(\Omega)$, and we have

$$\begin{aligned} E(v,p)_{(\Omega_i)} &\equiv (\operatorname{div} v, p)_{(\Omega_i)} = \sum_e (\operatorname{div} v, p)_{(\Omega_i \cap \Omega_e^*)} \\ &= \sum_e \int_{\partial(\Omega_i\cap\Omega_h^n)\setminus\partial\Omega_i} (p_+ - p_-) v \cdot n ds_n - \int_{\Omega_i\cap\Omega_e^*} v \cdot \nabla_x p dx. \end{aligned}$$

By $\left[\frac{\partial}{\partial x_l}\right] = \left[\sum_m \frac{\partial}{\partial y_m}\frac{\partial y_m}{\partial x_l}\right] = \left[\frac{\partial x_l}{\partial y_m}\right]^{-1}\frac{\partial}{\partial y_m} = (B_{F_i}^{-1})\left[\frac{\partial}{\partial y_m}\right]$ we can easily obtain: $x \in \Omega_i, (\nabla_x p)(x) = (B_{F_i}^{-1})\nabla_y \bar{p}\ \forall y \in K$. The change of variables by $x = F_i(y)$ yields

$$\begin{aligned}
E(v,p)_{(\Omega_i)} &= \sum_e \int_{\partial(\Omega_i\cap\Omega_e^*)\backslash\partial\Omega_i}(p_+ - p_-)\bar{v}\cdot n ds s_x - \int_{\Omega_i\cap\Omega_e^*} v\nabla_x p ds \\
&= \sum_e \text{ sign } (J_{F_i})\oint_{\Gamma_e}(\bar{p}_+ - \bar{p}_-)\bar{v}\cdot(\bar{n}J_{F_i}(B_{F_i}^{-1})^T)ds_y - \int_{K_e}\bar{v}(B_{F_i}^{-1})\nabla_y\bar{p}|J_{F_i}|dy \\
&= \frac{|J_{F_i}|}{h}\sum_e \oint_{\Gamma_e}(\bar{p}_+ - \bar{p}_-)\bar{v}\cdot(\bar{n}(hB_{F_i}^{-1})^T)ds_u - \int_{K_e}\bar{v}((hB_{F_i}^{-1})\nabla_y\bar{p})dy
\end{aligned}$$

where $\Gamma_e = F_i^{-1}(\partial(\Omega_i\cap\Omega_e^*)\backslash\partial\Omega_i), K_e = F_i^{-1}(\Omega_i\cap\Omega_e^*)$, and n is the outward normal to Γ_e. Let

$$\bar{E}(\bar{V},\bar{p})_{(K)} = \sum_e \int_{\Gamma_e}(\bar{p}_+ - \bar{p}_-)\bar{v}\cdot(n(hB_{F_i}^{-1})^T)ds_y - \int_{K_e}\bar{v}\cdot((hB_{F_i}^{-1})\nabla_y\bar{p})dy. \tag{3.2}$$

Note that there exist constants $\tilde{C}_1$ and $\tilde{C}_2$ independent of h such that

$$\tilde{C}_1 \le h^n/|J_{F_i}|^2 \le \tilde{C}_2 \quad \forall i = 1,2,\cdots. \tag{3.3}$$

By (3.1)–(3.3), we get

$$\tilde{C}_1 h^{n-1}\bar{E}(\bar{v},\bar{p})_{(K)} \le E(v,p)_{(\Omega_i)} \le \tilde{C}_2 h^{n-1}\bar{E}(\bar{v},\bar{p})_{(K)}. \tag{3.4}$$

Let us define

$$\begin{aligned}
&\|\bar{v}\|^2_{\bar{U}(K)} = \|\bar{v}\|^2_{0,K} + \|h(B_{F_i}^{-1}\nabla_y\bar{v})\|^2_{0,K}, \\
&\|\bar{p}\|^2_{\bar{V}(K)} = \sum_e \|hB_{F_i}^{-1}\nabla y\bar{p}\|^2_{K_e} + \|\bar{p}_+ - p_-\|^2_{0,\Gamma_e}.
\end{aligned} \tag{3.5}$$

Using the change of variables and (3.3), we obtain that there exist constants C_1, C_2; $\tilde{C}_1, \tilde{C}_2$ independent of h such that

$$\begin{aligned}
&(C_1 h^{n/2-1}\|\bar{v}\|_{\bar{U}(K)} \le \|v\|_{1,h,\Omega_i} \le C_2 h^{n/2-1}\|\bar{v}\|_{\bar{U}(K)}), \\
&(\tilde{C}_1 h^{n/2}\|\bar{p}\|_{\bar{V}(K)} \le \|p\|_{0,h,\Omega_i} \le C_2 h^{n/2}\|\bar{p}\|_{\bar{V}(K)}).
\end{aligned} \tag{3.6}$$

By (3.4), (3.6), we get

$$\beta_1 W(\Omega_i) \le \bar{W}(K) \le \beta_2 W(\Omega_i), \tag{3.7}$$

where β_1 and β_2 are constants independent of h, and $\bar{W}(K)$ is as follows:

$$\bar{W}(K) = \inf_{\substack{\bar{p}\in\bar{V}(K)\\ |\bar{p}|_{\bar{V}(K)}\neq 0}} \sup_{\substack{\bar{v}\in\bar{U}(K)\\ \|\bar{v}\|_{\bar{U}(K)}\neq 0}} \frac{\bar{E}(\bar{v},\bar{p})_{(K)}}{\|\bar{v}\|_{\bar{U}(K)}|\bar{p}|_{\bar{V}(K)}},$$

$$\bar{U}(K) = U_h(\Omega_i)\circ F_i(y), \quad \bar{V}(K) = V_h(\Omega)|_{\Omega_i}\circ F_i(y).$$

If $\{x_k^{(i)}\}$ and $x_{k_0}^{(i)}$ denote the vertices of $\Omega_i (i = 1, 2, \cdots,)$, let $\tilde{x}_k^{(i)} = (x_k^{(i)} - x_{k_0}^{(i)})/h$. It is easy to verify that B_{F_i} is a function of $\{x_k^{(i)}\}_k$ and $x_{k_0}^i$ such that

$$h \cdot B_{F_i}^{-1}(x_k^{(i)}) = B_{F_i}^{-1}(\tilde{x}_k^{(i)}), \quad i = 1, 2, \cdots, \tag{3.8}$$

which implies that the set function $\bar{W}(K)$ can be regarded as an ordinary function of variables $\{\tilde{x}_k^{(i)}\}$, defined over a bounded subset S_F in a finite dimensional Euclidean space, where

$$S_F = \{\{\tilde{x}_k^{(i)}\}_k : |\tilde{x}_k^{(i)}| \le 1, \quad |J_{F_i}| \le C\}.$$

Thus

$$\min_{\forall \Omega_i \in \mathcal{T}_h} W(\Omega_i) \ge \frac{1}{\beta_2} \min_{\{x_h^{(i)}\}_k \in S_F} \bar{W}(K). \tag{3.9}$$

So far, we need only to prove that $\bar{W}(K)$ attains its positive infimum on S_F. In fact, by (3.7), there exists $\bar{W}(K) > 0 \ \forall \tilde{x}_k^{(i)} \in S_F$; on the other hand, it is clear that $\bar{W}(K)$ is a continuous function of variables $\{\tilde{x}_k^{(i)}\}_k$. Every continuous function defined on a bounded closed set always attains its infimum on this set. Therefore there is a constant $\beta_0 > 0$ independent of h such that

$$\min_{\forall \Omega_i \in \mathcal{T}_h} W(\Omega_i) \ge \beta_0.$$

Theorem 3.1. *For $(U_h(\Omega), V_h(\Omega))$, if the element test of rank non-deficiency is passed, then there exists a constant C independent of h such that*

$$\sup_{v \in U_h(\Omega)} \frac{(\operatorname{div} v, p)_{(\Omega)}}{\|v\|_{1,h,\Omega}} \ge C \|p\|_{0,h,\Omega}, \quad \forall p \in V_h(\Omega).$$

Proof. By the definition of element test of rank non-deficiency, for every $\Omega_i \in \mathcal{T}_h$ and $p \in V_h(\Omega)$, if $(\operatorname{div} v, p)_{(\Omega_i)} = 0 \ \ \forall v \in U_h(\Omega_i)$, then $p =$ constant in Ω_i. Thus by Lemma 3.3, there exist constants $\{\beta(\Omega_i)\}$ such that for $\Omega_i \in \mathcal{T}_h$,

$$\sup_{v \in U_h(\Omega_i)} \frac{(\operatorname{div} v, p)_{(\Omega_i)}}{\|v\|_{1,h,\Omega_i}} \ge \beta(\Omega_i) |p|_{0,h,\Omega_i} \quad \forall p \in V_h(\Omega), \tag{3.10}$$

which implies that $W(\Omega_i)$ is always positive for every $\Omega_i \in \mathcal{J}_h$. In virtue of Lemma 3.4, there is a positive constant β_0 independent of h and Ω_i such that

$$\sup_{v \in U_h(\Omega_i)} \frac{(\operatorname{div} v, p)_{(\Omega_i)}}{\|v\|_{1,h,\Omega_i}} \ge \beta_0 |p|_{0,h,\Omega_i}, \quad \forall p \in V_h(\Omega), \quad \forall \Omega_i \in \mathcal{T}_h. \tag{3.11}$$

In virtue of [8], by (3.11), there exist two constants C_1 and C_2 independent of h such that for every $p \in V_h(\Omega), \forall \Omega_i \in \mathcal{T}_h \exists \ v_{\Omega_i} \in U_h(\Omega_i)$ such that

$$(\operatorname{div} v_{\Omega_i}, p)_{(\Omega_i)} \ge C_1 |p|_{0,h,\Omega_i}^2, \quad \|v_{\Omega_i}\|_{1,h,\Omega_i} \le C_2 |p|_{0,h,\Omega_i}. \tag{3.12}$$

Let $v = \sum_i v_{\Omega_i} \in U_h(\Omega)$ (by $U_h(\Omega_i) \subset U_h(\Omega)$). Thus

$$\int_\Omega \operatorname{div} vpdx = \sum_i \int_{\Omega_i} \operatorname{div} v_{\Omega_i} pdx \geq C_1 \sum_i |p|^2_{0,h,\Omega_i} = C_1 \|p\|^2_{0,h,\Omega}. \tag{3.13}$$

By (3.12), we have

$$\|v\|^2_{0,h,\Omega_i} = \|v_{\Omega_i}\|^2_{1,h,\Omega_i} \leq C_2^2 |p|^2_{0,h,\Omega_i},$$

$$\|v\|^2_{1,h,\Omega} = \sum_i \|v\|^2_{1,h,\Omega_i} \leq C_2^2 \sum_i |p|^2_{0,h,\Omega_i} = C_2^2 \|p\|^2_{0,h,\Omega_i},$$

Then

$$\|v\|_{1,h,\Omega} \leq C_2 \|p\|_{0,h,\Omega}. \tag{3.14}$$

By (3.13) and (3.14), we get

$$\int_\Omega \operatorname{div} vpdx \geq C \|v\|_{1,h,\Omega} \|p\|_{0,h,\Omega}.$$

Therefore, we obtain

$$\sup_{v \in U_h(\Omega)} \frac{(\operatorname{div} v, p)_{(\Omega)}}{\|v\|_{1,h,\Omega_i}} \geq C \|p\|_{0,h,\Omega_i}, \quad \forall p \in V_h(\Omega).$$

From the above discussion, it is not difficult to find out that the element test of rank non-deficiency has some limitation, and it is not convenient to use Theorem 3.1 for determining the stability of some combinations of finite element spaces. But it is easy to see from the following discussion that this shortage can be avoided by introducing the patch test of rank non-deficiency. That is to say, if the patch test of rank non-deficiency is passed, the result of Theorem 3.1 is also correct.

Let us first introduce the definition of "patch" and "patch class", and then give the appropriate result.

Definition 3.2. *For a regular partition $\mathcal{T}_h$, a union of all elements attached to the common vertex is said to be a "patch".*

Definition 3.3. *For a regular partition $\mathcal{T}_h$, the set of patches is said to be of the patch class T_h (patch-Ne) if the following conditions are satisfied:*

1) *For any two $\tilde{\Omega}^e, \Omega^r \in \mathcal{T}_h$ (patch-Ne), there exist two sets of elements $\{\Omega_i^e\}$ and $\{\tilde{\Omega}_i^r\}$ such that*

$$\tilde{\Omega}^e = \bigcup_{i=1}^{N_e} \tilde{\Omega}_i^e, \quad \tilde{\Omega}^r = \bigcup_{i=1}^{N_e} \tilde{\Omega}_i^r.$$

2) *There is a reference polygon $K_{Ne} = \bigcup_{i=1}^{Ne} K_{Ne,i}$ independent of h, where $K_{Ne,i}$ ($i = 1, 2, \cdots, Ne$) are triangles independent of h. For each $M \in \mathcal{T}_h$ (patch-Ne), there exists a mapping $F_M : K_{Ne} \xrightarrow{\text{onto}} M$ satisfying the conditions:*

(i) *F_M is continuous and one-to-one.*

(ii) $F_M(K_{Ne}) = M, M_j = F_M(K_{Ne,j}), j = 1, 2, \cdots, Ne$, *are the triangles in* M.

(iii) $F_M|_K = F_{M_j} \cdot F_K^{-1}$ *where* F_{M_j} *and* F_K *are the affine invertible mappings from the usual unit reference triangle (or triangular pyramid) onto* M_j *and* $K_{Ne,j}$. *The shape of a patch is shown in Figure* 3.1.

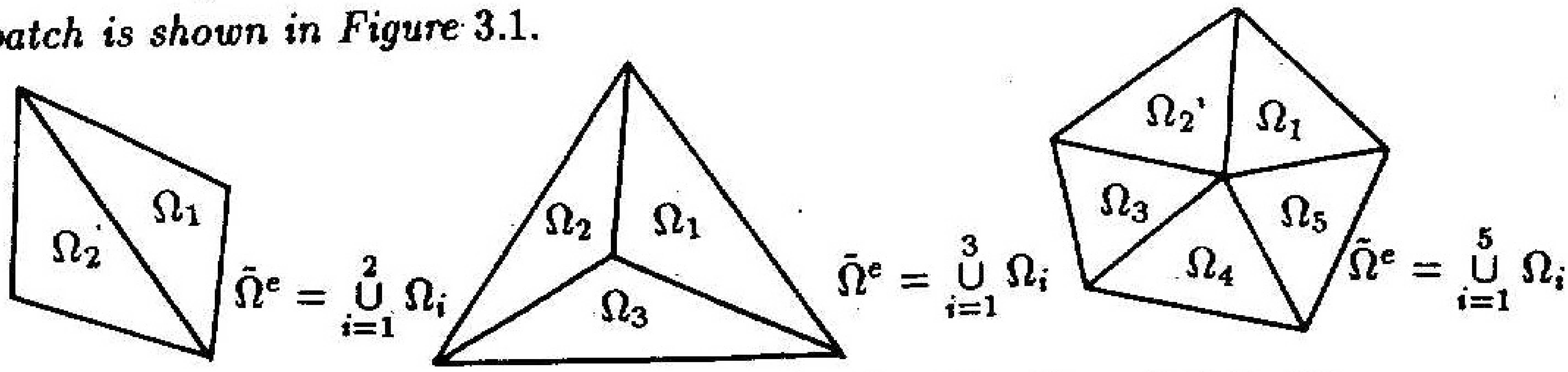


Fig. 3.1. A two-dimensional example of patch. $Ne = 2, 3, 5$, etc.

Definition 3.1′. *A pair* (U_h, V_h) *of element subspaces is said to pass the patch test of rank non-deficiency if for a regular partition* $\mathcal{T}_h, 0 \le h \le 1$, *there exists a set* $\{\tilde{\Omega}^e\}$ *of different element patches such that*

1) $\bigcup_e \tilde{\Omega}^e = \tilde{\Omega}$;

2) *for* $\tilde{\Omega}^e \in \{\tilde{\Omega}^e\}$ *and* $p \in V_h(\Omega)$, *if* $(\operatorname{div} v, p)_{(\tilde{\Omega}^e)} = 0\ \forall v \in U_h(\tilde{\Omega}^e)$ *implies* $p =$ *constant in* $\tilde{\Omega}_e$, *where* $U_h(\tilde{\Omega}^e) = U_h(\Omega)|_{\tilde{\Omega}_e} \bigcap (H_0^1(\tilde{\Omega}^e))^n$.

Theorem 3.1′. *For* (U_h, V_h), *if the patch test of rank non-deficiency is passed, then there exists a constant* C *independent of* h *such that*

$$\sup_{v \in U_h(\Omega)} \frac{(\operatorname{div} v, p)_{(\Omega)}}{\|V\|_{1,h,\Omega}} \ge C\|P\|_{0,h,\Omega}, \quad \forall p \in V_h(\Omega).$$

Remark 4. Theorem 3.1′ can be proved with the same method of proof for Theorem 3.1.

Remark 5. The stability of various combinations of finite element spaces can be determined with Theorem 3.1 or Theorem 3.1′.

Remark 6. For the above results, we have given proofs for $\mathcal{T}_j$ being a regular triangulation. In fact, these results are also correct for quadrilateral subdivision.

§4. The Stability of Linear-Constant and Other Finite Element Spaces

In this section, first a new simple combination of piecewise linear velocity and piecewise constant pressure is constructed. Then, its stability and some other finite element spaces in [6, 9, 15] are discussed with the theory developed in Section 3.

4.1. The linear-constant elements.

Let Ω be a bounded convex polygonal (or polyhedral) domain in $R^n (n = 2, 3)$. $\mathcal{J}_h = \{\Omega_i\}$ is a regular triangulation of Ω, and every element Ω_i is divided into $n + 1$ microelements with equal measure. Let $\{\Omega_{it}\}(t = 1, 2, \cdots, n+1)$ be these microelements.

Then $\Omega_i = \bigcup_{i=1}^{n+1} \Omega_{it}$ (see Fig. 4.1) where $\{\Omega_{it}\}$, $t = 1, 2, \cdots, n+1$, are microelements formed by the connecting lines of barycenter O_i to any two (or three) vertices and one edge (or face) of Ω_i.

Let $\mathcal{T}_h^* = \{\Omega_h^n\}$ be the dual subdivision of $\mathcal{T}_h$ where Ω_h^* denotes a simply connected subdomain of Ω, which consists of two adjacent microelements with one common edge (or face) which belong to distinct elements respectively.

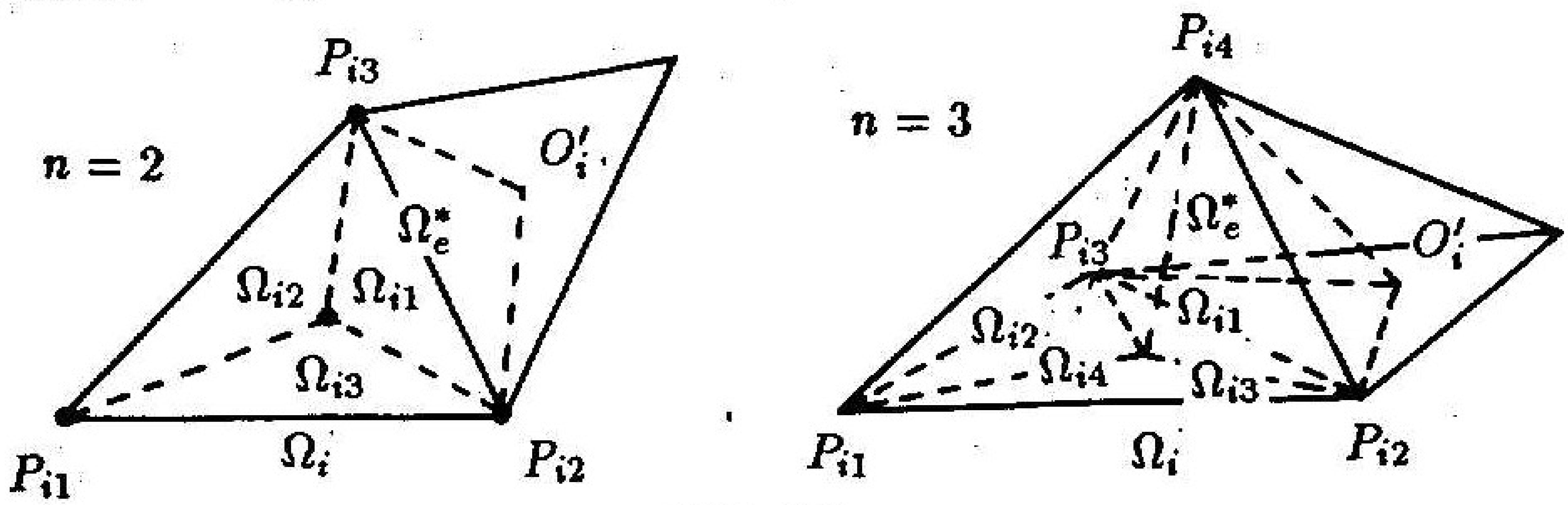


Fig. 4.1.

For example, as shown in Fig. 4, $\Omega_e^* = \{$ quadrilateral formed by $O_iP_{12}O_i'P_{13}$ for $n = 2$ and $\Omega_e^* =$ hexahedron formed by $I_iP_{12}P_{13}P_{14}O_i'\}$ for $n = 3$. Let

$$U_h^{(1)}(\Omega) = \Big\{v \in [H_0^1(\Omega)]^n : v|_{\Omega_{it}} \in (P_1(x))^n, \bigcup_{i=1}^{n} \Omega_{it} = \Omega_i, \Omega = \bigcup_i \Omega_i\Big\},$$

$$V_h^{(0)}(\Omega) = \Big\{q \in L_0^2(\Omega) : q|_{\Omega_e^*} = \text{const.}, \Omega_e^* \in \mathcal{T}_h^*\Big\},$$

where $P_m(x)$ denotes the space of polynomials of degree $\le m$.

For every $\Omega_i \in \mathcal{T}_h$, take the value of velocity on vertices and barycenter of Ω_i as the freedoms. Obviously, $\forall v \in U_h(\Omega), v|_{\Omega_i}$ is uniquely defined by these freedoms.

Let us now check that $\{U_h^{(1)}(\Omega), V_h^{(0)}(\Omega)\}$ passes the element test of rank non-deficiency, i.e. $\forall p \in V_h(\Omega), \forall \Omega_i \in \mathcal{T}_h, \int_{\Omega_i} \operatorname{div} v \cdot p d\Omega = 0 \; \forall v \in U_h(\Omega_i)$ implies $p =$const. in Ω_i. In fact, as $p \in V_h(\Omega) = \{q \in L_0^2(\Omega) : q|_{\Omega_e^*} = \text{const. } \Omega_e^* \in \mathcal{T}_h^*\}$, we can let $P|_{\Omega_{it}} = p_t$, where $p_t (t = 1, 2, \cdots, n+1)$ are constants. Then it suffices to prove $p_1 = p_2 = \cdots p_{n+1}$.

$$\int_{\Omega_i} \operatorname{div} v \cdot p d\Omega = 0, \quad \forall v \in U_h^{(1)}(\Omega_i)$$

implies

$$\int_{\Omega_i} \operatorname{div} v \cdot p \, d\Omega = \sum_{t=1}^{n+1} p_t(\operatorname{div} v)|_{\Omega_{it}} = 0, \quad \forall v \in U_h^{(1)}(\Omega_i)$$

where $U_h^{(1)}(\Omega_i) = U_h^{(1)}(\Omega)|_{\Omega_i} \cap (H_0^1(\Omega_i))^n$, and $(\operatorname{div} v)|_{\Omega_{it}} (t = 1, 2, \cdots, n+1)$ are constants. Choose $v \in U_h^{(1)}(\Omega)$ such that

For $n = 2, v(p_{it}) = 0$ $(t = 1, 2, 3)$, and $v(o_i) = (1, 0)$ or $v(o_i) = (0, 1)$;

For $n = 3, v(p_{it}) = 0$ $(t = 1, 2, 3, 4)$ and $v(o_i) = (1, 0, 0)$ or $v(o_i) = (0, 1, 0)$ or $v(o_i) = (0, 0, 1)$.

The condition $\sum_{t=1}^{n+1} p_t(\operatorname{div} v)|_{\Omega_{it}} = 0$ yields the following equations:

$$\text{For } n=2, \begin{cases} p_1(y^2-y^3)+p_2(y^3-y^1)+p_3(y^1-y^2)=0; \\ p_1(x^2-x^3)+p_2(x^3-x^1)+p_3(x^1-x^2)=0; \end{cases} \tag{4.1}$$

$$\text{For } n=3, \begin{cases} p_1(x^2-x^3)+p_2(x^3-x^1)+p_3(x^4-x^1)+p_4(x^1-x^2)=0, \\ p_1(y^2-y^3)+p_2(y^3-y^1)+p_3(y^4-y^1)+p_4(y^1-y^2)=0, \\ p_1(z^2-z^3)+p_2(z^3-z^1)+p_3(z^4-z^1)+p_4(z^1-z^2)=0, \end{cases} \tag{4.2}$$

where we have written $p_{it} = (x^t, y^t)$ for $n = 2$, and $p_{it} = (x^t, y^t, z^t)$ for $n = 3$. From (4.1) and (4.2), we have

$$p_1 = p_2 = p_3; \quad p_1 = p_2 = p_3 = p_4.$$

Therefore p = const. in Ω_i. By Theorem 3.1, $\{U_h^{(1)}(\Omega), V_h^{(0)}(\Omega)\}$ satisfies the new stability inequality.

4.2. The second-order approximations.

Let $\lambda_j (j = 1, \cdots, n+1)$ be the barycentric coordinates of $\Omega_i \in \mathcal{T}_h, X = \operatorname{span}\{\lambda_i : 1 \le i \le n+1; \lambda_i\lambda_j, 1 \le i \le j \le n+1; \lambda_1\lambda_2 \cdot \lambda_{n+1}\}$ (see Fig. 4.2)

$$\begin{cases} \tilde{U}_h^{(2)}(\Omega) = \{v \in (H_0^1(\Omega))^n : v|_{\Omega_i} \in X^n, \Omega_i \in \mathcal{T}_h)\}, \\ V_h^{(1)}(\Omega) = \{q \in L_0^2(\Omega) \cap C(\Omega) : q|_{\Omega_i} \in P_1(x), \Omega_i \in \mathcal{T}_h\}. \end{cases}$$

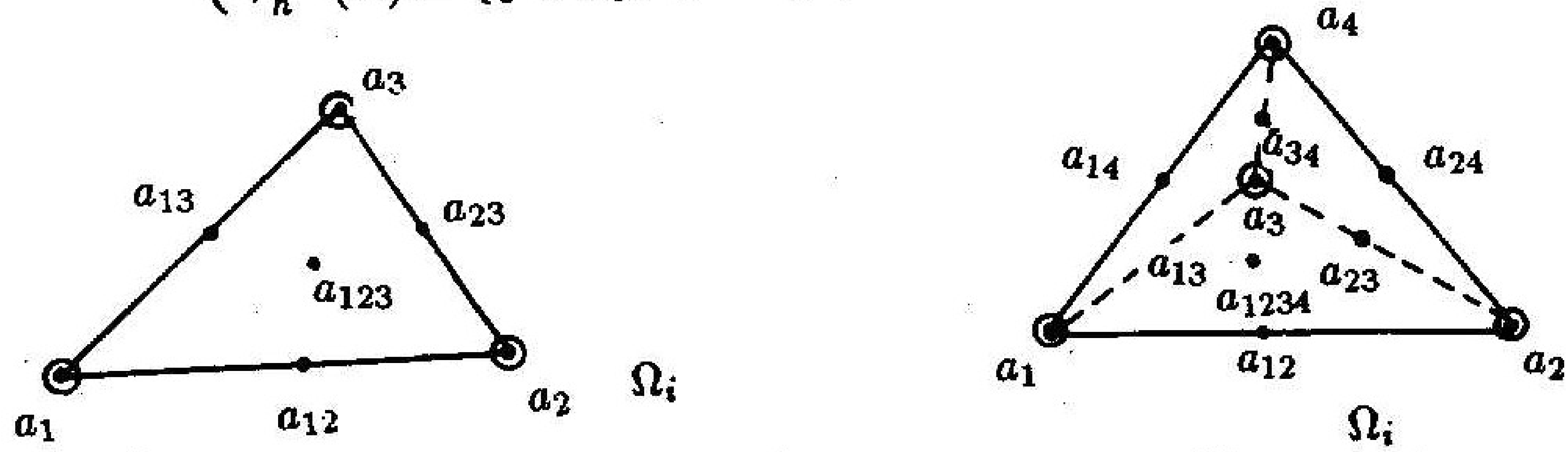


Fig 4.2. Degrees of freedom for $\tilde{U}_h^{(2)}|_{\Omega_i(\cdot)}$ and for $V_h^{(1)}|_{\Omega^i(\circ)}$,

$$a_{ij} = \frac{1}{2}(a_i, a_j),\ 1 \le i < j \le n+1,\ a_{1\cdots n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1} a_i.$$

For $\{\tilde{U}_h^{(2)}(\Omega), V_h^{(1)}(\Omega)\}$, [15] discussed its stability by another but complicated method. With our method, it is easy to know that $\{\tilde{U}_h^{(2)}(\Omega), V_h^{(1)}(\Omega)\}$ is stable.

In fact, $\forall p \in V_h^{(1)}(\Omega), \forall \Omega_i \in \mathcal{T}_h$, by $(\operatorname{div} v, p)_{\Omega_i} = 0\ \forall v \in \tilde{U}_h^{(2)}(\Omega_i)$, we get

$$(v, \nabla p)_{(\Omega_i)} = 0, \quad \forall v \in U_h^{(2)}(\Omega_i).$$

Choosing $v \in \tilde{U}_h^{(2)}(\Omega_i)$ such that

For $n = 2$, $v(a_i) = v(a_{ij}) = 0 (1 \le i \le j \le 3)$ and $v(a_{1,2,3}) = (1,0)$ or $v(a_{1,2,3}) = (0,1)$;

For $N = 3$, $v(a_i) = v(a_{ij}) = 0 (1 \le i \le j \le 4)$ and $v(a_{1,2,3,4}) = (1,0,0)$ or $v(a_{1,2,3,4}) = (0,1,0)$ or $v(a_{1,2,3,4}) = (0,0,1)$.
Then the conditions $(v, \nabla p)_{(\Omega_i)} = 0$ and $\nabla p|_{\Omega_i} = \text{const.}$ give $\nabla p = 0$ in Ω_i. Therefore, by Theorem 3.1 $\{\tilde{U}_h^{(2)}(\Omega), V_h^{(1)}(\Omega)\}$ satisfies the new stability inequality.

Quadratic-linear elements (Fig. 4.3)

$$U_h^{(2)}(\Omega) = \{v \in (H_0^1(\Omega))^2 : v|_{\Omega_i} \in [P_2(x)]^2, \quad \forall \Omega_i \in \mathcal{T}_h\},$$

$$V_h^{(1)}(\Omega) = \{q \in L_0^2(\Omega) \cap C(\Omega) : q|_{\Omega_i} \in P_1(x), \quad \forall \Omega_i \in \mathcal{T}_h\}.$$

$$a_{ij} = \frac{1}{2}(a_i + a_j) \quad 1 \le i < j \le 3$$

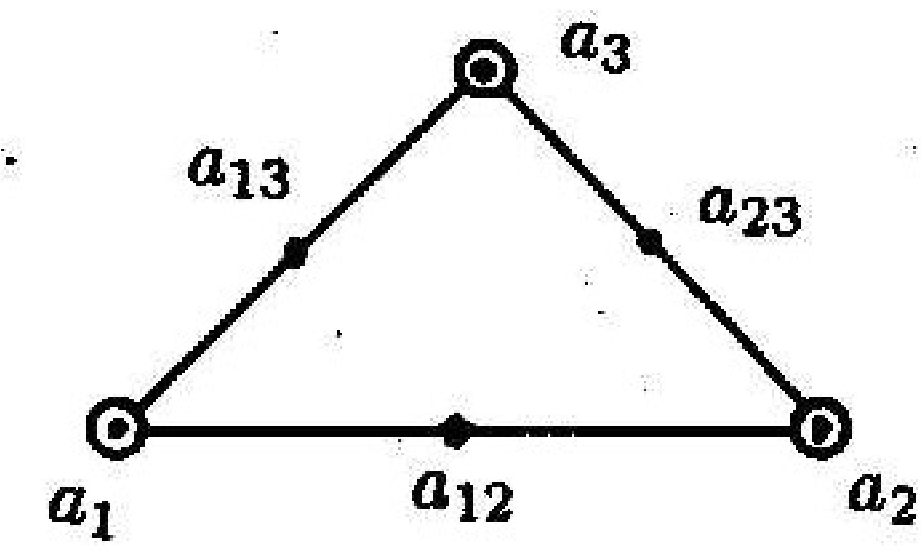
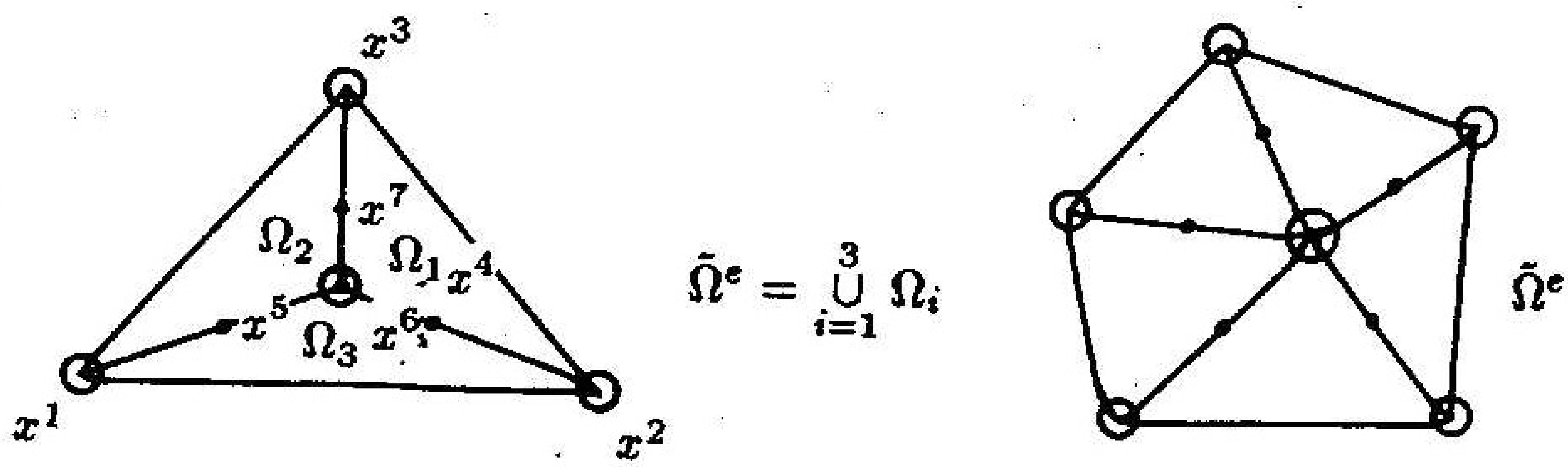


Fig. 4.3. Degrees of freedom for $U_h|_{\Omega_i(\cdot)}$ and for $V_h|_{\Omega_i(\circ)}$.

Let us check if $\{U_h^{(2)}(\Omega), V_h^{(1)}(\Omega)\}$ satisfies rank non-deficiency in triangle and pentagon patches.



Triangle element patch peutagon element patch

Fig. 4.4

In order to avoid unnecessary technical details, we only verify that $\{U_h^{(2)}(\Omega), V_h^{(1)}(\Omega)\}$ satisfies rank non-deficiency in triangle patches. For the pentagon patches, the discussion is similar.

Obviously, $\forall v \in U_h^{(2)}(\tilde{\Omega}^e) = U_h^{(2)}(\Omega)|_{\tilde{\Omega}_e} \cap (H_0^1(\tilde{\Omega}^e))^2$, v is uniquely defined by the values $v_i = v(x_i) (i = 1, \cdots, 4)$. For

$$p \in V_h^{(1)}(\Omega), \text{ by } (\operatorname{div} v, p)_{(\tilde{\Omega}^e)} = 0, \quad \forall v \in U_h^{(2)}(\tilde{\Omega}^e)$$

we have

$$\int_{\tilde{\Omega}^e} \operatorname{div} v \cdot p dx = -\int_{\tilde{\Omega}^e} \cdot \nabla p ds = 0, \quad \forall v \in U_h^{(2)}(\tilde{\Omega}^e).$$

Choose $v \in U_h^{(2)}(\tilde{\Omega}^e)$ such that $v(x^4) = v(x^6) = v(x^7) = 0$ and $v(x^5) = (1,1)$. Then the condition gives

$$\text{meas }(\Omega_3)(\nabla p)|_{\Omega_2} + \text{ meas }(\Omega_2)(\nabla p)|_{\Omega_2} = 0. \tag{4.3}$$

Similarly, we can get

$$\text{meas }(\Omega_2)(\nabla p)|_{\Omega_2} + \text{ meas }(\Omega_i)(\nabla p)|_{\Omega_i} = 0, \tag{4.4}$$

$$\text{meas }(\Omega_3)(\nabla p)|_{\Omega_3} + \text{ meas }(\Omega_1)(\nabla p)|_{\Omega_1} = 0. \tag{4.5}$$

At last, we choose $v \in U_h^{(2)}(\tilde{\Omega}^e)$ such that

$$v(x^5) = v(x^6) = v(x^7) = 0 \quad \text{and} \quad v(x^4) = (1,1).$$

The condition $\int_{\tilde{\Omega}^e} v\nabla p ds = 0$ gives

$$\sum_{i=1}^{3} \text{ meas }(\Omega_i)(\nabla p)|_{\Omega_i} = 0.$$

Combining (4.3)–(4.6), we obtain

$$(\nabla p)|_{\Omega_i} = 0, \quad 1 \le i \le 3.$$

Since $p \in C(\Omega)$, $p|_{\tilde{\Omega}^e} = \text{const}$.

4.3. Revisory linear-linear elements.

Let $\lambda_j = \lambda_j(x)(1 \le j \le n+1)$ be the barycentric coordinates of any point $x \in \Omega$, with respect to the $n+1$ vertices $a_j, 1 \le j \le n+1$, of a triangular element $\Omega_i \in \mathcal{T}_h$. (see Fig. 4.5)

$$X' = \text{Span}\{\lambda_i, 1 \le i \le n+1; \ \ \lambda_1\lambda_1 \cdots \lambda_{n+1}\},$$

$$U_h^{(1)}(\Omega) = \{v \in (H_0^1(\Omega)^n : v|_{\Omega_i} \in (X')^n, \quad \forall \Omega_i \in \mathcal{T}_h\},$$

$$V_h^{(1)}(\Omega) = \{q \in L_0^2(\Omega) : q|_{\Omega_i} \in P_1(x), \quad \forall \Omega_i \in \mathcal{T}_h\}.$$

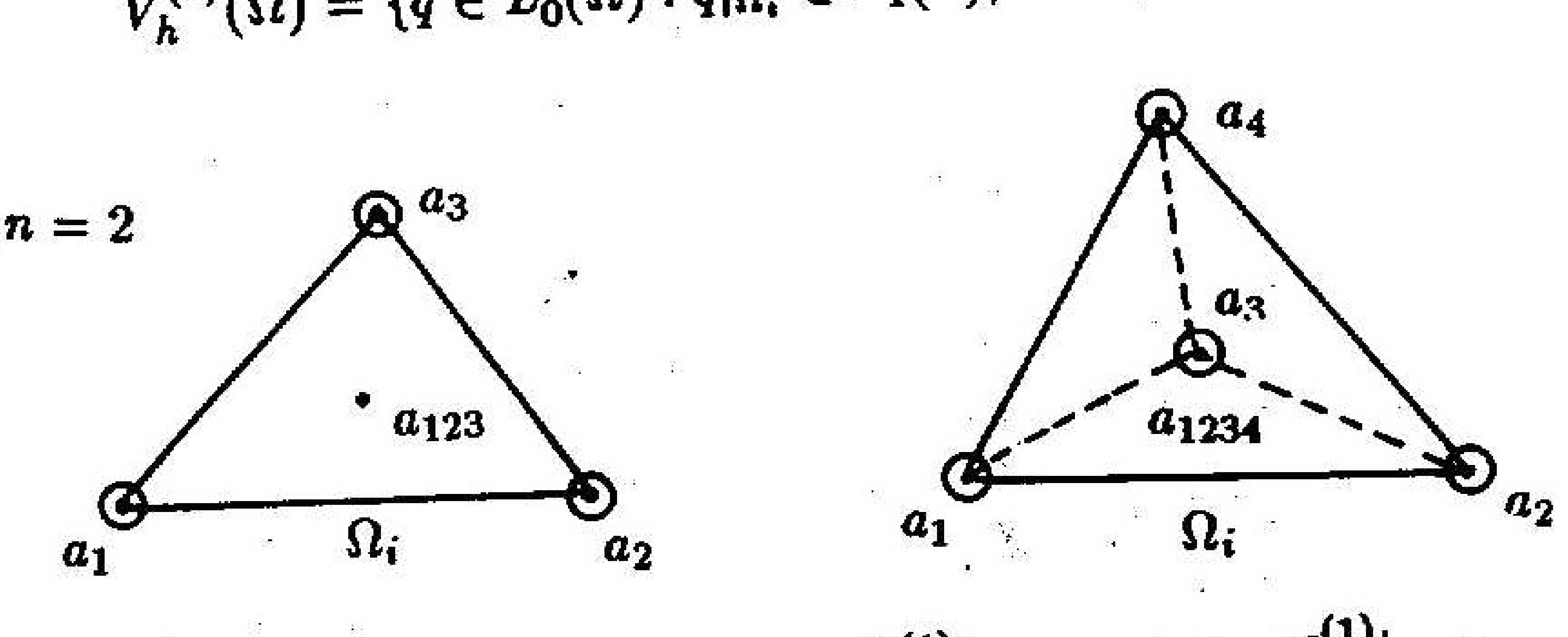


Fig. 4.5. Degrees of freedom for $\tilde{U}_h^{(1)}|_{\Omega_i(\cdot)}$ and for $V_h^{(1)}|_{\Omega_i(\circ)}$

For $\{\tilde{U}_h^{(1)}(\Omega), V_h^{(1)}(\Omega)\}$, [15] also discussed its stability with another method. With an our method, it is easy to know that $\{\tilde{U}_h^{(1)}(\Omega), V_h^{(1)}(\Omega)\}$ is stable. In fact, by

$(\operatorname{div} v, p)_{(\Omega_i)} = \int_{\Omega_i} v \cdot \nabla p dx$, $\forall v \in U_h^{(1)}(\Omega)$, $p \in V_h^{(1)}(\Omega)$ implies that, $\forall p \in V_h^{(1)}(\Omega)$, $\forall \Omega_i \in \mathcal{T}_h$, if $(\operatorname{div}, v, p)_{(\Omega_i)} = 0$ $\forall v \in \tilde{U}_h^{(1)}(\Omega_i)$, we have $\int_{\Omega_i} \lambda_1\lambda_2\lambda_3 \nabla p d\Omega = 0$; then $p =$ constant in Ω_i. Therefore, by Theorem 3.1, $\{\tilde{U}_h^{(1)}(\Omega), V_h^{(1)}(\Omega)\}$ satisfies the new stability inequality established in this paper, i.e. $\{\tilde{U}_h^{(1)}(\Omega), V_h^{(1)}(\Omega)\}$ is stable.

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