Journal of Computational Mathematics, Vol.10, No.3, 1992, 204–208.

DETECTING CODIMENSION TWO BIFURCATIONS WITH A PURE IMAGINARY PAIR AND A SIMPLE ZERO EIGENVALUE*1)

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Abstract

An extended system for codimension two bifurcation with a pure imaginary pair and a simple zero eigenvalue is proposed . Its regularity is proved. An efficient algorithm for solving the extended system is constructed. Finally, some results on the axial dispersion problem in a tubular non-adiabatic reactor is given.

§1. Introduction

We consider a nonlinear evolution problem with two parameters

$$\frac{dx}{dt} = F(\lambda, \mu, x)$$

where λ, μ are real parameters, $x \in X$, a Hilbert space and F is a smooth mapping from $R \times R \times X$ to X.

As well known, codimension two bifurcations with a pure imaginary pair and a simple zero eigenvalue imply that chaotic motions may happen nearby. The unfoldings of these local bifurcations contain secondary global bifurcations involving homoclinic orbits (see [1] for details.)

The following assumptions are made in the paper:

(1) There is a solution family of $x = x(\lambda, \mu)$ near λ^0, μ^0 with $x^0 = x(\lambda^0, \mu^0)$;

(2) The Frechet derivative $F_x(\lambda, \mu, x(\lambda, \mu))$ has a pair of simple complex conjugate eigenvalues

$$r(\lambda, \mu) = u(\lambda, \mu) \pm iw(\lambda, \mu)$$

and a simple real eigenvalue $z(\lambda, \mu)$. We have

$$u(\lambda^0, \mu^0) = 0, \quad w(\lambda^0, \mu^0) = w^0 > 0, \quad z(\lambda^0, \mu^0) = 0.$$

The eigenvector corresponding to $w^0 i$ is $\phi_1^0 + i\phi_2^0$ and the real eigenvector corresponding to $z(\lambda, \mu)$ is $\phi(\lambda, \mu)$ with $\phi(\lambda^0, \mu^0) = \phi^0$. Let

$$\ker F_x^0 = \{c\phi^0 |\ c \in \mathbb{R}\}.$$

There exists a nontrivial $\psi^0 \in X$ such that Range $F_x^0 = \{x \in X | \langle \psi^0, x\rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ is an inner product.

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* Received October 12, 1988.

1) The Project Supported by National Natural Science Foundation of China.

(3) $F_\lambda^0 \notin \text{Range}\, F_x^0$, i.e. $\langle \psi^0, F_\lambda^0 \rangle \neq 0$;

(4) F_x^0 has no eigenvalues of the form $kiw^0, k \neq \pm 1$.

There are two ways for detecting the codimension two bifurcations. One way is to find the cross point of the Hopf bifurcation branch and the folds branch with respect to when μ varies. We can get these branches by using the algorithms in [2] and [3]. The other way is to determine the codimension two bifurcations directly. In Section 2, we propose an extended system for the codimension two bifurcations, which is regular. An efficient algorithm for solving the extended system is given in Section 3. Finally, we use the above methods to get the codimension two bifurcations in the axial dispersion problem in a tubular non-adiabatic reactor.

§2. An Extended System

We propose an extended system for the codimension two bifurcations as follows:

$$G(y) = \begin{bmatrix} F(\lambda,\mu,x) \\ F_x(\lambda,\mu,x)\phi \\ e\phi - 1 \\ [F_x^2(\lambda,\mu,x) + w^2 I]p \\ \langle p,p\rangle - 1 \\ \langle q,p\rangle \end{bmatrix} = 0 \tag{2.1}$$

where $y = (\lambda,\mu,w,x,\phi,p)$, q is a constant vector with nonzero projection on span $\{\phi_1^0,\phi_2^0\}$, $e \in X^*$ is chosen later on.

There is a unique vector $p^0 \in \text{Span}\,\{\phi_1^0,\phi_2^0\}$ such that $y^0 = (\lambda^0,\mu^0,w^0,x^0,\phi^0,p^0)$ is the isolated solution of (2.1). The Frechet derivative $G_y(y^0)$ is

$$\begin{bmatrix} F_x^0 & 0 & 0 & F_\lambda^0 & F_\mu^0 & 0 \\ F_{xx}^0\phi^0 & F_x^0 & 0 & F_{\lambda x}^0\phi^0 & F_{\mu x}^0\phi^0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 \\ F_{xx}^0F_x^0p^0 + F_x^0F_{xx}^0p^0 & 0 & F_x^{0^2} + w^{0^2}I & F_{\lambda x}^0F_x^0p^0 + F_x^0F_{\lambda x}^0p^0 & F_{\mu x}^0F_x^0p^0 + F_x^0F_{\mu x}^0p^0 & 2w^0p^0 \\ 0 & 0 & 2p^0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \end{bmatrix}$$

Let

$$N([(F_x^0)^2 + w^{0^2}I]^*) = \text{Span}\,\{\psi_1^0,\psi_2^0\}$$

with

$$\langle \psi_i^0, \phi_j^0 \rangle = \delta_{ij}, \qquad i,j = 1,2\,.$$

We can get

$$\psi_3^0 = (d_1^2 + d_2^2)^{-1}(d_1\psi_1^0 + d_2\psi_2^0), \qquad \psi_4^0 = (d_1^2 + d_2^2)^{-1}(d_2\psi_1^0 - d_1\psi_2^0)$$

with $\langle \psi_3^0, p^0 \rangle = 1$, $\quad \langle \psi_4^0, p^0 \rangle = 0$, where $p^0 = d_1\phi_1^0 + d_2\phi_2^0$. Let $g_1; g_2$ be respectively the unique solution of

$$\begin{cases} F_x^0 g_1 = QF_\lambda^0, \\ eg_1 = 0; \end{cases} \qquad \begin{cases} F_x^0 g_2 = QF_\mu^0, \\ eg_2 = 0 \end{cases}$$

where Q is the project operator onto Range F_x^0.

Theorem 1. *If condition* (1) - (4) *are satisfied and the determinant*

$$\begin{vmatrix} 0 & \langle\psi^0, F_\lambda^0\rangle & \langle\psi^0, F_\mu^0\rangle \\ \langle\psi^0, F_{xx}^0\phi^0\phi^0\rangle & \langle\psi^0, F_{\lambda x}^0\phi^0 - F_{xx}^0\phi^0 g_1\rangle & \langle\psi^0, F_{\mu x}\phi^0 - F_{xx}^0\phi^0 g_2\rangle \\ \langle\psi_4^0, F_1^0\rangle & \langle\psi_4^0, F_2^0\rangle & \langle\psi_4^0, F_3^0\rangle \end{vmatrix} \neq 0$$

where

$$F_1^0 = F_x^0(F_{xx}^0 p^0\phi^0) + F_{xx}^0(F_x^0 p^0)\phi^0,$$
$$F_2^0 = F_x^0 F_{\lambda x}^0 p^0 + F_{\lambda x}^0 F_x p^0 - F_x^0(F_{xx}^0 p^0 g_1) - F_{xx}^0(F_x^0 p^0)g_1,$$
$$F_3^0 = F_x^0 F_{\mu x}^0 p^0 + F_{\mu x}^0 F_x p^0 - F_x^0(F_{xx}^0 p^0 g_2) - F_{xx}^0(F_x^0 p^0)g_2.$$

Then $G_y(y^0)$ is regular.

Proof. The mapping $G_y(y^0)$ is one to one and onto from $R\times R\times R\times X\times X\times X$ to itself. The open mapping theorem ensures the conclution.

Direct calculations lead to the following lemmas.

Lemma 1. $\langle\psi_4^0, F_2^0\rangle = 2w^0\dfrac{\partial u(\lambda^0,\mu^0)}{\partial\lambda} - \xi\langle\psi_4^0, F_1^0\rangle.$

Lemma 2. $\langle\psi_4^0, F_3^0\rangle = 2w^0\dfrac{\partial u(\lambda^0,\mu^0)}{\partial\mu} - \eta\langle\psi_4^0, F_2^0\rangle.$

Lemma 3. $\langle\psi_4^0, F_1^0\rangle = w^0\{\langle\psi_1^0, F_{xx}^0\phi_1^0\phi^0\rangle + \langle\psi_2^0, F_{xx}^0\phi_2^0\phi^0\rangle\}.$

Instead of Theorem 1 we have

Theorem 2. *If the conditions* (1) - (4) *are satisfied and the determinant*

$$\begin{vmatrix} 0 & \langle\psi^0, F_\lambda^0\rangle & \langle\psi^0, F_\mu^0\rangle \\ \langle\psi^0, F_{xx}^0\phi^0\phi^0\rangle & \langle\psi^0, F_{xx}\phi^0 - F_{xx}^0\phi^0 g_1\rangle & \langle\psi^0, F_{\mu x}^0\phi^0 - F_{xx}^0\phi^0 g_2\rangle \\ \langle\psi_1^0, F_{xx}^0\phi_1^0\phi^0\rangle + \langle\psi_2^0, F_{xx}^0\phi_2^0\phi^0\rangle & 2\dfrac{\partial u(\lambda^0,\mu^0)}{\partial\lambda} & 2\dfrac{\partial u(\lambda^0,\mu^0)}{\partial\mu} \end{vmatrix} \neq 0,$$

then $G_y(y^0)$ is regular.

§3. An Efficient Algorithm

Let

$$A = F_x, B_1 = F_{xx}\phi, \quad C_1 = -F, \quad C_2 = -F_x\phi, \quad C_3 = -[F_x^2 + w^2 I]p,$$
$$C_4 = 1 - \langle p,p\rangle, \quad C_5 = -\langle q,p\rangle, \quad D_1 = F_\lambda, \quad D_2 = F_\mu, \quad D_3 = F_{\lambda x}\phi,$$
$$D_4 = F_{\mu x}\phi, \quad A_w = F_x^2 + w^2 I, \quad B_w = F_{xx}F_x p + F_x F_{xx} p,$$
$$D_5 = F_{\lambda x}F_x p + F_x F_{\lambda x} p, \quad D_6 = F_{\mu x}F_x p + F_x F_{\mu x} p$$

which are all valued $(\lambda^k, \mu^k, w^k, x^k, \phi^k, p^k)$ of the k-th iteration. Since $G_y(y^0)$ is regular, we can use the Newton method to solve the extended system (2.1). The Newton iteration for (2.1) is

$$A\delta x^k + D_1\delta\lambda^k + D_2\delta\mu^k = C_1 \quad (3.1)$$

$$B_1\delta x^k + A\delta\phi^k + D_3\delta\lambda^k + D_4\delta\mu^k = C_2 \quad (3.2)$$

$$e\phi^k - 1 = 0, \quad (3.3)$$

$$B_w\delta x^k + A_w\delta p^k + D_5\delta\lambda^k + D_6\delta\mu^k + 2w^k p^k\delta w^k = C_3, \quad (3.4)$$

$$\langle 2p^k, \delta p^k\rangle = C_4, \quad (3.5)$$

$$\langle q^k, \delta p^k\rangle = C_5. \quad (3.6)$$

In real computation (2.1) needs to be discretized. We may as well suppose that X is already an n-dimensional space E^n. We always take

$$e\phi - 1 = \phi_r - 1 = 0$$

where ϕ_r expresses the r-th component of ϕ. For convenience we shall choose $r = 1$.

The main idea in our algorithm is to use the strategy in [4]. Let

$$A_1 = (D_1|\tilde{A})$$

which is A with the first column replaced by D_1. Although A is singular at the codimension two bifurcations, A_1 is nonsingular as $F^0_\lambda \notin \text{Range}\, F^0_x$. Let

$$\delta s^T = (\delta\lambda, \delta x_2 - \delta x_1\phi_2^k, \cdots, \delta x_n - \delta x_1\phi_n^k,$$

$$\delta t^T = (\delta\lambda, \delta\phi_2, \cdots, \delta\phi_n),$$

(3.1), (3.2) can be rewritten as

$$A_1\delta s = C_1 + C_2\delta x_1 - D_2\delta\mu, \quad (3.7)$$

$$A_1\delta t = C_2 - B_1\delta x + (D_1 - D_3)\delta\lambda - D_4\delta\mu. \quad (3.8)$$

From (3.7) to (3.8) we can express $\delta s, \delta t$ successively in terms of δx_1 and $\delta\mu$ by solving six linear systems with the same matrix A_1. From the first component of $\delta s, \delta t$ we get the expression $\delta\lambda$ and $\delta\mu$ in terms of δx_1. Substituting $\delta x, \delta\lambda, \delta\mu$ into (3.4) and solving it with (3.5), (3.6) we can obtain $\delta p, \delta x_1, \delta w$. Finally we get $\delta\lambda, \delta\mu, \delta w, \delta x, \delta\phi, \delta p$. So far one step of the Newton iteration is completed.

§4. A Numerical Example

The axial dispersion problem in a tubular non-adiabatic reactor is considered. It can be described by

$$\frac{\partial y}{\partial t} = \frac{1}{Pe_y}\frac{\partial^2 y}{\partial z^2} - \frac{\partial y}{\partial z} + Da(1-y)\exp(\theta/(1+\theta/\gamma)),$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pe_\theta}\frac{\partial^2 \theta}{\partial z^2} - \frac{\partial \theta}{\partial z} + BDa(1-y)\exp(\theta/(1+\theta/\gamma)) - \beta(\theta - \theta_c) \quad (4.1)$$

with the boundary condition

$$z = 0: \quad Pe_y * y = \frac{\partial y}{\partial z}, \qquad Pe_\theta * \theta = \frac{\partial \theta}{\partial z},$$
$$z = 1: \quad \frac{\partial y}{\partial z} = \frac{\partial \theta}{\partial z} = 0.$$

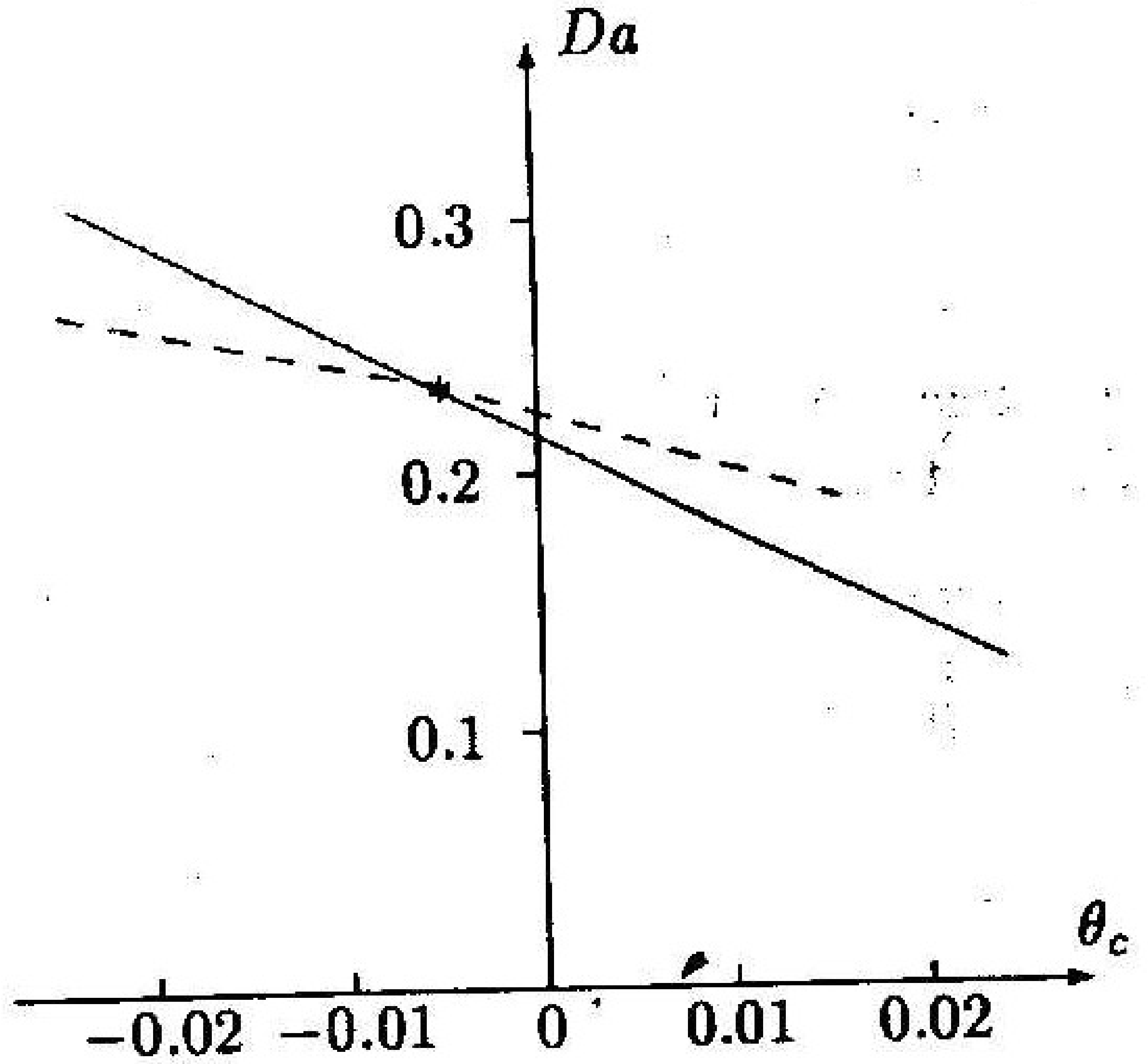

Fig. 1

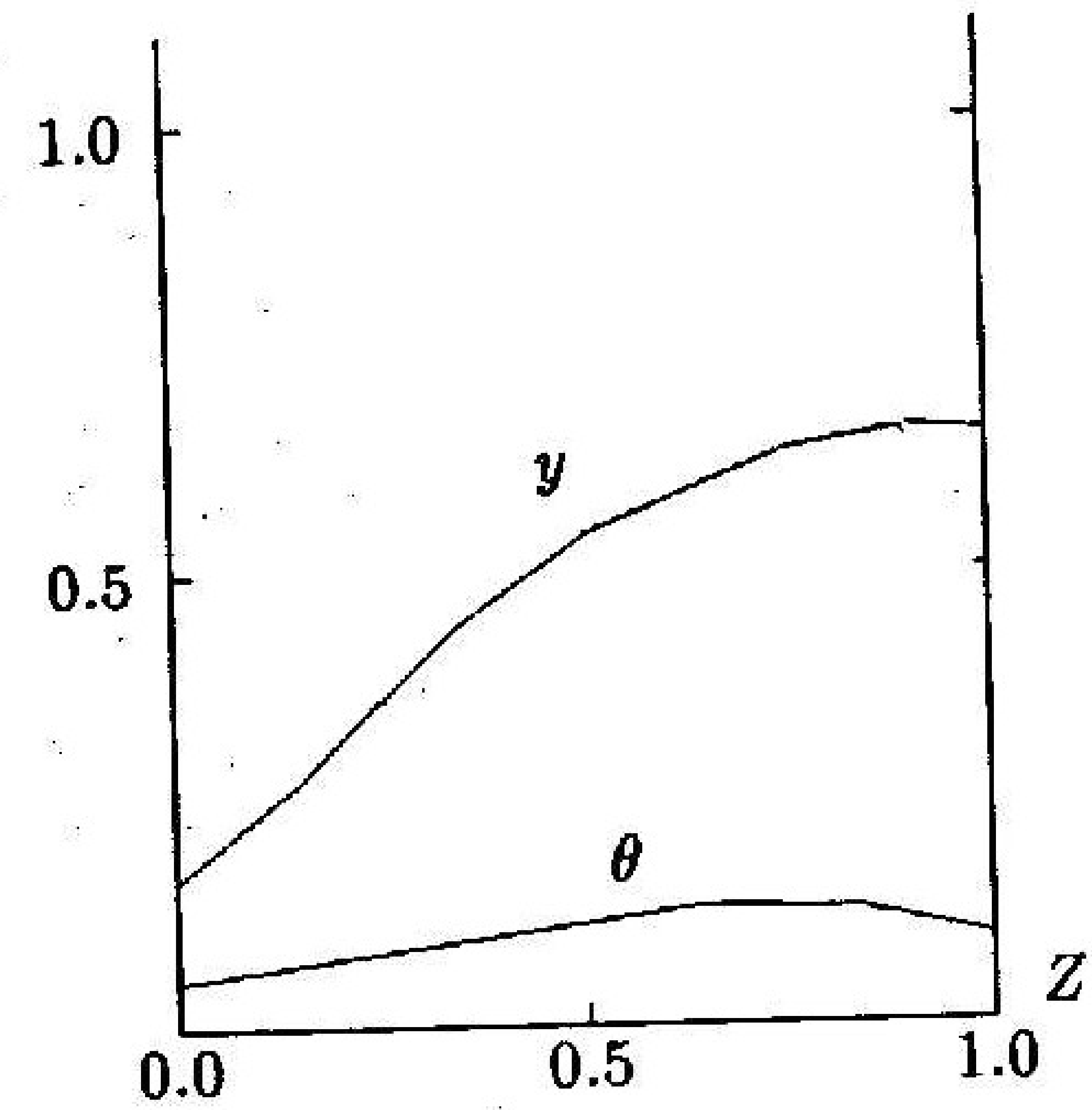

Fig. 2

- - - -Folds branch
——— Hopf bifurcation branch
* The codimension two bifurcation

Fixing $Pe_y = 5, Pe_\theta = 5, B = 0.5, \beta = 3, \gamma = 25$, we have a nonlinear problem with two parameters Da, θ_c. The numerical results are drawn in Fig.1. The folds branch and Hopf bifurcation branch are computed by using the algorithms in [2] and [3]. By using the algorithm in Section 3 we have directly computed the codimension two bifurcation with a pure imaginary pair and a simple zero eigenvalue. The numerical results are

$$Da = 0.232, \qquad \theta_c = -0.005, \qquad w = 0.0617,$$

and y, θ are drawn in Fig.2.

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