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RATIONAL INTERPOLATION FOR STIELTJES FUNCTIONS*[1)]

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Abstract

The continuity conclusions about rational Hermite interpolating functions are given under some conditions. On that basis, we establish the convergence results for the paradiagonal sequences of the rational interpolants for Stieltjes functions and Hamburger functions.

§1. Introduction

Let $\{x_i\}_{i=0}^{\infty} \subset [a,b]$ and $f \in C[a,b]$ be properly smooth. Given integers m, n, we consider the following problem: Find $R \in R_{mn} = \{u = p/q : \ p \in H_m, \ q \in H_n\}$ such that

$$R^{\sigma_i}(x_i) = f^{\sigma_i}(x_i), \qquad i = 0, 1, \cdots, m+n, \tag{1.1}$$

where H_l denotes the class of all polynomials of degree at most l and $\sigma_i + 1$ is the multiplicity of x_i in $\{x_0, x_1, \cdots, x_i\}$. Relating to the above problem, we introduce a linearized problem as follows. Find $(P, Q) \in H_m \times H_n$ such that

$$(Qf - P)^{(\sigma_i)}(x_i) = 0, \qquad i = 0, 1, \cdots, m+n. \tag{1.2}$$

Complete results about the solvability of the two problems can be found in [5]. For our purpose in this paper, we introduce the following conclusions.

Theorem 1.1 ([5],[6]). (i) *Problem* (1.1) *is solvable iff*

$$\begin{aligned} &\operatorname{rank} C(m-1, n-1, x_0, x_1, \cdots, x_{m+n}) \\ &\quad = \operatorname{rank} C(m-1, n-1, x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+m}), \\ &\quad i = 0, 1, \cdots, m+n. \end{aligned} \tag{1.3}$$

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(ii) *Let* (P^*, Q^*) *be a solution of problem* (1.2) *with minimum degree. Then* $\partial P^* = m$ *if and only if the matrix* $C(m-1, n, x_0, \cdots, x_{m+n})$ *is nonsingular;* $\partial Q^* = n$ *if and only if* $C(m, n-1, x_0, \cdots, x_{m+n})$ *is nonsingular.*

(iii) *The interpolation operator* T_{mn}, *for which* $T_{mn}(x_0, \cdots, x_{m+n}, f) = (P^*, Q^*)$, *is continuous at* $(x_0, \cdots, x_{m+n}, f)$ *if and only if* (P^*, Q^*) *is non-degenerate.*

The matrices $C(p, q, t_0, \cdots, t_k)$ used in the above theorem are defined as

$$C(p,q,t_0,\cdots,t_k) = \begin{bmatrix} v^{\sigma_0}(p,q,t_0) \\ v^{\sigma_1}(p,q,t_1) \\ \vdots \\ v^{\sigma_k}(p,q,t_k) \end{bmatrix},$$

where

$$v(p,q,t) = [1, t, \cdots, t^p, f(t), tf(t), \cdots, t^q f(t)]$$

and $\sigma_i + 1$ is the multiplicity of t_i in $\{t_0, t_1, \cdots, t_i\}$.

Let

$$H(m,i,j,t_0,\cdots,t_{m+j}) = \begin{bmatrix} f_{0,m} & f_{1,m} & \cdots & f_{i,m} \\ f_{0,m+1} & f_{1,m+1} & \cdots & f_{i,m+1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{0,m+j} & f_{1,m+j} & \cdots & f_{i,m+j} \end{bmatrix},$$

where f_{ij} is the divided difference of f at $t_i, t_{i+1}, \cdots, t_j$. Then we have

$$\operatorname{rank} C(p,q,t_0,\cdots,t_k) = p + 1 + \operatorname{rank}\ H(p+1, q, k-p-1, t_0, \cdots, t_k). \tag{1.4}$$

Hence the conclusions (i) and (ii) in Theorem 1.1 can be restated as follows:

(i) Problem (1.1) is solvable if and only if

$$\begin{aligned}&\operatorname{rank} H(m, n-1, n, x_0, \cdots, x_{m+n}) \\ &\quad = \operatorname{rank} H(m, n-1, n-1, x_0, \cdots, x_{i-1}, x_{i+1}, \cdots x_{m+n}), \quad i = 0, 1, \cdots, m+n.\end{aligned}$$

(ii) $\partial(P^*) = m \iff H(m, n, n, x_0, \cdots, x_{m+n})$ is nonsingular; $\partial(Q^*) = n \iff H(m+1, n-1, n-1, x_0, \cdots, x_{m+n})$ is nonsingular.

For the Cauchy interpolation problem, i.e., the interpolation points x_i are mutually distinct, the continuity results of the interpolation function $u_{mn} := R$ to f and the conclusions about the position of poles of u_{mn} are obtained by Braess in [2],[3]. In this paper, we first generalize these results to the Hermite case by a similar approach, and then establish convergence results for paradiagonal sequences of the rational interpolations for Stieltjes functions and Hamburger functions. For Padé approximants, similar convergence results can be found in [1].

§2. The Continuity of the Rational Interpolants

Let $l := m - n + 1 \geq 0, f \in C^{m+n-1}[a,b]$. Set

$$M_{mn}(x,f) = \begin{bmatrix} D^{m-n+1}f & D^{m-n+2}f & \cdots & D^m f \\ D^{m-n+2}f & D^{m-n+3}f & \cdots & D^{m+1}f \\ \cdots & \cdots & \cdots & \cdots \\ D^m f & D^{m+1}f & \cdots & D^{m+n-1}f \end{bmatrix},$$

where $D^k f(x) = f^{(k)}(x)/k!$. Then we have

Lemma 2.1. *If $M_{mn}(x,f)$ is definite in $[a,b]$, then*

(i) ([3], p.121) *$Q^2 f - P$ has at most $m+n-1$ zeros in $[a,b]$ counting multiplicities, whenever $Q \in H_{n-1} \setminus \{0\}, P \in H_{m+n-2}$;*

(ii) *for any $\{x_i\}_{i=0}^{m+n-1} \subset [a,b]$, the matrix $H(m,n-1,n-1,x_0,\cdots,x_{m+n-1})$ is nonsingular;*

(iii) *for any $\{x_i\}_{i=0}^{m+n} \subset [a,b]$, $H(m,n,n,x_0,\cdots,x_{m+n})$ or $H(m+1,n-1,n-1,x_0,\cdots,x_{m+n})$ is nonsingular.*

Proof. The proof of (i) can be found in [3], p.123. In order to prove (ii), consider $(fQ^2 - P)_{0,m+n-1}$ with $P \in H_{m+n-2}, Q \in H_{n-1} \setminus \{0\}$. Using the formula

$$(fg)_{0,k} = \sum_{i=0}^{k} f_{ik} g_{0i}, \tag{2.1}$$

we have

$$\begin{aligned}(fQ^2 - P)_{0,m+n-1} &= \sum_{i=0}^{n-1} (fQ)_{i,m+n-1} Q_{0i} = \sum_{i=0}^{n-1} \sum_{j=0}^{m+n-1-i} f_{i,i+j} Q_{i+j,m+n-1} Q_{0i} \\ &= \sum_{k=m}^{m+n-1} \sum_{i=0}^{n-1} f_{ik} Q_{k,m+n-1} Q_{0i} = \alpha_1 H(m,n-1,n-1,x_0,\cdots,x_{m+n-1}) \alpha_2^T,\end{aligned} \tag{2.2}$$

where

$$\alpha_1 = [Q_{m,m+n-1}, Q_{m+1,m+n-1}, \cdots, Q_{m+n-1,m+n-1}],$$

$$\alpha_2 = [Q_{00}, Q_{01}, \cdots, Q_{0,n-1}].$$

Since $(fQ^2 - P)_{0,m+n-1} = D^{m+n-1}(fQ^2 - P)(\xi) > 0$ (see [3], p.124), $H(m,n-1,n-1,x_0,\cdots,x_{m+n-1})$ is nonsingular.

Assume (iii) is not true. Then (1.4) implies that problem (1.2) has two solutions (P_1,Q_1) and (P_2,Q_2) with $\partial P_1 \leq m-1, \partial Q_1 = n, \partial P_2 = m, \partial Q_2 \leq n-1$. This contradicts $P_1 Q_2 = P_2 Q_1$ (see [5]).

Lemma 2.2. *If $M_{m+1,n}(x,f)$ is definite in $[a,b]$, then* *(i) $Q^2f - P$ has at most $m+n$ zeros in $[a,b]$ counting multiplicities, whenever $Q \in H_{n-1} \setminus \{0\}, P \in H_{m+n-2}$,*

(ii) for any $\{x_i\}_{i=0}^{m+n} \subset [a,b]$, the matrix $H(m+1, n-1, n-1, x_0, \cdots, x_{m+n})$ is nonsingular.

Proof. The proof of the lemma is the same as that of (i) and (ii) of Lemma 2.1.

Lemma 2.3. *If $\{1, x, \cdots, x^{m-1}, f(x), xf(x), \cdots, x^{n-1}f(x)\}$ spans an E-Haar subspace of $C[a,b]$ on $[a,b]$, then the conclusions (ii) and (iii) of Lemma 2.1 are true.*

Proof. In this case, $C(m-1, n-1, x_0, \cdots, x_{m+n-1})$ is nonsingular. Then (1.4) implies the required conclusion (ii) of Lemma 2.1. Since conclusion (iii) of Lemma 2.1 is derived from (ii), it holds, too.

Theorem 2.4. *If $\{1, x, \cdots, x^{m-1}, f(x), xf(x), \cdots, x^{n-1}f(x)\}$ spans an E-Haar subspace of $C[a,b]$ and $M_{m+1,n}(x,f)$ is definite on $[a,b]$, or if $M_{mn}(x,f)$ is definite on $[a,b]$, then for any $\{x_i\}_{i=0}^{m+n} \subset [a,b]$, the solution of problem (1.1) exists and is continuous in $[\alpha_{mn}, \beta_{mn}]$, where*

$$\alpha_{mn} = \min\{x_0, \cdots, x_{m+n}\}, \qquad \beta_{mn} = \max\{x_0, \cdots, x_{m+n}\}.$$

Proof. It follows from Theorem 1.1 and Lemmas 2.1–3 that the solution of problem (1.1) exists and is non-degenerate for any $\{x_i\}_{i=0}^{m+n} \subset [a,b]$. From these facts the continuity of the solution can be proved (see [3]).

From the proof of Theorem 2.4, we know that in order to have the continuity of the rational interpolant u_{mn}, it is sufficient for f to have properties (i) of Lemma 2.2 and (ii), (iii) of Lemma 2.1. Since property (i) implies (iii), Theorem 2.4 can be generalized as follows:

Theorem 2.5. *Suppose f has the following properties:*

(i) $Q^2f - P$ has at most $m+n$ zeros in $[a,b]$ counting multiplicities, whenever $Q \in H_{n-1} \setminus \{0\}, P \in H_{m+n-2}$. (ii) For any $\{x_i\}_{i=0}^{m+n-1} \subset [a,b]$, the matrix $H(m, n-1, n-1, x_0, \cdots, x_{m+n-1})$ is nonsingular. Then for any $\{x_i\}_{i=0}^{m+n} \subset [a,b]$, the solution of problem (1.1) exists and is continuous in $[\alpha_{mn}, \beta_{mn}]$.

§3. The Position of Poles

Assume $0 \in [a,b]$ and $n \geq 1$. Set

$$R^p_{mn}[a,b] = \Big\{u(x) = p(x) + x^l \sum_{k=1}^{n} \frac{a_k}{1+t_kx} : \; p(x) \in \Pi_{l-1}, a_k \geq 0,$$
$$t_k \in (-b^{-1}, -a^{-1}), k = 1, 2, \cdots, n, t_k \text{ are mutually distinct}\Big\}.$$

Then $R^p_{mn}[a,b] \subset R_{mn}$ and the set has the following properties.

Lemma 3.1. (i) *$u = p/q \in R^p_{mn}[a,b]$ is non-degenerate (i.e., p and q are irreducible, and $\partial p = m$ or $\partial q = n$) iff $a_k > 0, k = 1,2,\cdots,n$.*

(ii) *If $u \in R^p_{mn}[a,b]$, then $M_{mn}(x,u)$ is nonnegative definite, and $M_{mn}(x,u)$ is positive definite iff u is non-degenerate.*

(iii) *Set $u = p + x^l \sum_{k=1}^{n} \frac{a_k}{1+t_k x}$. Then $a_k t_k \le 0$ $(k = 1,2,\cdots,n)$ iff $M_{m+1,n}(x,u)$ is nonnegative definite on $[a,b]$; $a_k t_k < 0 (k = 1,2,\cdots,n)$ iff $M_{m+1,n}(x,u)$ is positive definite on $[a,b]$.*

(iv) *Let $\{y_i\}_{i=0}^{l-1} \subset [a,b]$. Then*

$$R^p_{mn}[a,b] = \Big\{u(x) = \tilde{p}(x) + \textstyle\prod_{i=0}^{l-1}(x-y_i) \displaystyle\sum_{k=1}^{n} \frac{\tilde{a}_k}{1+t_k x} :$$
$$\tilde{p}(x) \in H_{l-1}, \tilde{a}_k \ge 0, t_k \in (-b^{-1}, -a^{-1}) \text{ are distinct mutually}\Big\}. \tag{3.1}$$

(v) *$R^p_{mn}[a,b]$ is a closed set in R_{mn} in the sense of Chebyshev norm.*

(vi) *If $u \in R^p_{mn}[a,b]$ is non-degenerate, then u is an interior point of $R^p_{mn}[a,b]$, i.e., there exists an ϵ, such that*

$$\{R : \|u - R\| < \epsilon, \ R \in R_{mn}\} \subset R^p_{mn}[a,b].$$

Proof. (i) If $a_k = 0$ for some k, then u is evidently degenerate. If, on the contrary, $a_k \ne 0$ for $k = 1,2,\cdots,n$, then u is non-degenerate for $t_k \ne 0, k = 1,2,\cdots,n$. If $t_k = 0 \in (-b^{-1}, -a^{-1})$ for some k, then the degrees of the numerator and the denorminator of $u(x)$ are m and $n-1$ respectively. Therefore, $u(x)$ is non-degenerate.

(ii) For $k \ge l$,

$$D^k\left(\frac{x^l}{1+tx}\right) = \sum_{j=0}^{k} D^{k-j}(1+tx)^{-1} D^j x^l = \sum_{j=0}^{l} \frac{(-t)^{k-j}}{(1+tx)^{k-j+1}} \frac{l! x^{l-j}}{(l-j)!j!}$$
$$= \frac{(-t)^{k-l}}{(1+tx)^{k-l+1}} (1 + \frac{-tx}{1+tx})^l = \frac{(-t)^{k-l}}{(1+tx)^{k+1}}. \tag{3.2}$$

Then

$$M_{mn}(x, \frac{x^l}{1+tx}) = \frac{1}{(1+tx)^{l+1}} b_t^T b_t, \quad b_t = \left(1, \frac{-t}{1+tx}, \cdots, (\frac{-t}{1+tx})^{n-1}\right).$$

Therefore

$$M_{mn}(x,u) = \sum_{k=1}^{n} \frac{a_k}{(1+t_k x)^{l+1}} b_{t_k}^T b_{t_k}.$$

From this we get the conclusion (ii).

(iii) Similarly,

$$M_{m+1,n}(x, \frac{x^l}{1+tx}) = \frac{-t}{(1+tx)^{l+2}} b_t^T b_t, \quad M_{m+1,n}(x,u) = \sum_{k=1}^{n} \frac{-a_k t_k}{(1+t_k x)^{l+2}} b_{t_k}^T b_{t_k}.$$

Hence the conclusion (iii) holds.

(iv) Let $p/q \in R_{mn}^p[a,b]$ and $q(x) = \prod_{k=1}^{n}(1+t_k x)$. Since $w(x) := \prod_{i=0}^{l-1}(x-y_i)$ and $q(x)$ are coprime, there exist $p_1 \in \Pi_{n-1}$, $q_1 \in \Pi_{l-1}$ such that

$$p(x) = p_1(x)w(x) + q_1(x)q(x).$$

Then

$$\frac{p(x)}{q(x)} = q_1(x) + w(x)\frac{p_1(x)}{q_1(x)} = q_1(x) + w(x)\sum_{k=1}^{n}\frac{a_k}{1+t_k x}.$$

Now we shall prove that $a_k \ge 0$. If $k \ge l$, it follows from (3.2) that

$$D^k\left(\frac{w(x)}{1+tx}\right) = \frac{(-t)^k w(-t^{-1})}{(1+tx)^{k+1}} = \frac{(-t)^{k-l}\prod_{i=0}^{l-1}(1+ty_i)}{(1+tx)^{k+1}}.$$

Hence

$$M_{mn}(x, \frac{p}{q}) = \sum_{k=1}^{n} \frac{a_k \prod_{i=0}^{l-1}(1+t_k y_i)}{(1+t_k x)^{l+1}} b_{t_k}^T b_{t_k}.$$

Since $M_{mn}(x, p/q)$ is nonnegative definite, $a_k \ge 0$.

On the other hand, we can prove in the same manner that the elements in the set of the right side of (3.1) belong to $R_{mn}^p[a,b]$.

(v) Let $u_i(x) \in R_{mn}^p[a,b]$ and $u_i \Longrightarrow u^* \in C[a,b]$ ("$\Longrightarrow$" means "converge uniformly"). We shall show that $u^* \in R_{mn}^p[a,b]$. Set

$$u_i(x) = p_i(x) + w(x)\sum_{k=1}^{n}\frac{a_k^{(i)}}{1+t_k^{(i)}x}, \tag{3.3}$$

where $w(x) = \prod_{j=1}^{l}(x-y_j)$ and $a < y_1 < y_2 < \cdots < y_l < b$. Since $\|u_i\| := \max_{x\in[a,b]}|u_i(x)|$ are bounded and $p_i(y_j) = u_i(y_j)$ for $j = 1,2,\cdots,l$, $\|p_i\|$ are bounded. Hence there is an M such that

$$\left\|\frac{a_k^{(i)}w(x)}{1+t_k^{(i)}x}\right\| \le M, \quad i = 1,2,\cdots. \tag{3.4}$$

Then $a_k^{(i)} \le M/|w(0)|$. Now we may assume (if necessary, passing to subsequences)

$$a_k^{(i)} \to a_k, \quad t_k^{(i)} \to t_k (t_k \text{ may be } \pm\infty), \quad k = 1, 2, \cdots, n; \text{ and } p_i \Longrightarrow p. \tag{3.5}$$

If $t_k = -b^{-1}$ or $-a^{-1}$, say $-b^{-1}$, then (3.4) implies $a_k = 0$ for $|t_k| < \infty$. Therefore

$$\overline{\lim_{i\to\infty}} \frac{a_k^{(i)} w(x)}{1 + t_k^{(i)} x} = \begin{cases} \overline{\lim\limits_{i\to\infty}} \dfrac{a_k^{(i)} w(b)}{1 + t_k^{(i)} b} \ge 0, & x = b, \\ 0, & x \in [a, b). \end{cases} \tag{3.6}$$

If $t_k = -b^{-1} = -\infty$, then

$$\lim_{i\to\infty} \frac{a_k^{(i)} w(x)}{1 + t_k^{(i)} x} = \begin{cases} a_k w(b) \ge 0, & x = b = 0, \\ 0, & x \in [a, b). \end{cases} \tag{3.7}$$

Since $u_i \Longrightarrow u^* \in C[a, b]$, $\left\| \dfrac{a_k^{(i)} w(x)}{1 + t_k^{(i)} x} \right\| \longrightarrow 0$. Hence

$$u_i(x) \Longrightarrow p(x) + w(x) \sum_{\substack{k=1,2,\cdots,n \\ t_k \in (-b^{-1}, -a^{-1})}} \frac{a_k}{1 + t_k x} = u^* \in R^p_{mn}[a, b].$$

(vi) If the required conclusion does not hold, then there exists a sequence $u_i = p_i/q_i \in (R_{mn} \bigcup C[a, b]) \setminus R^p_{mn}[a, b]$, such that $u_i \Longrightarrow u = p/q$. We may assume

$$p_i \Longrightarrow \tilde{p}, \qquad q_i \Longrightarrow \tilde{q}, \tag{3.8}$$

where $u = \tilde{p}/\tilde{q}$. Since u is non-degenerate and q has at least $n - 1$ distinct real zeros, q_i has at least the same number of zeros and has no zeros in $[a, b]$ for i big enough. Hence q_i has only real zeros and then u_i can be expressed as

$$u_i(x) = p_i(x) + x^l \sum_{k=1}^{n} \frac{a_k^{(i)}}{1 + t_k^{(i)} x}$$

with $-t_k^{(i)^{-1}} \notin [a, b]$. From (3.8), we have $D^k u_i(x) \Longrightarrow D^k u(x)$. Then $M_{mn}(x, u_i) \Longrightarrow M_{mn}(x, u)$. This implies $M_{mn}(x, u_i)$ is positive definite if i is sufficiently large. Hence $u_i \in R^p_{mn}[a, b]$, a contradiction.

Theorem 3.2. (i) *If $M_{mn}(x, f)$ is positive definite on $[\alpha_{mn}, \beta_{mn}]$ and $0 \in [\alpha_{mn}, \beta_{mn}]$, the solution u_{mn} of problem (1.1) belongs to $R^p_{mn}[\alpha_{mn}, \beta_{mn}]$.*

(ii) *Moreover, if $M_{m+1,n}(x, f)$ is positive (or negative) definite, then all poles of u_{mn} are contained in (β_{mn}, ∞) (or $(-\infty, \alpha_{mn})$).*

(iii) *If* $a < \alpha_{mn} \le 0 \le \beta_{mn} < b$ *and* $M_{m+1,n+1}(x, f)$ *is positive in* $[a, b]$, *then* $u_{mn} \in R^p_{mn}[a, b]$ *and*

$$(-1)^{m+n+1}[f(x) - u(x)] > 0, \quad x \in [a, \alpha_{mn}), \tag{3.9}$$

$$f(x) - u(x) > 0, \quad x \in (\beta_{mn}, b]. \tag{3.10}$$

Proof. (i) Choose a non-degenerate $u_0 \in R^p_{mn}[\alpha_{mn}, \beta_{mn}]$ and put $f_\lambda = (1-\lambda)u_0 + \lambda f$. Then $M_{mn}(x, f_\lambda)$ is positive definite for $\lambda \in [0,1]$. Therefore the interpolation function $u_\lambda(x)$ to f_λ is continuous and non-degenerate. From conclusions (v) and (vi) of Lemma 3.1, and the continuity of u_λ with respect to λ, it follows that $u_\lambda(x) \in R^p_{mn}[\alpha_{mn}, \beta_{mn}]$ for $\lambda \in [0,1]$. Then (i) is obtained.

(ii) Suppose $M_{m+1,n}(x, f)$ is positive definite. Choose $u_0 \in R^p_{mn}[\alpha_{mn}, \beta_{mn}]$ such that $M_{m+1,n}(x, u_0)$ is also positive definite, i.e., all poles of u_0 are in (β_{mn}, ∞). Hence $M_{m+1,n}(x, f_\lambda)$ is positive definite for $\lambda \in [0,1]$. Since $u_\lambda = p_\lambda / q_\lambda \in R^p_{mn}[\alpha_{mn}, \beta_{mn}]$ and $\partial q_\lambda = n$, all zeros of q_λ are contained in (β_{mn}, ∞) for $\lambda \in [0,1]$.

(iii) Since the positive definiteness of $M_{m+1,n+1}(x, f)$ implies that $M_{mn}(x, f)$ has the same property, interpolant $u_{mn} = P/Q$ to f is in $R^p_{mn}[\alpha_{mn}, \beta_{mn}]$. From Lemma 2.1 it follows that $g_{mn} = Q^2 f - PQ$ has exactly $m + n + 1$ zeros. Since $D^{m+n+1} g_{mn}(x) > 0$ for $x \in [a, b]$, we can prove by induction that

$$(-1)^{m+n+1-k} D^k g_{mn}(x) > 0, \qquad x \in [a, \alpha_{mn}),$$

$$D^k g_{mn}(x) > 0, \qquad x \in (\beta_{mn}, b].$$

Taking $k = 0$, we get (3.9) and (3.10), and then Q has no zeros in $[a, \alpha_{mn}) \cap (\beta_{mn}, b]$. Hence $P/Q \in R^p_{mn}[a, b]$.

§4. Convergence for Stieltjes Functions

Lemma 4.1. *Let* $f(z)$ *be a Stieltjes function:*

$$f(z) = \int_0^\infty \frac{d\mu(t)}{1 + tz} \tag{4.1}$$

where $\mu(t)$ *is a bounded, non-decreasing function, taking infinitely many different values. Let* $p(z) \in H_l$ *be the polynomial interpolant of* $f(z)$ *at* $x_0, x_1, \cdots, x_l \in [0, \infty)$. *Then* $g(z) = [f(z) - p(z)] / \prod_{i=0}^{l} (x_i - z)$ *is also a Stieltjes function represented by*

$$g(z) = \int_0^\infty \frac{t^{l+1} d\mu(t)}{\prod_{i=0}^{l} (1 + t x_i)(1 + tz)}. \tag{4.2}$$

Proof. Let

$$h(z) = p(z) + \prod_{i=0}^{l}(z - x_i)\int_0^\infty \frac{t^{l+1}d\mu(t)}{\prod_{i=0}^{l}(1+tx_i)(1+tz)}.$$

From (3.2), we have

$$D^{l+1}f(z) = \int_0^\infty \frac{(-t)^{l+1}}{(1+tx)^{l+2}}\,d\mu(z),$$

$$D^{l+1}h(z) = \int_0^\infty \frac{t^{l+1}}{\prod_{i=0}^{l}(1+tx_i)}\frac{(-t)^{l+1}\prod_{i=0}^{l}(t^{-1}+x_i)}{(1+tx)^{l+2}}\,d\mu(t) = \int_0^\infty \frac{(-t)^{l+1}}{(1+tx)^{l+2}}\,d\mu(t).$$

Hence $D^{l+1}(f-h)(z) = 0$. Since $h(x_i) = f(x_i)$ for $i = 0, 1, \cdots, l$, $h(z) = f(z)$.

Lemma 4.2. *Let*

$$f(z) = \int_0^{R^{-1}} \frac{1}{1+tz}\,d\mu(t), \qquad R \geq 0$$

and

$$X_R = \begin{cases} (-R,\infty), & R > 0, \\ [0,\infty), & R = 0. \end{cases}$$

Then

(i) *if $m-n+1 \geq 0$ is even, $M_{mn}(x,f)$ is positive definite for $x \in X_R$;*

(ii) *if $m-n+1 \geq 0$ is odd, $M_{mn}(x,f)$ is negative definite for $x \in X_R$.*

Proof. Since

$$M_{mn}(x,f) = \int_0^{R^{-1}} M_{mn}(x, \frac{1}{1+tx})\,d\mu(t) = \int_0^{R^{-1}} \frac{(-t)^{m-n+1}}{(1+tx)^{m-n+2}} b_t^T b_t\,d\mu(t),$$

where

$$b_t = \left[1, \frac{-t}{1+tx}, \left(\frac{-t}{1+tx}\right)^2, \cdots, \left(\frac{-t}{1+tx}\right)^{n-1}\right]$$

and $\mu(t)$ is not concentrated on a finite set, the required conclusions hold.

Theorem 4.3. *Let*

$$f(z) = \int_0^{R^{-1}} \frac{1}{1+tz}\,d\mu(t), \qquad R \geq 0,$$

and $\{x_i\}_{i=0}^{\infty} \subset X_R$, $\gamma = \sup \beta_{mn} < \infty$. Then for any given $l \geq 0$,

(i) *the sequence of the interpolants* $u_{n-1+l,n}$ *to* f *is uniformly bounded as* $n \longrightarrow \infty$ *on the domain* $\mathcal{D}(\Delta)$;

(ii) *the sequence of* $u_{n-1+l,n}$ *is equicontinuous on* $\mathcal{D}(\Delta)$,

where $\mathcal{D}(\Delta)$ *is a bounded region of the complex plane which is at least at a distance* Δ *from* $(-\infty, -R)$.

Proof. Let

$$\delta^{-1} = \inf_{\substack{t\in[0,R^{-1}]\\ z\in\mathcal{D}(\Delta)}} |1+tz|. \tag{4.3}$$

Then $\delta^{-1} > 0$. Put

$$u_{n-1+l,n}(z) = p(x) + \prod_{i=0}^{l-1}(z-x_i)\sum_{k=1}^{n}\frac{a_k}{1+t_k z} = p(x) + w(x)q_n(x),$$

where $p(x) \in \prod_{l-1}$ is the interpolating polynomial of f at $x_0, \cdots, x_{l-1}$. It follows from Lemma 4.2 and Theorem 3.2 that $a_k > 0, t_k \in [0, R^{-1})$.

(i) Taking $\tau > \max\{0, \gamma\}$, we have by Theorem 3.2 that

$$\begin{aligned} &w(\tau)q_n(\tau) = u_{n-1+l,n}(\tau) - p(\tau) < f(\tau) - p(\tau), \\ &q_n(\tau) < [f(\tau) - p(\tau)]/w(\tau) = M_0. \end{aligned} \tag{4.4}$$

It follows that

$$|u_{n-1+l,n}(z)| \le \|p\| + \|w\| \sum_{k=1}^{n} \frac{a_k}{1+t_k\tau}\frac{1+t_k\tau}{|1+t_k z|} \le \|p\| + \|w\|M_0 \sup_{\substack{t\in[0,R^{-1}]\\ z\in\mathcal{D}(\Delta)}} \frac{1+t\tau}{|1+tz|}. \tag{4.5}$$

For $R = 0$, since

$$\left|\frac{c+td}{|1+tz|} - \frac{d}{|z|}\right| \le \frac{|cz-d|}{|1+tz||z|} \le \frac{(cr+d)\delta}{\Delta},$$

where $c \ge 0, d > 0$ and $r = \sup_{z\in\mathcal{D}(\Delta)} |z|$, then

$$\sup \frac{c+td}{|1+tz|} \le \sup\left|\frac{c+td}{|1+tz|} - \frac{d}{|z|}\right| + \sup\frac{d}{|z|} \le \frac{(cr+d)\delta + d}{\Delta}.$$

If $R > 0$, then

$$\sup\frac{c+td}{|1+tz|} \le (c+R^{-1}d)\sup\frac{1}{|1+tz|} = \delta(c+R^{-1}d).$$

Therefore there exists a constant $M(c,d)$, such that

$$\sup_{\substack{t\in[0,R^{-1}]\\ z\in\mathcal{D}(\Delta)}} \frac{c+td}{|1+tz|} \le M(c,d). \tag{4.6}$$

Statement (i) follows from (4.5) and (4.6) by taking $c = 1$, $d = \tau$.

(ii) From the proof of (4.5), we have

$$|q_n(z)| \le M_0 M(1,\tau).$$

On the othor hand,

$$\begin{aligned}
|q_n(z_1) - q_n(z_2)| &\le |z_1 - z_2| \sum_{k=1}^{n} \frac{|a_k t_k|}{|1+t_k z_1||1+t_k z_2|} \\
&= |z_1 - z_2| \sum_{k=1}^{n} \frac{a_k}{1+t_k\tau} \cdot \sup \frac{|t||1+t\tau|}{|1+tz_1||1+tz_2|} \\
&\le |z_1 - z_2| M_0 \sup \frac{|1+t\tau|}{|1+tz|} \sup \frac{t}{|1+tz|} \\
&\le |z_1 - z_2| M_0 M(1,\tau) M(0,1).
\end{aligned}$$

Then

$$|u_{n-1+l,n}(z_1) - u_{n-1+l,n}(z_2)| \le [\|p'\| + M_0 M(1,\tau)(\|w\| M(0,1) + \|w'\|)]\, |z_1 - z_2|.$$

Hence conclusion (ii) holds.

Theorem 4.4. *Let*

$$f(z) = \int_0^{R^{-1}} \frac{1}{1+tz}\, d\mu(t), \qquad R \ge 0$$

$\{x_i\}_{i=0}^{\infty} \subset X_R$, *and* $\gamma = \sup \beta_{mn} < \infty$. *Then for any given* $l \ge 0$,

(i) *the sequence of interpolants* $u_{n-1+l,n}$ *of* f *converges uniformly on* $\mathcal{D}(\Delta)$ *to an analytic function* $f^{(l)}(z)$;

(ii) *if* $-R$ *is not a limit point of* $\{x_i\}_{i=0}^{\infty}$, *then* $f^{(l)}(z)$ *is analytic in the complex plane cut by* $(-\infty, -R]$ *and* $f^{(l)}(z) = f(z)$.

Proof. (i) Since the sequence $\{u_{n-1+l,n}\}$ is uniformly bounded and equicontinuous on $\mathcal{D}(\Delta)$, it follows from Arzela's theorem (see [1], p. 175) that there exists a subsequence which converges uniformly to a continuous function $f^{(l)}(z)$ defined on $\mathcal{D}(\Delta)$. By Weierstrass' theorem ([4], p.95), we assert that $f^{(l)}(z)$ is analytic on $\mathcal{D}(\Delta)$. From Theorem 3.2, we have, for $x > \gamma$,

$$u_{n-1+l,n}(x) < u_{n+l,n+1} < f(x), \qquad \text{for even } l,$$

$$u_{n-1+l,n}(x) > u_{n+l,n+1} > f(x), \qquad \text{for odd } l.$$

Hence $\{u_{n-1+l,n}\}$ converges uniformly to $f^{(l)}(x)$ for $x > \gamma$. By the uniqueness theorem of analytic functions, we know that $\{u_{n-1+l,n}\}$ converges uniformly to $f^{(l)}(z)$ on $\mathcal{D}(\Delta)$.

(ii) By the definition, $f(z)$ is an analytic function in the complex plane except on $(-\infty, -R]$. Since the domain $\mathcal{D}(\Delta)$ can be arbitrarily big and Δ can be arbitrarily small, $f^{(l)}(z)$ and $f(z)$ have the same analytic structure. If $\{x_i\}_{i=0}^{\infty}$ contains only finite different points, then there exists an $x^* > -R$ such that there are infinitely many x_i in $\{x_i\}_{i=0}^{\infty}$ coinciding with x^*. Hence $D^k f^{(l)}(x^*) = D^k f(x^*)$ for $k = 0, 1, \cdots$. This means that $f^{(l)}(z)$ and $f(z)$ have the same power series expansion at x^*. Then $f^{(l)}(z) = f(z)$ in a neighborhood of x^* and furthermore $f^{(l)}(z) = f(z)$ for $z \notin (-\infty, -R]$. If $\{x_i\}_{i=0}^{\infty}$ contains infinitely many different points, then there exists a subsequence $\{x_{i_j}\}$ such that x_{i_j} are mutually distinct and $x_{i_j} \longrightarrow x^* > -R$. Since $f^{(l)}(x_{i_j}) = f(x_{i_j})$ for $j = 0, 1, \cdots$, it follows from the uniqueness theorem of analytic functions that $f^{(l)}(z) = f(z)$ for $z \notin (-\infty, -R]$.

§5. Convergence for Hamburger Functions

Lemma 5.1. *Let $f(z)$ be a Hamburger function defined by*

$$f(z) = \int_{-R_2^{-1}}^{R_1^{-1}} \frac{d\mu(t)}{1+tz}, \qquad R_1 > 0,\ R_2 > 0, \tag{5.1}$$

where $\mu(t)$ is a bounded, non-decreasing function taking infinitely many different values. Then for $x \in (-R_1, R_2)$ and even integers $m - n + 1$, the matrix $M_{mn}(x, f)$ is positive definite.

The proof of the lemma is the same as that of Lemma 4.2.

Following the proof of Theorem 4.3 and Theorem 4.4, we can establish the following theorem.

Theorem 5.2. *Let $f(z)$ be a Hamburger function defined as (5.1), $\{x_i\} \subset (-R_1, R_2)$. Suppose $\tau = \inf \alpha_{mn} > -R_1$, or $\gamma = \sup \beta_{mn} < R_2$. Then for any given even integer $l \geq 0$,*

(i) *the sequence of the interpolants $u_{n-1+l,n}$ to f is uniformly bounded as $n \longrightarrow \infty$ on the domain $\mathcal{D}_1(\Delta)$;*

(ii) *the sequence of $u_{n-1+l,n}$ is equicontinuous on $\mathcal{D}_1(\Delta)$;*

(iii) *the sequence of $u_{n-1+l,n}$ converges uniformly on $\mathcal{D}_1(\Delta)$ to an analytic function $f^{(l)}(z)$;*

(iv) *if $[\tau, \gamma] \subset (-R_1, R_2)$, then $f^{(l)}(z)$ is analytic in the complex plane except on $(-\infty, -R_1] \cup [R_2, \infty)$, and $f^{(l)}(z) = f(z)$ for $z \notin [-\infty, -R_1) \cup [R_2, \infty)$, where $\mathcal{D}_1(\Delta)$ is a bounded region of the complex plane which is at least at a distance Δ from $(-\infty, -R_1) \cup (R_2, \infty)$.*

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