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ON THE SUFFICIENT CONDITIONS FOR THE SOLUBILITY OF ALGEBRAIC INVERSE EIGENVALUE PROBLEMS *1)

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Abstract

With the help of Brouwer's fixed point theorem and the relations of the eigenvalues and diagonal elements of a Hermitian matrix, we give some new sufficient conditions for the solubility of algebraic inverse eigenvalue problems.

§1. Introduction

We are interested in solving the following inverse eigenvalue problems:

Problem A (Additive inverse eigenvalue problem). Given an $n \times n$ Hermitian matrix $A = [a_{ij}]$, and n real numbers $\lambda_1, \cdots, \lambda_n$, find a real $n \times n$ diagonal matrix $D = \text{diag }(c_1, \cdots, c_n)$ such that the matrix $A + D$ has eigenvalues $\lambda_1, \cdots, \lambda_n$.

Problem M (Multiplicative inverse eigenvalue problem). Given an $n \times n$ positive Hermitian matrix $A = [a_{ij}]$, and n positive real numbers $\lambda_1, \cdots, \lambda_n$, find an $n \times n$ positive definite diagonal matrix $D = (c_1, \cdots, c_n)$ such that the matrix DA has eigenvalues $\lambda_1, \cdots, \lambda_n$.

Problem G (General inverse eigenvalue problem). Given $n+1$ complex $n \times n$ Hermitian matrix $A = [a_{ij}]$, $A_k = [a_{ij}^{(k)}]$, $k = 1, \cdots, n$, and n real numbers $\lambda_1, \cdots, \lambda_n$, find n real numbers $c_1, \cdots, c_n$, such that the matrix $A(c) = A + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, \cdots, \lambda_n$.

A number of sufficient conditions for those problems to have a solution have been discovered by many authors (see [1]-[3], [5]-[7]) . In the present paper we shall give some new sufficient conditions for those three problems to have solutions. These results are not contained in the presently known results and are better than the known results in some aspects.

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Notation and Definitions. Throughout this paper we use the following notation. $\mathbb{R}^n$ is the set of all n-dimensional real column vectors and ϕ is an empty set. The superscripts T and H are for transpose and conjugate transpose, respectively. $\rho(A)$ and $\mathrm{tr}(A)$ denote the spectral radius and the trace of a matrix A, respectively. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of a Hermitian matrix A, respectively. The norm $\| \quad \|_\infty$ stands for max-norm of a vector and maximum row sum matrix norm.

For arbitrary $n \times n$ matrices $B = [b_{ij}]$ and vectors $b = (b_1, \cdots, b_n)^T \in \mathbb{R}^n$, let

$$d(b) = \min_{i \neq j} |b_i - b_j|$$

and

$$B^{(0)} = B - \mathrm{diag}\,(b_{ii}).$$

Without loss of generality we can assume that $a_{jj}^{(k)} = \delta_{kj}$ for $k, j = 1, 2, \cdots, n$ in Problem G (see [1]), that $a_{ii} = 1$ for $i = 1, 2, \cdots, n$ in Problem M, and that $a_{ii} = 0$ for $i = 1, 2, \cdots, n$ in Problem A.

§2. Main Results

Theorem 1. *Suppose that $\lambda_n > \cdots > \lambda_1 > 0, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ in Problem M. If*

$$d(\lambda) \geq \lambda_n(\lambda_{\max}(A^{(0)}) - \lambda_{\min}(A^{(0)})), \tag{2.1}$$

then there exist n real numbers $c_1, \cdots, c_n$ such that the matrix DA has eigenvalues $\lambda_1, \cdots, \lambda_n$, where $D = \mathrm{diag}\,(c_1, \cdots, c_n)$.

Theorem 2. *Suppose that $\lambda_n > \cdots > \lambda_1, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ in Problem G. If*

$$d(\lambda) \geq \lambda_n^* - \lambda_1^* + \lambda_0 \sum_{k=1}^{n} (\lambda_{\max}(A_k^{(0)}) - \lambda_{\min}(A_k^{(0)})), \tag{2.2}$$

then there exist n real numbers $c_1, \cdots, c_n$ such that the matrix $A + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, \cdots, \lambda_n$, where

$$\lambda_1^* = \lambda_{\min}\Big(A^{(0)} - \sum_{k=1}^{n} a_{kk} A_k^{(0)}\Big), \quad \lambda_n^* = \lambda_{\max}\Big(A^{(0)} - \sum_{k=1}^{n} a_{kk} A_k^{(0)}\Big),$$

$$\lambda_0 = \max\{|\lambda_1|, |\lambda_n|\}.$$

Applying Theorem 2 to the additive inverse eigenvalue problem, we get the following corollary.

Corollary. *Suppose that* $\lambda_n > \cdots > \lambda_1, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ *in Problem A. If*

$$d(\lambda) \geq \lambda_{\max}(A) - \lambda_{\min}(A), \tag{2.3}$$

then there exist n real numbers $c_1, \cdots, c_n$ *such that the matrix* $D + A$ *has eigenvalues* $\lambda_1, \cdots, \lambda_n$*, where* $D = \operatorname{diag}(c_1, \cdots, c_n)$.

Remark 2.1. Condition (2.3) is weaker than the sufficient condition for the solubility of the additive inverse eigenvalue problem due to Laborde (see [5]).

§3. Proofs of Theorem 1 and Theorem 2

First we need the following definition.

Definition 1. *Let* $x = (x_1, \cdots, x_n)^T \in I\!R^n$ *and* $y = (y_1, \cdots, y_n)^T \in I\!R^n$ *be given. The vector y is said to majorize the vector x if*

$$\min\left\{\sum_{j=1}^{k} y_{i_j} \quad | 1 \leq i_1 < \cdots < i_k \leq n\right\} \geq \min\left\{\sum_{j=1}^{k} x_{i_j} \quad | 1 \leq i_1 < \cdots < i_k \leq n\right\}$$

for all $k = 1, 2, \cdots, n$ *with equality for* $k = n$*. If the vector y majorizes the vector x, we write* $y \succ x$ (see [4], p192).

Remark 3.1. If we arrange the entries of x and y in increasing order

$$x_{i_1} \leq \cdots \leq x_{i_n}, \quad y_{m_1} \leq \cdots \leq y_{m_n},$$

the defining inequalities can be restated in the equivalent form

$$\sum_{i=1}^{k} y_{m_i} \geq \sum_{j=1}^{k} x_{i_j}$$

for all $k = 1, 2, \cdots, n$ with equality for $k = n$.

The proofs of Theorems 1 and 2 will be based on the following lemmas.

Lemma 3.1. *Let* $x = (x_i) \in I\!R^n$ *and* $y = (y_i) \in I\!R^n$*. Then* $y \succ x$ *if and only if there exists a doubly stochastic matrix S such that* $y = Sx$ *(an* $n \times n$ *doubly stochastic matrix is a matrix which has* n^2 *nonnegative entries such that the sum of entries in every row and every column is* 1). (see [4], p197)

Lemma 3.2. *Let* $B = (b_{ij})$ *be an* $n \times n$ *Hermitian matrix. The vector of diagonal entries of B majorizes the vector of eigenvalues of B* (see [4], p193).

Lemma 3.3. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T \in \mathbb{R}^n$. *Set*

$$\mathcal{K}_\lambda = \{x \in \mathbb{R}^n \ : \ x \succ \lambda\}.$$

Then $\mathcal{K}_\lambda$ is a nonempty convex compact set in $\mathbb{R}^n$.

Proof. (1) Since $\lambda \in \mathcal{K}_\lambda$, $\mathcal{K}_\lambda \neq \phi$.

(2) For any $x \in \mathcal{K}_\lambda$, by Lemma 3.1 there exists a doubly stochastic matrix S such that $x = S\lambda$. Therefore

$$\|x\|_\infty \leq \|S\|_\infty \|\lambda\|_\infty = \|\lambda\|_\infty.$$

This shows that $\mathcal{K}_\lambda$ is bounded.

(3) Suppose that $x_m \in \mathcal{K}_\lambda$ and $x \in \mathbb{R}^n$, and $\lim_{m\to\infty} x_m = x$. By Lemma 3.1 there exists a doubly stochastic matrix S_m such that $x_m = S_m\lambda$ for every $m = 1, 2, \cdots$. Noting that $\|S_m\|_\infty = 1$ for $m = 1, 2, \cdots$, we know that there exists a subsequence $\{S_{m_i}\}_{i=1}^\infty$ of $\{S_m\}_{m=1}^\infty$ and an $n \times n$ real matrix S such that $\lim_{m\to\infty} S_{m_i} = S$. It is easy to prove that S is a doubly stochastic matrix . Therefore

$$x = \lim_{i\to\infty} x_{m_i} = \lim_{i\to\infty} S_{m_i}\lambda = S\lambda \in \mathcal{K}_\lambda.$$

It follows that $\mathcal{K}_\lambda$ is closed.

(4) Let $x, y \in \mathcal{K}_\lambda$. By Lemma 3.1 there exists doubly stochastic matrices S_1 and S_2 such that $S_1\lambda = x$ and $S_2\lambda = y$. Thus

$$tx + (1-t)y = (tS_1 + (1-t)S_2)\lambda, \quad 0 \leq t \leq 1.$$

Noting that $tS_1 + (1-t)S_2$ is a doubly stochastic matrix, we know that $tx + (1-t)y \in \mathcal{K}_\lambda$. This shows that $\mathcal{K}_\lambda$ is a convex set.

Now we prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Without loss of generality we can suppose that $A^{(0)} \neq 0$. Set

$$\mathcal{D}_\lambda = \{x = (x_i) \in \mathbb{R}^n : \quad x \in \mathcal{K}_\lambda, \quad x_1 \leq \cdots \leq x_n\}.$$

Using Lemma 3.3 we easily prove that $\mathcal{D}_\lambda$ is a nonempty convex compact set in $\mathbb{R}^n$.

Let $x \in \mathcal{D}_\lambda$. Then $x \in \mathcal{K}_\lambda$, i. e. , $x \succ \lambda$. By Definition 1 we have

$$\sum_{i=1}^k x_i \geq \sum_{i=1}^k \lambda_i, \qquad k = 1, 2, \cdots, n-1,$$

and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n \lambda_i.$$

It follows that

$$x_1 \geq \lambda_1 > 0$$

and

$$x_n = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} x_i \leq \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \lambda_i = \lambda_n.$$

Therefore

$$\lambda_n \geq x_i \geq \lambda_1, \qquad 1 \leq i \leq n. \tag{3.1}$$

Now let f be the continuous map with

$$f: \quad \mathcal{D}_\lambda \to I\!R^n \quad \text{and} \quad f(x) = (f_1(x), \cdots, f_n(x))^T$$

where $f_i(x) = \lambda_i + x_i - \lambda_i(x)$, $\lambda_i(x)$ are eigenvalues of the matrix $D^{\frac{1}{2}}AD^{\frac{1}{2}}$, $D = \text{diag}\,(x_i)$, $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$. Applying Courant-Fischer's theorem (see [4], p179), we have

$$x_i + \lambda_{\min}(D^{\frac{1}{2}}A^{(0)}D^{\frac{1}{2}}) \leq \lambda_i(x) \leq x_i + \lambda_{\max}(D^{\frac{1}{2}}A^{(0)}D^{\frac{1}{2}}), \tag{3.2}$$

for $i = 1, 2, \cdots, n$.

Since $A^{(0)} \neq 0$ and $\text{tr}(A^{(0)}) = 0$, $\lambda_{\max}(A^{(0)}) > 0$ and $\lambda_{\min}(A^{(0)}) < 0$. By Rayleigh-Ritz's theorem (see [4], p176), we have

$$\begin{aligned}\lambda_{\max}(D^{\frac{1}{2}}A^{(0)}D^{\frac{1}{2}}) &= \max_{y\neq 0} y^H D^{\frac{1}{2}}A^{(0)}D^{\frac{1}{2}}y \Big/ y^H y = \max_{y\neq 0} y^H A^{(0)}y \Big/ y^H D^{-1}y \\ &\leq \max_{y\neq 0} y^H y \Big/ y^H D^{-1}y \max_{y\neq 0} y^H A^{(0)}y \Big/ y^H y = x_n\lambda_{\max}(A^{(0)}) \leq \lambda_n\lambda_{\max}(A^{(0)})\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\lambda_{\min}(D^{\frac{1}{2}}\ A^{(0)}D^{\frac{1}{2}}) &= \min_{y\neq 0} y^H D^{\frac{1}{2}}A^{(0)}D^{\frac{1}{2}}y \Big/ y^H y \\ &= \min_{y\neq 0} y^H A^{(0)}y \Big/ y^H D^{-1}y = \min_{y\neq 0} y^H y \Big/ y^H D^{-1}y\ y^H A^{(0)}y \Big/ y^H y\,.\end{aligned} \tag{3.4}$$

Since $\lambda_{\min}(A^{(0)}) = \min_{y\neq 0} y^H A^{(0)}y \Big/ y^H y < 0$ and $0 < y^H y \Big/ y^H D^{-1}y \leq x_n \leq \lambda_n$ for any $y \neq 0$, we have for any $y \neq 0$

$$\begin{aligned} y^H y \Big/ y^H D^{-1}y\ y^H A^{(0)}y \Big/ y^H y &\geq y^H y \Big/ y^H D^{-1}y \min_{y\neq 0} y^H A^{(0)}y \Big/ y^H y \\ &\geq \lambda_n \min_{y\neq 0} y^H A^{(0)}y \Big/ y^H y = \lambda_n\lambda_{\min}(A^{(0)}).\end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$\lambda_{\min}(D^{\frac{1}{2}}A^{(0)}D^{\frac{1}{2}}) \geq \lambda_n\lambda_{\min}(A^{(0)}). \tag{3.6}$$

Applying (3.2), (3.3) and (3.6), we have

$$\begin{aligned} f_{i+1}(x) - f_i(x) &= (\lambda_{i+1} - \lambda_i) + (x_{i+1} - \lambda_{i+1}(x) - x_i + \lambda_i(x)) \\ &\geq d(\lambda) + (-\lambda_n \lambda_{\max}(A^{(0)}) + \lambda_n \lambda_{\min}(A^{(0)})) > 0 \end{aligned} \tag{3.7}$$

for $i = 1, 2, \cdots, n-1$. Note that $x = (x_1, \cdots, x_n)^T$ is the vector of diagonal entries of $D^{\frac{1}{2}} A D^{\frac{1}{2}}$. Applying Lemma 3.2 we know that

$$\lambda \succ \lambda(x),$$

that is

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} \lambda_i(x) \tag{3.8}$$

for all $k = 1, 2, \cdots, n$ with equality for $k = n$. It follows from (3.7) and (3.8) that

$$\begin{aligned} &\min \left\{ \sum_{j=1}^{k} f_{i_j}(x) \ : \ 1 \leq i_1 < \cdots < i_k \leq n \right\} = \textstyle\sum_{j=1}^{k} f_j(x) \\ &= \sum_{i=1}^{k} (\lambda_i + x_i - \lambda_i(x)) \geq \sum_{i=1}^{k} \lambda_i = \min \left\{ \sum_{j=1}^{k} \lambda_{i_j} \ : \ 1 \leq i_1 < \cdots < i_k \leq n \right\} \end{aligned} \tag{3.9}$$

for all $k = 1, 2, \cdots, n$ with equality for $k = n$. From (3.7) and (3.9) we know that $f(x) \in \mathcal{D}_\lambda$. Applying Brouwer's fixed-point theorem, we thus have $x \in \mathcal{D}_\lambda$ such that $f(x) = x$, that is, $\lambda(x) = \lambda$. In other words, Problem M has a solution $x \in \mathcal{D}_\lambda$. The proof of Theorem 1 is completed.

Proof of Theorem 2. Let

$$a = (a_{11}, \cdots, a_{nn})^T$$

and

$$\mathcal{D}_\lambda(a) = \{x = (x_i) \in I\!R^n \ : x + a \in \mathcal{K}_\lambda, \quad x_1 + a_{11} \leq \cdots \leq x_n + a_{nn}\}.$$

Using Lemma 3.3 we easily prove that $\mathcal{D}_\lambda(a)$ is a nonempty convex compact set in $I\!R^n$.

For any $x \in \mathcal{D}_\lambda(a)$, let

$$A(x) = A_0 + \sum_{k=1}^{n} x_k A_k$$

and let its eigenvalues be $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$. Now let f be the continuous map with

$$f : \mathcal{D}_\lambda(a) \to I\!R^n \quad \text{and} \quad f(x) = (f_1(x), \cdots, f_n(x))^T$$

where $f_i(x) = \lambda_i + x_i - \lambda_i(x)$ for $i = 1, 2, \cdots, n$.

For any $x \in \mathcal{D}_\lambda(a)$ we have $x + a \in \mathcal{K}_\lambda$. It implies that

$$\lambda_1 \le x_i + a_{ii} \le \lambda_n$$

for all $i = 1, 2, \cdots, n$. Therefore

$$|x_i + a_{ii}| \le \max\{|\lambda_1|, |\lambda_n|\} = \lambda_0. \tag{3.10}$$

Moreover, applying Courant-Fischer's theorem , we have

$$x_i + a_{ii} + \lambda_{\min}([A(x)]^{(0)}) \le \lambda_i(x) \le x_i + a_{ii} + \lambda_{\max}([A(x)]^{(0)}) \tag{3.11}$$

for $i = 1, 2, \cdots, n$.

Now let $x \in \mathcal{D}_\lambda(a)$. We shall prove that $f(x) \in \lambda_n(a)$. A given $x \in \lambda_n(a)$ may satisfy

(i) $x_i + a_{ii} \ge 0$ for $i = 1, 2, \cdots, n$, or

(ii) $x_i + a_{ii} \le 0$ for $i = 1, 2, \cdots, n$, or

(iii) $x_1 + a_{11} \le 0$ and $x_n + a_{nn} \ge 0$.

Without loss of generality we can assume that x satisfies condition (iii). Then there exists $j_0 \in \{1, 2, ..., n\}$ such that $x_{j_0} + a_{j_0 j_0} \le 0$ but $x_{j_0+1} + a_{j_0+1, j_0+1} \ge 0$. We thus have

$$\begin{aligned}
\lambda_{\max}([A(x)]^{(0)}) &= \max_{\|y\|_2=1} y^H [A(x)]^{(0)} y = \max_{\|y\|_2=1} y^H (A^{(0)} + \sum_{k=1}^{n} x_k A_k^{(0)}) y \\
&= \max_{\|y\|_2=1} [y^H (A^{(0)} - \sum_{k=1}^{n} a_{kk} A_k^{(0)}) y + \sum_{k=1}^{n} (a_{kk} + x_k) y^H A_k^{(0)} y] \\
&\le \max_{\|y\|_2=1} y^H (A^{(0)} - \sum_{k=1}^{n} a_{kk} A_k^{(0)}) y + \sum_{k=1}^{n} \max_{\|y\|_2=1} (a_{kk} + x_k) y^H A_k^{(0)}) y \\
&= \lambda_n^* - \sum_{k=1}^{j_0} |a_{kk} + x_k| \min_{\|y\|_2=1} y^H A_k^{(0)}) y + \sum_{k=j_0+1}^{n} |a_{kk} + x_k| \max_{\|y\|_2=1} y^H A_k^{(0)}) y \\
&= \lambda_n^* - \sum_{k=1}^{j_0} |a_{kk} + x_k| \lambda_{\min}(A_k^{(0)}) + \sum_{k=j_0+1}^{n} |a_{kk} + x_k| \lambda_{\max}(A_k^{(0)}).
\end{aligned}$$

Note that $\lambda_{\max}(A_k^{(0)}) \ge 0$ and $\lambda_{\min}(A_k^{(0)}) \le 0$. Applying (3.10) we get

$$\lambda_{\max}([A(x)]^{(0)}) \le \lambda_n^* - \lambda_0 \sum_{k=1}^{j_0} \lambda_{\min}(A_k^{(0)}) + \lambda_0 \sum_{k=j_0+1}^{n} \lambda_{\max}(A_k^{(0)}). \tag{3.12}$$

Similarly, we can prove that

$$\lambda_{\min}([A(x)]^{(0)}) \geq \lambda_1^* - \lambda_0 \sum_{k=1}^{j_0} \lambda_{\max}(A_k^{(0)}) + \lambda_0 \sum_{k=j_0+1}^{n} \lambda_{\min}(A_k^{(0)}). \tag{3.13}$$

From (3.11) , (3.12) and (3.13), we have

$$x_i + a_{ii} + \lambda_1^* - \lambda_0 \sum_{k=1}^{j_0} \lambda_{\max}(A_k^{(0)}) + \lambda_0 \sum_{k=j_0+1}^{n} \lambda_{\min}(A_k^{(0)}) \leq \lambda_i(x)$$

$$\leq x_i + a_{ii} + \lambda_n^* - \lambda_0 \sum_{k=1}^{j_0} \lambda_{\min}(A_k^{(0)}) + \lambda_0 \sum_{k=j_0+1}^{n} \lambda_{\max}(A_k^{(0)}) \tag{3.14}$$

for all $i = 1, 2, \cdots, n$. Therefore we have

$$(f_{i+1,i+1}(x) + a_{i+1,i+1}) - (f_i(x) + a_{ii}) = (\lambda_{i+1} + x_{i+1} - \lambda_{i+1}(x) + a_{i+1,i+1})$$

$$-(\lambda_i + x_i - \lambda_i(x) + a_{ii}) \geq d(\lambda) + [-\lambda_n^* + \lambda_0 \sum_{k=1}^{j_0} \lambda_{\min}(A_k^{(0)})$$

$$-\lambda_0 \sum_{k=j_0+1}^{n} \lambda_{\max}(A_k^{(0)}) + \lambda_1^* - \lambda_0 \sum_{k=1}^{j_0} \lambda_{\max}(A_k^{(0)}) + \lambda_0 \sum_{k=j_0+1}^{n} \lambda_{\min}(A_k^{(0)})]$$

$$= d(\lambda) - [\lambda_n^* - \lambda_1^* + \lambda_0 \sum_{k=1}^{n} (\lambda_{\max}(A_k^{(0)}) - \lambda_{\min}(A_k^{(0)}))] \geq 0 \tag{3.15}$$

for $i = 1, 2, \cdots, n-1$.

Note that $x + a$ is the vector of diagonal entries of $A(x)$. Applying Lemma 3.2 we have

$$\sum_{i=1}^{k} (x_i + a_{ii}) \geq \sum_{i=1}^{k} \lambda_i(x) \tag{3.16}$$

for all $k = 1, \cdots, n$ with equality for $k = n$.

Combining (3.15) with (3.16), we get

$$\min \left\{ \sum_{j=1}^{k} (f_{i_j}(x) + a_{i_j,i_j}) : 1 \leq i_1 < \cdots < i_k \leq n \right\}$$

$$\geq \min \left\{ \sum_{j=1}^{k} \lambda_{i_j} \quad : \quad 1 \leq i_1 < \cdots < i_k \leq n \right\}$$

for all $k = 1, \cdots, n$ with equality for $k = n$. Therefore $f(x) \in \mathcal{D}_\lambda(a)$. Applying Brouwer's fixed-point theorem we thus have $x \in \mathcal{D}_\lambda(a)$ such that $f(x) = x$, that is, $\lambda(x) = \lambda$. In other words, Problem G has a solution $x \in \mathcal{D}_\lambda(a)$. The proof of Theorem 2 is completed.

§4. Examples

Example 1. Problem G, where

$$A = I + (1 + \tfrac{1}{16})B, \quad A_k = e_k e_k^T + \frac{1}{48}B, \qquad k = 1,2,3,$$
$$\lambda_1 = -4, \quad \lambda_2 = 0, \quad \lambda_3 = 4$$

where I is the 3×3 identity matrix, e_k is the kth column of I, and

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{4.1}$$

Then

$$\lambda_1^* = -1, \quad \lambda_3^* = 2, \quad d(\lambda) = 4, \quad \lambda_0 = 4,$$
$$\lambda_{\min}(A_k^{(0)}) = -\frac{1}{48}, \quad \lambda_{\max}(A_k^{(0)}) = \frac{1}{24}, \quad k = 1,2,3,$$

and

$$d(\lambda) = 4 > \frac{15}{4} = (-\lambda_1^* + \lambda_3^*) + \lambda_0 \sum_{k=1}^{3} (-\lambda_{\min}(A_k^{(0)}) + \lambda_{\max}(A_k^{(0)})).$$

By Theorem 2, we know that there exist 3 real numbers c_1, c_2, c_3 such that $c_1 \leq c_2 \leq c_3$, $1 + c_1 \geq -4$, $2 + c_1 + c_2 \geq -4$, $3 + c_1 + c_2 + c_3 = 0$, and $A + \sum_{k=1}^{3} c_k A_k$ has eigenvalues $-4, 0, 4$. But the matrices and the numbers above do not satisfy the hypothesis of Theorem 1 in [7] and Theorem 6 in [1]. Therefore, Theorem 2 when used for real symmetric matrices is not contained in Theorem 1 of [7] and Theorem 6 of [1].

Example 2. Let

$$A = B, \quad \lambda_1 = -3, \quad \lambda_2 = 0, \quad \lambda_3 = 3,$$

where B is defined by (4.1). Then it satisfies condition (2.3) but not any conditions due to Laborde [5], Morel [6] and Hadeler ([2], [3]).

Example 3. Let

$$A = I + 0.15B, \qquad \lambda = (1, 10, 19)^T,$$

where B is defined by (4.1). Then

$$\lambda_{\min}(A^{(0)}) = -0.15, \quad \lambda_{\max}(A^{(0)}) = 0.3.$$

and

$$d(\lambda) = 9 > 8.55 = \lambda_3(\lambda_{\max}(A^{(0)}) - \lambda_{\min}(A^{(0)})).$$

Applying Theorem 1 we know that there exist 3 real numbers c_1, c_2, c_3 such that diag $(c_1, c_2, c_3)A$ has eigenvalues 1, 10, 19. But using the sufficient condition for the solubility of Problem M due to Hadeler [2], we cannot tell whether Problem M with the data above has a solution. In fact, for the matrix above there are no positive numbers $\lambda_1, \lambda_2, \lambda_3$ such that the matrix A above and the numbers $\lambda_1, \lambda_2, \lambda_3$ satisfy the sufficient condition of Hadeler.

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