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ON THE STABILITY OF BIQUADRATIC-BILINEAR VELOCITY-PRESSURE FINITE ELEMENTS *1)

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Abstract

In this paper, the stability of biquadratic-bilinear velocity-pressure finite elements are discussed for the stationary Stokes problem. It is proved that there exist constants $c, c' > 0$ independent of h such that

$$c'h \geq \inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{\vec{v}_h \in V_h \\ \vec{v}_h \neq 0}} \frac{(\operatorname{div} \vec{v}_h, q_h)}{\|q_h\|_h |\vec{v}_h|_1} \geq ch$$

Hence a question in [1] is answered.

§1. Introduction

Let us consider the mixed finite element methods for the stationary Stokes problem.It is well known[1],[2] that many usual finite elements do not satisfy the discrete classical Babuska-Brezzi condition and there exist the checker-board modes. G.F. Carey and R. Krishnan[1] proposed the analogy condition,i.e.,there exists β_h dependent on h,such that

$$\sup_{\substack{\vec{v}_h \in V_h \\ \vec{v}_h \neq 0}} \frac{(\operatorname{div} \vec{v}_h, q_h)}{|\vec{v}_h|_1} \geq \beta_h \|q_h\|_h, \quad \forall q_h \in Q_h \tag{1.1}$$

where $V_h \in [H_0^1(\Omega)]^2$ and $Q_h \in L_0^2(\Omega)$ are two finite element spaces, $H_0^1(\Omega)$ denotes the usual Sobolev space with the seminorm $|\cdot|_1$, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, and

$$\|q_h\|_h = \inf_{q_h^* \in KerB_h^*} \|q_h + q_h^*\|_0 \tag{1.2}$$

$$KerB_h^* = \{q_h \in Q_h : (q_h, div\vec{v}_h) = 0, \forall \vec{v}_h \in V_h\} \tag{1.3}$$

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Obviously, it is important to estimate the order of β_h in condition (1.1). G. F. Carey et al[1] have got $\beta_h = O(h)$ and asked whether this estimate is optimal. In this paper, we discuss the biquadratic-bilinear velocity-pressure (Q_2/Q_1) finite elements and prove that the result $\beta_h = O(h)$ in (1.1) is optimal. In addition, it is shown that the result $\beta_h = O(h^{\frac{1}{2}})$ in [4] is not true.

§2. Q_2/Q_1 Finite Elements

Assume the bounded region Ω is just the unit square $0 < x, y < 1$. Let J_h be a subdivision of Ω, i.e., Ω is divided into subsquares of side $h = \frac{1}{N}$ ($N > 1$ integer). Denote subsquares

$$K_{ij} = [ih, (i+1)h] \times [jk, (j+1)h]$$

for $i, j = 0, 1, 2, \cdots, N-1$.

Define the Q_2/Q_1 finite element spaces:

$$V_h = \{\vec{v}_h \in (C^0(\bar{\Omega}))^2; \vec{v}_h|_{K_{ij}} \in [Q_2(K_{ij})]^2\} \cap [H_0^1(\Omega)]^2, \tag{2.1}$$

$$Q_h = \{q_h \in L_0^2(\Omega); q_h|_{K_{ij}} \in Q_1(K_{ij})\}, \ i, j = 0, 1, 2, \cdots, N-1 \tag{2.2}$$

where $Q_2(K)$ and $Q_1(K)$ denote the biquadratic polynomial space and bilinear polynomial space on K and $L_0^2(\Omega) = \left\{q \in L_2(\Omega), \int_\Omega q dx = 0\right\}$.

For two finite element spaces V_h, and Q_h defined by (2.1) and (2.2) we shall show the following result.

Theorem 1. *There exist constants $c, c' > 0$ independent of h such that*

$$c'h \geq \inf_{q_h \in Q_h} \sup_{\vec{v}_h \in V_h} \frac{(\operatorname{div} \vec{v}_h, q_h)}{|\vec{v}_h|_1 \|q_h\|_h} \geq ch.$$

The inequality on the right has been proved in [1]. Here we will deal with only the inequality on the left. Let P_{ij}^e denote the standard biquadratic basis function,equal to 1 at A_{ij}^e and zero at the other nodes,where $A_{ij}^e (e = 1, 2, \cdots, 9)$ denote the nodes in K_{ij} (see Fig.1)

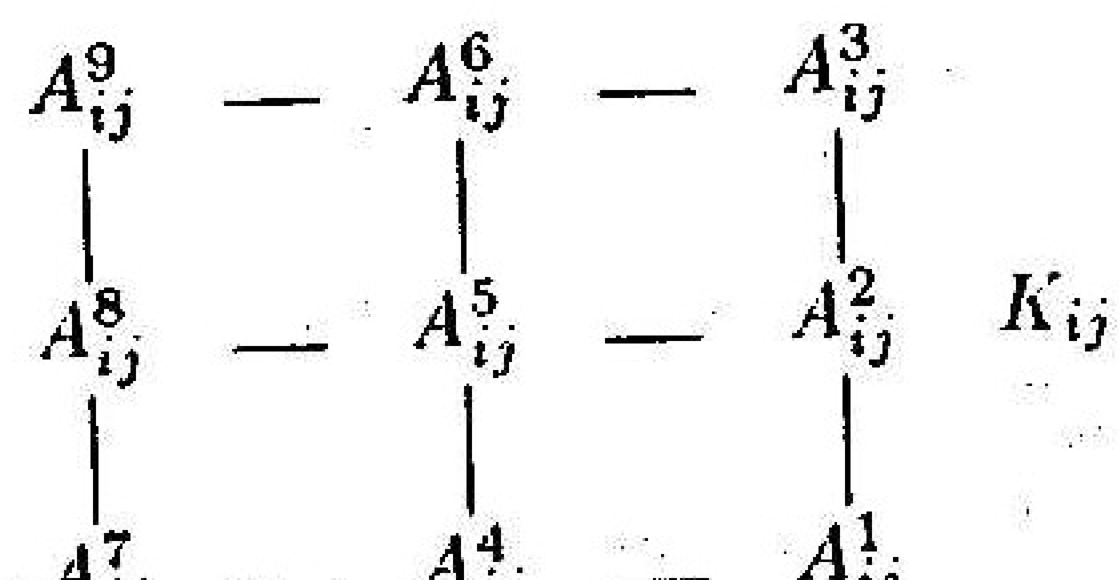

Fig.1

Let $\vec{v}_h = (z_h, w_h)$,then we can write

$$z_h = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{e=1}^{9} z_h(A_{ij}^e) p_{ij}^e, \tag{2.3}$$

$$w_h = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{e=1}^{9} w_h(A_{ij}^e) p_{ij}^e. \tag{2.4}$$

To complete the proof of the theorem we need to prove two lemmas.

Lemma 1. *If z_h is as defined by* (2.3), *then*

$$[z_h(A_{ij}^1) - 2z_h(A_{ij}^2) + z_h(A_{ij}^3)]^2 \le 12 \iint_{K_{ij}} (\frac{\partial z_h}{\partial y})^2 dxdy \le 12|z_h|_{1,K_{ij}}^2$$

for all $i, j = 1, 2, \cdots, N-1$.

Proof. By scaling argument and simple computing.

Lemma 2. *If*

$$\tilde{q}_h \in Q_h, \tilde{q}_h(B_{ij}^1) = \tilde{q}_h(B_{ij}^3) = i+1, \tilde{q}_h(B_{ij}^2) = \tilde{q}_h(B_{ij}^4) = -(i+1)$$

for all $i, j = 0, 1, \cdots, N-1$, where B_{ij}^e denotes the Gauss point of element K_{ij} with 2×2 Gauss quadrature (*see* Fig.2)

B_{ij}^4	B_{ij}^3
B_{ij}^1	B_{ij}^2

Fig.2

then

$$\|\tilde{q}_h\|_h = \sqrt{\frac{1}{12}(N^2 - 1)}$$

Proof. [1] showed that if $q_h^* \in KerB_h^*$,then

$$q_h^*(B_{ij}^1) = q_h^*(B_{ij}^3) = a, \quad q_h^*(B_{ij}^2) = B_h^*(B_{ij}^4) = b$$

where a, b are constants. Thus,

$$\|\tilde{q}_h\|_h^2 = \inf_{a,b \in R} \{\sum_{i,j=0}^{N-1} \frac{h^2}{4} 2[(i+1-a)^2 + (-i-1-b)^2]\}$$

$$= \inf_{a,b \in R} \{\sum_{i=1}^{N} \frac{h}{2} [(i-a)^2 + (i+b)^2]\}$$

Taking $a = \frac{N+1}{2}, b = -\frac{N+1}{2}$, we get

$$\|\tilde{q}_h\|_h^2 = 2 * \frac{h}{2} N(N+1)\frac{1}{12}(N-1) = \frac{1}{12}(N^2-1).$$

The proof is completed.

Finally we return to the proof of Theorem 1. A calculation shows that, if $\tilde{q}_h$ is defined as by Lemma 2,then

$$\begin{aligned}(\operatorname{div}\vec{v}_h, \tilde{q}_h) &= \sum_{i,j=0}^{N-1} h[-\frac{1}{3}w_h(A_{ij}^1) + \frac{2}{3}w_h(A_{ij}^2) - \frac{1}{3}w(A_{ij}^3)] \\ &\le \frac{1}{3}\{\sum_{i,j=0}^{N-1}[w_h(A_{ij}^1) - 2w_h(A_{ij}^2) + w_h(A_{ij}^3)]^2\}^{\frac{1}{2}} \\ &\le \frac{1}{3}|w_h|_{1,\Omega} \le \frac{1}{3}|\vec{v}_h|_{1,\Omega}.\end{aligned}$$

By Lemma 2

$$(\operatorname{div}\vec{v}_h, \tilde{q}_h) \le \frac{1}{3}|\vec{v}_h|_{1,\Omega}\|\tilde{q}_h\|_h \frac{\sqrt{12}}{\sqrt{N^2-1}} \le c'h|\vec{v}_h|_{1,\Omega}\|\tilde{q}_h\|_h$$

Hence the inequality on the left of Theorem 1 holds.

Remark. By Theorem 1 we know that the estimate $\beta_h = O(h)$ in (1.1) is not improvable. Hence $\beta_h = O(h^{\frac{1}{2}})$ in [4] is not true.

References

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