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THE ENTROPY CONDITION FOR IMPLICIT TVD SCHEMES*[1)]

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Abstract

A class of implicit trapezoidal TVD schemes is proven to satisfy a discrete convex entropy inequality and the solution sequence of such implicit trapezoidal schemes converges to the physically relevant solution for genuinely nonlinear scalar conservation laws. The results are extended for a class of generalized implicit one-leg TVD schemes.

§1. Introduction

We consider a hyperbolic conservation law:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}, \quad t>0, \tag{1.1}$$
$$u(x,0)=u^0(x), \quad -\infty<x<\infty.$$

If there exists a convex function $V(u)$ and a differentiable function $F(u)$ such that $V'f'=F'$, the admissible weak solution of (1.1), satisfies

$$\frac{\partial V(u)}{\partial t}+\frac{\partial F(u)}{\partial x}\le 0 \tag{1.2}$$

in a weak sense, then inequality (1.2) is called the entropy inequality. The pair (V,F) is called the entropy pair. It is well known that the weak solution of (1.1) safisfying the entropy inequality (1.2) for all entropy pairs is unique. The weak admissible solution satisfying the entropy inequality is called the entropy solution.

In the genuinely nonlinear case, where f is, say, strictly convex, if the admissible weak solution of (1.1) satisfies one special convex entropy inequality (1.2), the weak solution is unique (DiPerna [1]).

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The idea of Total Variation Diminishing Schemes was proposed by Harten[2]. The solutions of the conservative TVD scheme have a subsequence which converges to the weak solution of (1.1) in a weak sence. Unfortunately, the limit weak solution is not always unique. Hence, it is necessary to impose an entropy inequality to the TVD scheme to ensure the convergence of the solution sequence to a unique physically relevant solution.

The entropy inequality for first order accurate TVD schemes have been widely studied[3–6,8,9]. Osher[4] showed the entropy inequality for semidiscrete, second order accurate generalized MUSCL schemes. Osher and Chakrayarthy[5] constructed semidiscrete, second order accurate TVD schemes, and proved that those schemes satisfy a semi-discrete entropy inequality. Osher and Tadmor[6] considered the entropy inequality for fully discrete second order accurate explicit TVD schemes. The result given in [6] has the strict CFL limitation. As to first order accurate explicit TVD schemes, the limitation of CFL number is also strict[6,8,9]. Hence, we should consider the implicit TVD scheme to relax the limitation. Our discussion is motivated by papers [4–6, 10].

In Section 2, we give some preliminary results for implicit TVD schemes. In Section 3, we establish the entropy inequality for implicit trapezoidal TVD schemes. In Section 4 and 5, two examples of second order accurate implicit trapezoidal TVD schemes are presented. In the last section we give some explanation about our results.

§2. Preliminary Results on Implicit TVD Schemes

We consider the semidiscrete approximation to (1.1) in conservative form:

$$\frac{du_j}{dt}+\frac{1}{\Delta x}\left(h_{j+\frac{1}{2}}-h_{j-\frac{1}{2}}\right)=0, \tag{2.1}$$

$$u_j(0)=\frac{1}{\Delta x}\int_{\Gamma_j}u^0(x)dx,$$

where

$$h_{j+\frac{1}{2}}=h(u_{j-l+1},\cdots,u_{j+l},f,\Delta x),\quad \lambda=\frac{\Delta t}{\Delta x}, \tag{2.2}$$

$$\Gamma_j=\{x:(j-1/2)\Delta x\le x<(j+1/2)\Delta x\}.$$

The corresponding discrete entropy inequality is written as

$$\frac{dU(u_j(t))}{dt}+\frac{1}{\Delta x}\left(F_{j+\frac{1}{2}}-F_{j-\frac{1}{2}}\right)\le 0 \tag{2.3}$$

where U is the convex function of u, and

$$F(w) = F(w, w, \cdots, w, f, \Delta x) = \int^{w} U' f' dw \tag{2.4}$$

(U, F) is called the consistent numerical entropy pair.

The consistent scheme (2.1) is called E-scheme[3] and the numerical flux h is called E-flux if

$$(u_{j+1} - u_j)\Big(h_{j+\frac{1}{2}} - f(u)\Big) \leq 0, \quad \forall j \tag{2.5}$$

for all u between u_{j+1} and u_j.

Theorem[6]**.** *Consider semidiscrete approximation* (2.1) *and assume the entropy inequality* (2.3) *holds for all consistent entropy pairs* (U, F)*. Then the numerical flux* h *is an* E*-flux in any separating interval.*

E-scheme has at most first order accuracy[3]. Hence, if we want to impose an entropy inequality on the second order accurate TVD scheme, the entropy inequality should be special. Osher[4–6] considered special discrete entropy inequalities for second order accurate TVD schemes.

In this paper, we consider the fully discretized implicit TVD scheme. The implicit TVD scheme was originally developed by Harten[2], and further developed by Yee et al[11,12].

Consider a family of conservative schemes of the form:

$$\begin{aligned} &u_j^{n+1} + \lambda\eta\Big(h(u_{j-l+1}^{n+1}, \cdots, u_{j+l}^{n+1}) - h(u_{j-l}^{n+1}, \cdots u_{j+l-1}^{n+1})\Big) \\ &\quad = u_j^n - \lambda(1-\eta)\Big(h(u_{j-l+1}^n, \cdots, u_{j+l}^n) - h(u_{j-l}^n, \cdots, u_{j+l-1}^n)\Big), \quad 0 \leq \eta \leq 1. \end{aligned} \tag{2.6}$$

Using the notation of [2], we let

$$(L \cdot u)_j = u_j + \lambda\eta\Big(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}\Big), \tag{2.7}$$

$$(R \cdot u)_j = u_j - \lambda(1-\eta)\Big(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}\Big). \tag{2.8}$$

Define the total variation of a mesh function u:

$$TV(u) = \sum_{j=-\infty}^{+\infty} |u_{j+1} - u_j| = \sum_{j=-\infty}^{+\infty} \Big|\Delta_{j+\frac{1}{2}} u\Big|, \tag{2.9}$$

where

$$\Delta_{j+\frac{1}{2}} u = u_{j+1} - u_j.$$

Let $BV(R) = \{u : TV(u) < \infty\}$, the bounded variation function space.

Now (2.6) can be rewritten as

$$L \cdot u^{n+1} = R \cdot u^n. \tag{2.10}$$

Scheme (2.10) is called TVD (total variation diminishing) if $TV(u^{n+1}) \le TV(u^n)$.

Harten[2] gave the following sufficient condition for (2.10) to be TVD if

$$TV(R \cdot u) \le TV(u), \tag{2.11a}$$

$$TV(L \cdot u) \ge TV(u). \tag{2.11b}$$

Assume the numerical flux h in (2.6) is Lipschitz continuous and (2.6) can be written as

$$\begin{aligned} u_j^{n+1} &- \lambda\eta(C^-_{j+1/2}\Delta_{j+1/2}u - C^+_{j-1/2}\Delta_{j-1/2}u)^{n+1} \\ &= u_j^n + \lambda(1-\eta)(C^-_{j+1/2}\Delta_{j+1/2}u - C^+_{j-1/2}\Delta_{j-1/2}u)^n \end{aligned} \tag{2.12}$$

where

$$C^{\pm}_{j\mp 1/2} = C^{\pm}(u_{j-l-1}, \cdots, u_{j+l}). \tag{2.13}$$

If we consider the periodic boundary condition, the sufficient conditions for (2.12) to be TVD are

$$+\infty > C > \lambda C^{\mp}_{j+1/2} \ge 0, \quad \forall j, \tag{2.14}$$

$$\lambda(1-\eta)\left(C^+_{j+1/2} + C^-_{j+1/2}\right) \le 1. \tag{2.15}$$

From now on, we take $\eta = \dfrac{1}{2}$ in (2.6), namely, using the trapezoidal rule to discretize the semidiscrete system (2.1), which is at least second order accurate in the time direction if the numerical flux is smooth (e.g. differentiable numerical flux). The trapezoidal TVD scheme reads:

$$u_j^{n+1} + \frac{\lambda}{2}\left(h^{n+1}_{j+1/2} - h^{n+1}_{j-1/2}\right) = u_j^n - \frac{\lambda}{2}\left(h^n_{j+1/2} - h^n_{j-1/2}\right). \tag{2.16}$$

Lemma 2.1. *Assume* (2.14), (2.15) *are true. Then for the solution of scheme* (2.12) *the following inequality*

$$\sum_j |u_j^{n+1} - u_j^n| < C$$

is valid, where C is a constant independent of Δt and Δx.

With the conventional technique given in [7], we can extract the subsequence from the solution of TVD scheme (2.12), which converges to the weak solution of conservation laws in $L^1(R \times [0,T])$.

§3. The Entropy Inequality

We use the technique given in [10] to discuss the entropy inequality for trapezoidal TVD scheme (2.16) with periodic boundary condition.

We transform u_j^n to v_j^n via the formula

$$v_j^n = u_j^n + \frac{\lambda}{2}(h_{j+1/2}^n - h_{j-1/2}^n). \tag{3.1}$$

The scheme (2.16) is transformed to

$$v_j^{n+1} = v_j^n - \lambda(h_{j+1/2}^n - h_{j-1/2}^n) \tag{3.2}$$

and

$$u_j^{n+1} = \frac{1}{2}(v_j^{n+1} + v_j^n), \tag{3.3}$$

$$h_{j+1/2}^n = h(u_{j-l+1}^n, \cdots, u_{j+l}^n). \tag{3.4}$$

We multiply both sides of (3.2) by u_j^n:

$$u_j^n(v_j^{n+1} - v_j^n) = \frac{1}{2}[(v_j^{n+1})^2 - (v_j^n)^2] = -\lambda u_j^n(h_{j+1/2}^n - h_{j-1/2}^n). \tag{3.5}$$

That is

$$u_j^n(v_j^{n+1} - v_j^n) = -\lambda\Delta_+(u_j^n h_{j-1/2}^n) + \lambda\int_{u_j^n}^{u_{j+1}^n} h_{j+1/2}^n dw, \tag{3.6}$$

where Δ_+ denotes the forward difference. We define

$$F(u_j) = \int_0^{u_j} wf'(w)dw = u_j f(u_j) - \int_0^{u_j} f(w)dw. \tag{3.7}$$

Then

$$\Delta_+ F(u_j) = \Delta_+(u_j f(u_j)) - \int_{u_j}^{u_{j+1}} f(w)dw. \tag{3.8}$$

Hence, (3.6) can be rewritten as

$$\frac{1}{2}[(v_j^{n+1})^2 - (v_j^n)^2] + \Delta_+ F_A(u_j^n) = \lambda\int_{u_j^n}^{u_{j+1}^n}(h_{j+1/2}^n - f(w))dw \tag{3.9}$$

where

$$F_A(u_j^n) = F(u_j^n) + \lambda u_j^n(h_{j-1/2}^n - f(u_j^n)). \tag{3.10}$$

Similarly, using

$$\bar{v}_j^n = u_j^n - \frac{\lambda}{2}(h_{j+1/2}^n - h_{j-1/2}^n) \tag{3.11}$$

we get

$$\bar{v}_j^{n+1} = \bar{v}_j^n - \lambda(h_{j+1/2}^{n+1} - h_{j-1/2}^{n+1}) \tag{3.12}$$

and

$$\frac{1}{2}[(\bar{v}_j^{n+1})^2 - (\bar{v}_j^n)^2] + \Delta_+ F_A(u_j^{n+1}) = \lambda \int_{u_j^{n+1}}^{u_{j+1}^{n+1}} (h_{j+1/2}^n - f(w))dw. \tag{3.13}$$

We combine (3.9) and (3.13) to get

$$\frac{1}{2}[(u_j^{n+1})^2 - (u_j^n)^2] + \frac{1}{2}\Delta_+(F_A(u_j^n) + F_A(u_j^{n+1})) + \frac{\lambda^2}{8}[(h_{j+1/2}^{n+1} - h_{j-1/2}^{n+1})^2$$
$$- (h_{j+1/2}^n - h_{j-1/2}^n)^2] = [\phi(u_j^{n+1}) - \phi(u_j^n)] + \frac{1}{2}\Delta_+(F_A(u_j^{n+1}) + F_A(u_j^n))$$
$$\leq \frac{\lambda}{2}\Big[\int_{u_j^{n+1}}^{u_{j+1}^{n+1}} \left(h_{j+1/2}^{n+1} - f(w)\right) dw + \int_{u_j^n}^{u_{j+1}^n} (h_{j+1/2}^n - f(w))dw\Big] \tag{3.14}$$

where

$$\phi(u_j) = \frac{1}{2}(u_j)^2 + \frac{\lambda^2}{8}\left(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}\right)^2.$$

We call (3.14) the entropy inequality for the trapezoidal implicit TVD scheme (2.16).

Theorem 3.1. *If the semidiscrete approximation numerical flux* (2.1), $h_{j+1/2}$, *satisfies*

$$\lambda \int_{u_j}^{u_{j+1}} (h_{j+1/2} - f(w))dw \leq 0, \quad \forall j, \tag{3.15}$$

then the trapezoidal TVD scheme under assumption (2.14), (2.15) *with periodic boundary condition has a unique limit entropy solution for the genuinely nonlinear hyperbolic conservation law with initial value in BV.*

§4. The Fully Discrete Implicit Second Order Generalized MUSCL Scheme

In [4], Osher considered the convergence of semidiscrete generalized MUSCL schemes for the scalar conservation law. He showed that the generalized MUSCL scheme satisfying the entropy inequality converges to the unique physical solution. Here we prove that the solution of trapezoidal implicit second order accurate generalized MUSCL scheme is also convergent to the unique entropy solution for the genuinely nonlinear conservation law under the same assumption.

First, we introduce Osher's semidiscrete generalized MUSCL scheme[4].

Let

$$h(u_{j+1}, u_j) = h_{j+1/2}$$

be an arbitrary first order accurate numerical flux function. The generalized MUSCL scheme reads:

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta x}\Delta_- h\left(u_{j+1} - \frac{\Delta x}{2}s_{j+1},\ u_j + \frac{\Delta x}{2}s_j\right), \tag{4.1}$$

$$u_j(0) = \int_{\Gamma_j} u^0(x)dx/\Delta x$$

where Δ_- denotes the backward difference and $s_j(l)$ is the slope that satisfies

$$s_j = w_x(x_j, l) + O(\Delta x) \tag{4.2}$$

and

$$w(x_j, l) = \begin{cases} u_j(l), & \text{for } x < x_{j+1/2}, \\ u_{j+1}(l), & \text{for } x \geq x_{j+1/2}\ . \end{cases} \tag{4.3}$$

We discretize (4.1) as the trapezoidal conservative scheme:

$$u_j^{n+1} + \frac{1}{2}\lambda\Delta_- h\left(u_{j+1}^{n+1} - \frac{\Delta x}{2}s_{j+1}^{n+1},\ u_j^{n+1} + \frac{\Delta x}{2}s_j^{n+1}\right)$$
$$= u_j^n - \frac{1}{2}\lambda\Delta_- h\left(u_j^n - \frac{\Delta x}{2}s_{j+1}^n,\ u_j^n + \frac{\Delta x}{2}s_j^n\right). \tag{4.4}$$

Using Osher's technique[4], we can prove the following results for the conservative scheme (4.4).

Lemma 4.1. *Assume $h(u_{j+1}, u_j)$ is C^2 with Lipschitz continuous second partial derivatives in a neighborhood of $w(x_j, l) = u_j(l)$, and*

$$\frac{\Delta x s_{j+1}}{\Delta_+ u_j} = 1 + O(\Delta x).$$

Then the scheme (4.4) is at least second order accurate for the smooth function w.

Lemma 4.2. *If h is a E-flux corresponding to an E-scheme (see (2.5)), the scheme (4.4) is TVD if*

$$0 \leq \frac{\Delta x s_j}{\Delta_+ u_j},\quad \frac{\Delta x s_{j+1}}{\Delta_+ u_j} \leq 1,\quad \text{for all } j. \tag{4.5}$$

Lemma 4.3. *If h is a flux of a monotone scheme, the scheme is* TVD *if*

$$1 \geq \frac{s_{j+1} - s_j}{2\Delta_+ u_j} \Delta x \quad \text{near points where } h_1 \neq 0, \tag{4.6a}$$

$$1 \geq \frac{-s_{j+1} + s_j}{2\Delta_+ u_j} \Delta x \quad \text{near points where } h_0 \neq 0 \tag{4.6b}$$

where

$$h_1 = \frac{\partial h}{\partial u}(u, v), \quad h_0 = \frac{\partial h}{\partial v}(u, v).$$

Now, we give the convergence result for trapezoidal TVD scheme (4.4) :

Theorem 4.4. *The sequence of approximate solutions satisfying* (4.4) *converges a.e. to the unique solution of the scalar convex conservation law* (1.1) *with periodic boundary value provided that the initial data are in* BV, *that for each* j(4.6) *is true, and that, if* $u_j > u_{j+1}$,

$$-\Delta x s_j \leq 2 \max(\min((u_j - \tilde{u}_{j+1/2}), (\tilde{u}_{j-1/2} - u_j)), 0) \tag{4.7}$$

where

$$\tilde{u}_{j+1/2} = \frac{\displaystyle\int_{u_j}^{u_{j+1}} w f'(w) dw}{\Delta_+ f(u_j)} . \tag{4.8}$$

Proof. As in [4], if (4.7) is true, we can prove that

$$\lambda \int_{u_j}^{u_{j+1}} (h_{j+1/2} - f(w)) dw \leq 0 \tag{4.9}$$

with the assumptions (4.6), so that the entropy inequality is imposed on the scheme (4.4). Hence, the solution sequence of (4.4) is convergent to the unique solution of (1.1).

§5. Fully Discrete Osher-Chakravarthy's Second Order TVD Scheme

In [5], Osher and Chakravarthy constructed a class of semidiscrete second order accurate TVD schemes, and proved the convergence of the solution sequence of the scheme to the unique physical relevant solution. In this section, we discretize Osher-Chakravarthy's TVD semidiscrete schemes with trapezoidal rule, and prove similar convergence and uniqueness results.

Let

$$R_j^+ = \frac{(f(u_j) - h(u_j, u_{j-1}))^M}{(f(u_{j+1}) - h(u_{j+1}, u_j))^M}, \tag{5.1a}$$

$$R_j^- = \frac{(h(u_{j+1}, u_j) - f(u_j))^M}{(h(u_j, u_{j-1}) - f(u_{j-1}))^M}, \tag{5.1b}$$

$$\psi(R) = \max(0, \min(R, 1)) \tag{5.2}$$

where

$$(h(u_{j+1}, u_j) - f(u_j))^M = (h(u_{j+1}, u_j) - f(u_j))\left[1 + \frac{a_{j+1/2}^- \Delta_+ h_1(u_j, u_j)\Delta_+ u_j}{h(u_{j+1}, u_j) - f(u_j)}\right], \tag{5.3a}$$

$$(f(u_{j+1}) - h(u_{j+1}, u_j))^M$$

$$= (f(u_{j+1}) - h(u_{j+1}, u_j))\left[1 - \frac{a_{j+1/2}^+ \Delta_+ h_0(u_j, u_j)\Delta_+ u_j}{f(u_{j+1}) - h(u_{j+1}, u_j)}\right] \tag{5.3b}$$

where $h(u_{j+1}, u_j)$ is E-flux and $a_{j+1/2}^{\pm}$ are both positive numbers;

$$h_1(u, v) = \frac{\partial h}{\partial u}(u, v), \quad h_0(u, v) = \frac{\partial h}{\partial v}(u, v).$$

Osher-Chakravarthy's semidiscrete scheme reads[5]:

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta x}\Delta_- H^{ac}(u_{j+2}, u_{j+1}, u_j, u_{j-1}), \quad u_j(0) = \frac{1}{\Delta x}\int_{\Gamma_j} u(s)ds \tag{5.4}$$

where

$$H^{ac}(u_{j+2}, u_{j+1}, u_j, u_{j-1}) = h(u_{j+1}, u_j) - \frac{1}{2}\psi(R_{j+1}^-)(h(u_{j+1}, u_j) - f(u_j))^M$$
$$+ \frac{1}{2}\psi(R_j^+)(f(u_{j+1} - h(u_{j+1}, u_j)))^M. \tag{5.5}$$

The scheme (5.4) is TVD if $a_{j+1/2}^{\pm}$ are such that

$$\left|a_{j+1/2}^-(h_1(u_{j+1}, u_{j+1}) - h_1(u_j, u_j)\right| \le -\frac{h(u_{j+1}, u_j) - f(u_j)}{\Delta_+ u_j}, \tag{5.6a}$$

$$\left|a_{j+1/2}^+(h_0(u_{j+1}, u_{j+1}) - h_0(u_j, u_j)\right| \le \frac{f(u_{j+1}) - h(u_{j+1}, u_j)}{\Delta_+ u_j}. \tag{5.6b}$$

Namely,

$$-\frac{h(u_{j+1},u_j)-h(u_j,u_j)}{\Delta_+ u_j}\Big[1+\frac{1}{2}\frac{(h(u_{j+1},u_j)-h(u_j,u_j))^M}{(h(u_{j+1},u_j)-h(u_j,u_j))}\Big[\frac{\psi(R_j^-)}{R_j^-}-\psi(R_{j+1}^-)\Big]\Big]$$
$$=C_{j+1/2}^-\geq 0, \tag{5.7a}$$

$$\frac{h(u_j,u_j)-h(u_j,u_{j-1})}{\Delta\ u_j}\Big[1+\frac{1}{2}\frac{(h(u_j,u_j)-h(u_j,u_{j-1}))^M}{h(u_{j+1},u_j)-h(u_j,u_j)}\Big[\frac{\psi(R_j^+)}{R_j^+}-\psi(R_{j-1}^+)\Big]\Big]$$
$$=C_{j-1/2}^+\geq 0. \tag{5.7b}$$

The scheme (5.4) is convergent under (5.6) and the following assumption[5]:

$$\begin{aligned}\int_{u_j}^{u_{j+1}} f'(w)dw\Big(\Big(\frac{1}{2}\Delta_+u_j\Big)^2-\Big(w-\frac{1}{2}\big(u_{j+1}+u_j\big)\Big)^2\Big)\\ -a_{j+1/2}^-(\Delta_+u_j)^2[h_1(u_{j+1},u_{j+1})-h_1(u_j,u_j)]\\ -a_{j+1/2}^+(\Delta_+u_j)^2[h_0((u_{j+1},u_{j+1})-h_0(u_j,u_j)]\leq 0.\end{aligned} \tag{5.8}$$

Use the trapezoidal rule to discretize (5.4)

$$u_j^{n+1}+\frac{1}{2}\lambda\Delta_-H^{ac}\left(u_{j+2}^{n+1},u_{j+1}^{n+1},u_j^{n+1},u_{j-1}^{n+1}\right)=u_j^n-\frac{1}{2}\lambda\Delta_-H^{ac}\left(u_{j+2}^n,u_{j+1}^n,u_j^n,u_{j-1}^n\right). \tag{5.9}$$

Lemma 5.1. *Assume* (5.6) *is true, and*

$$\frac{\lambda}{2}(C_{j+1/2}^-+C_{j+1/2}^+)\leq 1. \tag{5.10}$$

The scheme (5.9) *is* TVD.

In [5] it is proved that, under condition (5.8), the following is true:

$$\int_{u_j}^{u_{j+1}}[H_{j+1/2}^{ac}-f(w)]dw\leq 0. \tag{5.11}$$

Therefore, according to our results, we have:

Theorem 5.2. *Under the assumptions* (5.6), (5.8), (5.10), *the solution sequence of* (5.9) *converges a.e. to the unique solution of the scalar convex conservation law* (1.1) *with periodic boundary value, provided that the initial data are in* BV.

§7. Discussion

In this paper, we do not discuss the numerical implementation of the implicit TVD scheme. The scheme is highly nonlinear and implicit. It should be linearized when used for practical computation. There is a rich literature on the linearized form of implicit TVD schemes. We refer the readers to [11, 12].

The semidiscrete approximation (2.1) can be fully discretized in the generalized one-leg form

$$u_j^{n+1} = u_j^n - \lambda \left[h\left(\bar{u}_{j-l+1}^{n+1/2}, \cdots, \bar{u}_{j+l}^{n+1/2}\right) - j\left(\bar{u}_{j-l}^{n+1/2}, \cdots, \bar{u}_{j+l-1}^{n+1/2}\right)\right] \tag{7.1}$$

where

$$\bar{u}_j^{n+1/2} = \eta u_j^{n+1} + (1-\eta) u_j^n .$$

The scheme (7.1) can be transformed into form (2.12) with the functional transformation

$$u_j^n = v_j^n - \lambda(1-\eta)(h_{j+1/2}^n(v_{j-l+1}^n, \cdots, v_{j+l}^n) - h_{j-1/2}^n(v_{j-l}^n, \cdots, v_{j+l-1}^n)). \tag{7.2}$$

The incremental form of (7.1) reads

$$\begin{aligned} u_j^{n+1} &- \eta\lambda(C_{j+1/2}^{-(n+1/2)} \Delta_+ u_j^{n+1} - C_{j-1/2}^{+(n+1/2)} \Delta_- u_j^{n+1}) \\ &= u_j^n + (1-\eta)\lambda(C_{j+1/2}^{-(n+1/2)} \Delta_+ u_j^n - C_{j-1/2}^{+(n+1/2)} \Delta_- u_j^n). \end{aligned} \tag{7.3}$$

The scheme (7.3) is TVD if

$$\infty > C > C_{j+1/2}^{\mp(n+1/2)} \geq 0, \quad (1-\eta)\lambda(C_{j+1/2}^{-(n+1/2)} + C_{j+1/2}^{+(n+1/2)}) \leq 1 \tag{7.4}$$

and if boundary conditions are such that (7.3) has the solution which is bounded at each n. We take $\eta = 1/2$,

$$u_j^{n+1} = u_j^n - \lambda \left[h\left(\hat{u}_{j-l+1}^{n+1/2}, \cdots, \hat{u}_{j+l}^{n+1/2}\right) - h\left(\hat{u}_{j-l}^{n+1/2}, \cdots, \hat{u}_{j+l-1}^{n+1/2}\right)\right] \tag{7.5}$$

where

$$\hat{u}_j^{n+1/2} = \frac{1}{2}(u_j^{n+1} + u_j^n).$$

The entropy inequality for (7.5) (see Section 3) is the following

$$\left(\frac{1}{2}(u_j^{n+1})^2 - \frac{1}{2}(u_j^n)^2\right) + \Delta_+ F_A(\hat{u}_j^{n+1/2}) \leq \lambda \int_{\hat{u}_j^{n+1/2}}^{\hat{u}_{j+1}^{n+1/2}} \left(h_{j+1/2}^{n+1/2} - f(w)\right) dw, \quad \forall j . \tag{7.6}$$

Therefore, we can extend all results about implicit trapezoidal TVD schemes to the one-leg form with all assumptions on the values

$$\hat{u}_j^{n+1/2} = \frac{1}{2}(u_j^{n+1} + u_j^n).$$

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