Journal of Computational Mathematics, Vol.10, No.2, 1992, 147–154.

SPLINE COLLOCATION APPROXIMATION TO PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS*

Zhang Li-qing
(*South China University of Technology, Guangzhou, China*)

Abstract

A spline collocation method is proposed to approximate the periodic solution of nonlinear ordinary differential equations. It is proved that the cubic periodic spline collocation solution has the same error bound $O(h^4)$ and superconvergence of the derivative at collocation points as that of the interpolating spline function. Finally a numerical example is given to demonstrate the effectiveness of our algorithm.

§1. Introduction

The numerical approximation to the periodic solution of an autonomous ordinary differential equation system has been brought into consideration for more than a decade. Many numerical methods like the shooting method, Newton method, the linear multistep method etc. have been used in approximating the periodic solutions[1–5], but hardly any rigorous analysis of the convergence and error estimate of numerical solutions is given. In this paper a spline collocation method is introduced to approximate the periodic solution of ODEs. It is proved that the cubic periodic spline collocation solution (including a periodic orbit and its period) has the same error bound $O(h^4)$ and superconvergence of the derivative at collocation points as that of the interpolating spline function.

Consider an autonomous ordinary differential equation system

$$\frac{dx}{dt} = f(x). \tag{1.1}$$

Finding a T-periodic solution of (1.1) is equivalent to solving a non-trivial solution of the following boundary value problem[2].

$$\frac{dx}{dt} = Tf(x), \quad \frac{dT}{dt} = 0\ ; \tag{1.2}$$

* Received November 23, 1988.

$$x(0) = x(1), \quad p(x) = 0, \tag{1.3}$$

where p is a functional of $C([0,1])$, which is known as a phase condition (Refer to [3] for further information). Here we choose

$$P(x(0)) = \begin{cases} x_k(0) - \alpha, \text{ if the } k\text{-th component of } x(0) \text{ is given,} \\ f_k(x(0)), \text{ if the } k\text{-th component of } x(t) \text{ takes its extremum at } t = 0. \end{cases} \tag{1.4}$$

Suppose $(x(t), T)$ is a solution of (1.2)-(1.3), $X(t)$ and $Y(t)$ are denoted respectively as the resolvents of

$$\frac{d\xi}{dt} = Tf'(x(t))\xi \tag{1.5}$$

and

$$\frac{d\xi}{dt} = -\frac{1}{3}Tf'(x(t))\xi. \tag{1.6}$$

If the matrix $J = \begin{pmatrix} I - X(1) & f(x(0)) \\ p'(x(0)) & 0 \end{pmatrix}$ is nonsingular, then the pair $(x(t), T)$ is a regular solution of (1.2)-(1.3)[3-4].

§2. The Cubic Spline Collocation Method

Let $\Delta = \{t_i\}_{i=1}^N (t_i = ih, h = \frac{1}{N})$ be a uniform partition of interval $[0,1]$, and $\{\phi_i(t)\}_{i=-1}^{N+1}$ be a cubic B-spline basis on mesh Δ. A pair $(x_h(t), T_h)$ is known as a cubic periodic spline collocation solution of (1.2)–(1.3), if $x_h(t) = \sum_{i=-1}^{N+1} c_i\phi_i(t)$ and T_h meets

$$\begin{cases} F_i(C_h) = x_h'(t_i) - T_h f(x_h(t_i)) = 0, \quad i = 1, \cdots, N, \\ F_{N+1}(C_h) = p(x_h(0)) = 0, \end{cases} \tag{2.1}$$

where $C_h = (c_1, \cdots, c_N, T_h)$, $c_{n+i} = c_i (i = -1, 0, 1), F_h = (F_1, \cdots, F_{N+1})$.

Lemma 2.1. *If $X(t)$ is a resolvent of (1.5), then*

$$f(x(0)) = X(1)\int_0^1 X(s)^{-1} f(x(s))ds. \tag{2.2}$$

Proof. Since $(x(t), T)$ is a solution of (1.2)–(1.3), $f(x(t))$ satisfies (1.5). So we have $f(x(t)) = X(t)f(x(0))$. Let $t = 1, f(x(0)) = f(x(1)) = X(1)f(x(0))$. Therefore

$$X(1)\int_0^1 X(s)^{-1} f(x(s))ds = X(1)\int_0^1 f(x(0))ds = X(1)f(x(0)) = f(x(0)).$$

Lemma 2.2. *If $A(t), f(t)$ are continuous, and $\mathbf{X}(t)$ is a resolvent of*

$$\frac{dx}{dt} = A(t)x, \tag{2.3}$$

then

$$\lim_{N\to\infty} \prod_{i=0}^{N-1} (I + A(t_i)h) = X(1), \quad h = 1/N, \tag{2.4}$$

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \prod_{i=k+1}^{N-1} (I + A(t_i)h) f(t_k)h = X(1)\int_0^1 X(s)^{-1} f(s) ds. \tag{2.5}$$

Proof. Let $w(t) = X(t)\int_0^1 X(s)^{-1}\phi(s)\,ds$. It is easy to see $w(t)$ satisfies

$$\frac{dw(t)}{dt} = A(t)w(t) + \phi(t), \quad w(0) = 0. \tag{2.6}$$

The Euler difference for (2.6) is

$$w_{i+1} - w_i = A(t_i)w_i h + \phi(t_i)h, \quad i = 0, \cdots, N-1. \tag{2.7}$$

By iteration,

$$w_N = \sum_{k=0}^{N-1} \prod_{i=k+1}^{N-1} (I + A(t_i)h)\phi(t_i)h.$$

So the relation (2.5) holds for the convergence of the Euler difference solution of (2.7). And (2.4) holds for the same reason.

Lemma 2.3.[8] *Suppose $F \in C^2(\omega, R^n)(\omega \subset R_n)$ and $J(u) = F'(u)$ is nonsingular at $u = u_0 \in \Omega$. If there exist positive constants δ and $\kappa (\kappa < 1)$ such that*

(1) $\Omega_\delta = \{u \mid \|u - u_0\| \le \delta\} \in \Omega$, (2) $\|J(u) - J(u_0)\| \le \kappa/M$, $\forall u \subset \Omega_\delta$,

(3) $\dfrac{M\gamma}{1-\kappa} \le \delta$

where γ and M are constants defined by

$$\|F(u_0)\| \le \gamma, \qquad \|J^{-1}(u_0)\| \le M$$

then there exists a unique solution $u = \bar{u}$ of equation $F(u) = 0$ in Ω_δ and

$$\|\bar{u} - u_0\| \le M\gamma/(1-\kappa).$$

§3. Existence and Error Estimate

Suppose $f \in C^2(R^n, R^n)$ and $x(t) \in C^5([0,1])$. Denote

$$\hat{x}_h(t) = \sum_{i=-1}^{N+1} \hat{c}_i \phi_i(t)$$

as the cubic periodic interpolating spline function of $x(t)$ on mesh Δ, and $\hat{C}_h = (\hat{c}_1, \cdots, \hat{c}_N, T)$. Consider the following system

$$J_h(\hat{C}_h)\xi_h = \phi_h \tag{3.1}$$

where $\xi_h = (\xi_1, \cdots, \xi_N, p), \phi_h = (\phi(t_1), \cdots, \phi(t_N), \tau)$. Rewrite (3.1) in component

$$\begin{cases} \dfrac{\xi_{k+1} - \xi_{k-1}}{2h} - T f_k' \dfrac{\xi_{k+1} + 4\xi_k + \xi_{k-1}}{6} = g(t_k), \quad k = 1, \cdots, N, \\ \xi_{N+1} = \xi_1, \quad \xi_N = \xi_0, \\ \dfrac{p_0(\xi_1 + 4\xi_0 + \xi_{N-1})}{6} = \tau \end{cases} \tag{3.2}$$

where $f_k' = f'(x(t_k))$, $g(t) = pf(x(t)) + \phi(t)$, $p_0 = p'(x(0))$. Let $\zeta_{k+1} = \xi_k$. Then the first equation of (3.2) becomes

$$\begin{pmatrix} \xi_{k+2} \\ \zeta_{k+2} \end{pmatrix} = A_k \begin{pmatrix} \xi_k \\ \zeta_k \end{pmatrix} + B_k \begin{pmatrix} g(t_{k+1}) \\ g(t_k) \end{pmatrix} \tag{3.3}$$

where

$$A_k = \begin{pmatrix} 1 + \frac{2T}{3} f_k' h & \frac{4h}{3} T f_k' \\ \frac{4h}{3} T f_k' & 1 + \frac{2T}{3} f_k' h \end{pmatrix} + O(h^2) = B^T \begin{pmatrix} 1 + 2hT f_k' & 0 \\ 0 & 1 - \frac{2}{3} hT f_k' \end{pmatrix} B$$
$$+ O(h^2),$$

$$B_k = \begin{pmatrix} 2hI & 0 \\ 0 & 2hI \end{pmatrix}, \quad B = \begin{pmatrix} I/\sqrt{2} & I/\sqrt{2} \\ I/\sqrt{2} & I/\sqrt{2} \end{pmatrix}.$$

Suppose N is an even number, $N = 2K$. By the iteration of (3.3) we have

$$\begin{pmatrix} \xi_N \\ \zeta_N \end{pmatrix} = D_N \begin{pmatrix} \xi_0 \\ \zeta_0 \end{pmatrix} + pH_N + E_N \tag{3.4}$$

where

$$D_N = \prod_{k=0}^{K-1} A_{2k}, \quad H_N = \sum_{k=0}^{K-1} \left(\prod_{j=k+1}^{K-1} A_{2j} \right) B_{2k} \begin{pmatrix} f(x(t_{2k+1})) \\ f(x(t_{2k})) \end{pmatrix},$$

$$E_N = \sum_{k=0}^{K-1} \left(\prod_{j=k+1}^{K-1} A_{2j} \right) B_{2k} \begin{pmatrix} \phi(t_{2k+1}) \\ \phi(t_{2k}) \end{pmatrix}.$$

From the second equation of (3.2) we have

$$p_0\left(\frac{2}{3}\xi_0 + \frac{1}{3}\zeta_0 + O(h)\right) = \tau. \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$G_N \begin{pmatrix} \xi_0 \\ \zeta_0 \\ p \end{pmatrix} = \begin{pmatrix} E_N \\ \tau \end{pmatrix}, \quad G_N = \begin{pmatrix} I - D_N & H_N \\ \left(\frac{2}{3}p_0 + O(h), \frac{1}{3}p_0 + O(h)\right) & O(h) \end{pmatrix}. \tag{3.6}$$

By lemmas 2.1 and 2.2, we know

$$\lim_{N\to\infty} G_N = G \tag{3.7}$$

where the matrix G is defined by

$$G = \begin{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - B^T \begin{pmatrix} X(1) & 0 \\ 0 & Y(1) \end{pmatrix} B & \begin{pmatrix} f(x(0)) \\ f(x(0)) \end{pmatrix} \\ \left(\frac{2}{3}p_0, \frac{1}{3}p_0\right) & 0 \end{pmatrix}.$$

It is easy to see that the matrix G is nonsingular provided the matrices $X(1)$ and $I - Y(1)$ are nonsingular. So from (3.7) we deduce that, if G is nonsingular, then there exists an N_0, such that for $N \geq N_0$, the matrix G_N is nonsingular and its inverse G_N^{-1} is uniformly bounded, i.e.

$$\|G_N^{-1}\| < M_0, \quad N \geq N_0 \quad \text{for some constant } M_0. \tag{3.8}$$

From (3.6) and (3.8) we have

$$\|(\xi_0, \zeta_0, p)^T\|_\infty \leq M_0 \|(E_N, \tau)^T\|_\infty. \tag{3.9}$$

On the other hand, the smoothness of f and the uniform boundedness of a periodic solution imply that $\prod_{i=k}^{K-1} A_i$ $(k = 0, \cdots, K-1)$ are uniformly bounded. So we have

$$\|E_N\|_\infty \leq \bar{M}_1 \|(\phi_1, \cdots, \phi_N)^T\|_\infty \leq \bar{M}_1 \|\phi_h\|_\infty \quad \text{for some } \bar{M}_1 > 0.$$

In summary, we obtain

Lemma 3.1. *Suppose $f \in C^2(R^n, R^n)$. If the matrix J and $I - Y(1)$ are nonsingular, then there exists an N_0, such that for $N \geq N_0$, the Jacobian $J_h(\hat{C}_h) = F_h'(\hat{C}_h)$ is nonsingular and its inverse is uniformly bounded in N. i.e. ,there exists a constant $M_1 > 0$, such that*

$$\|J_h^{-1}(\hat{C}_h)\| \leq M_1, \quad N \geq N_0. \tag{3.10}$$

Lemma 3.2. *If the conditions of the previous lemma are satisfied, then there exists a constant M_2 such that*

$$\|J_h(C_h) - J_h(\hat{C}_h)\|_\infty \leq M_2\|C_h - \hat{C}_h\|_\infty, \tag{3.11}$$

for any $C_h \in \Omega_\delta = \{C_h = (c_1, \cdots, c_N, \tau) \mid \|C_h - \hat{C}_h\| \leq \delta, \delta > 0\}$.

Proof. It is easily proved by use of the bandwidth property of $J_h(C_h)$ and the smoothness of f.

Theorem 3.3. *Suppose that $f \in C^2(R^n, R^n)$. The pair $(x(t), T)$ is a solution of (1.2)–(1.3), and $x(t) \in C^5([0,1])$. If the matrices J and $I - Y(1)$ are nonsingular, then there exists an N_0, such that for $N \geq N_0$, there exists a spline collocation solution of problem (1.2)–(1.3) and its coefficient $\bar{C}_h = (\bar{c}_1, \cdots, \bar{c}_N, \bar{T}_h)$ satisfies*

$$\|\bar{C}_h - \hat{C}_h\|_\infty \leq \text{const}\, h^4. \tag{3.12}$$

Proof. By Lemmas 3.1 and 3.2, there exist a positive integer N_0 and constants M_1, M_2 and δ_0, such that for $N \geq N_0$, $\delta \in (0, \delta_0]$,

$$\|J_h^{-1}(\hat{C}_h)\|_\infty \leq M_1, \tag{3.13}$$

$$\|J_h(C_h) - J_h(\hat{C}_h)\| \leq M_2\|C_h - \hat{C}_h\|_\infty. \tag{3.14}$$

By the superconvergence of the derivative of a cubic periodic spline interpolation function at interpolation points, we know there exists a constant M_3 such that

$$\|x'(t_i) - \hat{x}_h'(t_i)\|_\infty \leq M_3\|x\|_{5,\infty}h^4, \quad i = 1, \cdots, N. \tag{3.15}$$

Since $\hat{x}_h(t_i) = x(t_i)$ at interpolation points, we have

$$\begin{aligned} \|F_i(\hat{C}_h)\|_\infty &= \|\hat{x}_h'(t_i) - T \cdot f(x(t_i))\|_\infty = \|\hat{x}_h'(t_i) - x'(t_i)\|_\infty \\ &\leq M_3\|x\|_{5,\infty}h^4, \qquad i = 1, \cdots, N. \end{aligned} \tag{3.16}$$

On the other hand,

$$F_{N+1}(\hat{C}_h) = p(\hat{x}_h(0)) = p(x(0)) = 0.$$

So we have

$$\|F_h(\hat{C}_h)\|_\infty \leq M_3\|x\|_{5,\infty}h^4. \tag{3.17}$$

Now we choose $\delta_1 \in (0, \delta_0]$ such that $\kappa = M_1M_2\delta_1 < 1$. Then choose h so small that

$$\frac{M_1M_3\|x\|_{5,\infty}h^4}{1-\kappa} \leq \delta_1.$$

Using lemma 2.3, we obtain that equation (2.1) has a solution $\bar{C}_h = (\bar{c}_1, \cdots, \bar{c}_N, T_h)$ near C_h, and

$$\|\bar{C}_h - \hat{C}_h\|_\infty \leq \frac{M_1M_3\|x\|_{5,\infty}h^4}{1-\kappa}.$$

Theorem 3.4. *If the conditions in the above theorem are satisfied, then the cubic periodic spline collocation solution* $(\bar{x}_h(t), T_h)$ *has the following estimate*

$$\|D^k(\bar{x}_h(t) - x(t))\|_\infty < \text{const}\, h^{4-k}, \quad k = 0, 1, 2, \tag{3.19}$$

$$|T_h - T| < \text{const}\, h^4. \tag{3.20}$$

Furthermore, we have

$$|\bar{x}'_h(t_i) - x'(t_i)| < \text{const}\, h^4. \tag{3.21}$$

Proof. The estimate (3.20) is included in (3.12). For the partition Δ uniform, there exist constants α_k $(k = 0, 1, 2)$, such that

$$\sum_{i=-1}^{N+1} |D^k \phi_i(t)| < \frac{\alpha_k}{h^k}, \quad k = 0, 1, 2. \tag{3.22}$$

On the other hand, by use of the standard estimation of cubic periodic interpolation splines[8], there exist $A_k (k = 0, 1, 2)$, such that

$$\|D^k(\hat{x}_h(t) - x(t))\|_\infty \le A_k \|x\|_{4,\infty} h^{4-k}, \quad k = 0, 1, 2. \tag{3.23}$$

Using (3.12),(3.22) and (4.23), we have

$$\|D^k(\bar{x}_h(t) - x(t))\|_\infty \le \|D^k(\bar{x}_h(t) - \hat{x}_h(t))\|_\infty + \|D_k(\hat{x}_h(t) - x(t))\|_\infty$$
$$\le \max_{t\in[0,1]} \Big| \sum_{i=-1}^{N+1} |\bar{c}_i - \hat{c}_i| |D^k \phi_i(t)| \Big| + \|D^k(\hat{x}_h(t) - x(t))\|_\infty$$
$$\le (\text{const}\, \alpha_k + A_k \|x\|_{4,\infty}) h^{4-k}, \quad k = 0, 1, 2.$$

This proves (3.19). By definition,

$$|\bar{x}'_h(t_i) - x'(t_i)| = |T_h f(\bar{x}_h(t_i)) - T f(x(t_i))| \le |T_h (f(\bar{x}_h(t_i)) - f(x(t_i)))|$$
$$+ |(T_h - T) f(x(t_i))| \le |T_h f(\xi_i)| |\bar{x}_h(t_i) - x(t_i)| + |(T_h - T) f(x(t_i))|$$
$$= O(h^4), \quad i = 1, \cdots, N.$$

where ξ_i is the mean value. So the estimate (3.21) holds.

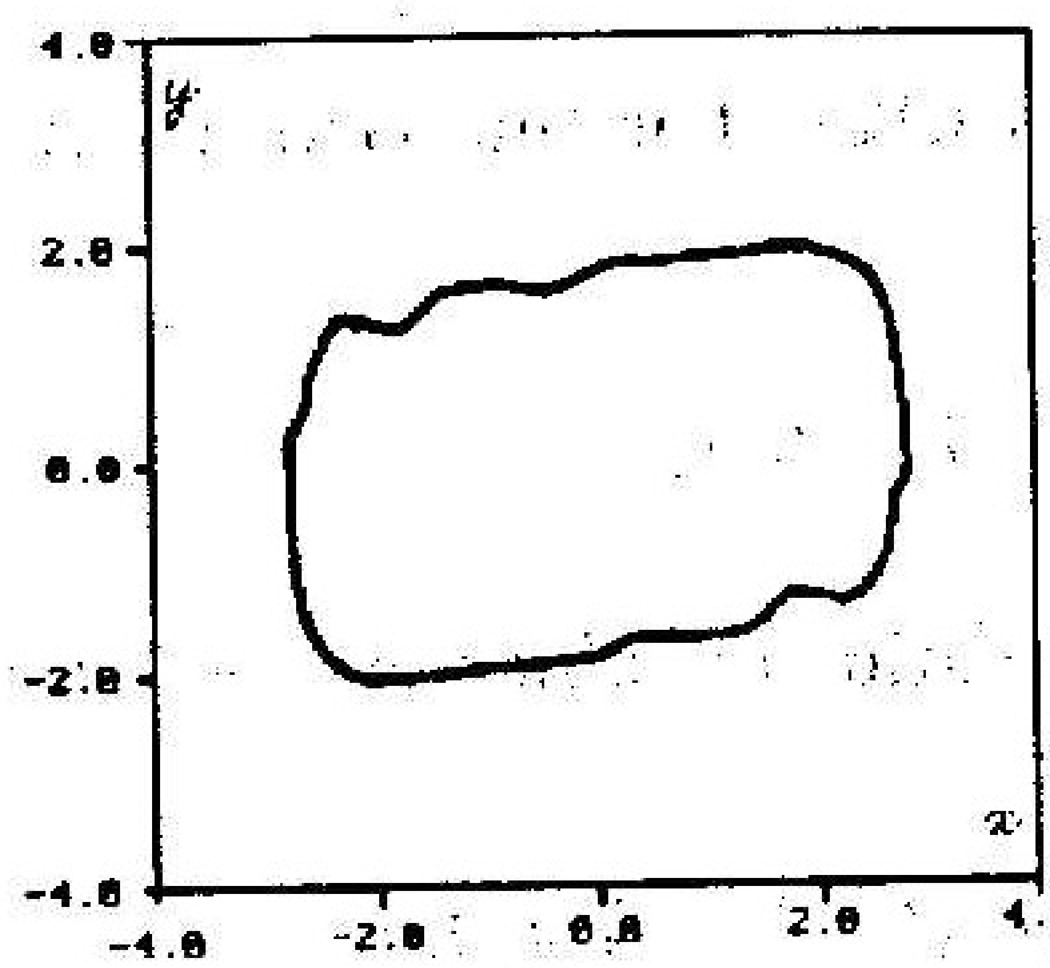
§4. A Numerical Example

The numerical implementation for the spline collocation method is to solve the nonlinear equation system (3.1). Usually the quasi-Newton method is used to solve the nonlinear system, and the homotopy method is used to treat the initial guess problem[2],[3],[6]. The numerical example given below is also computed by the Newton method and the homotopy method.

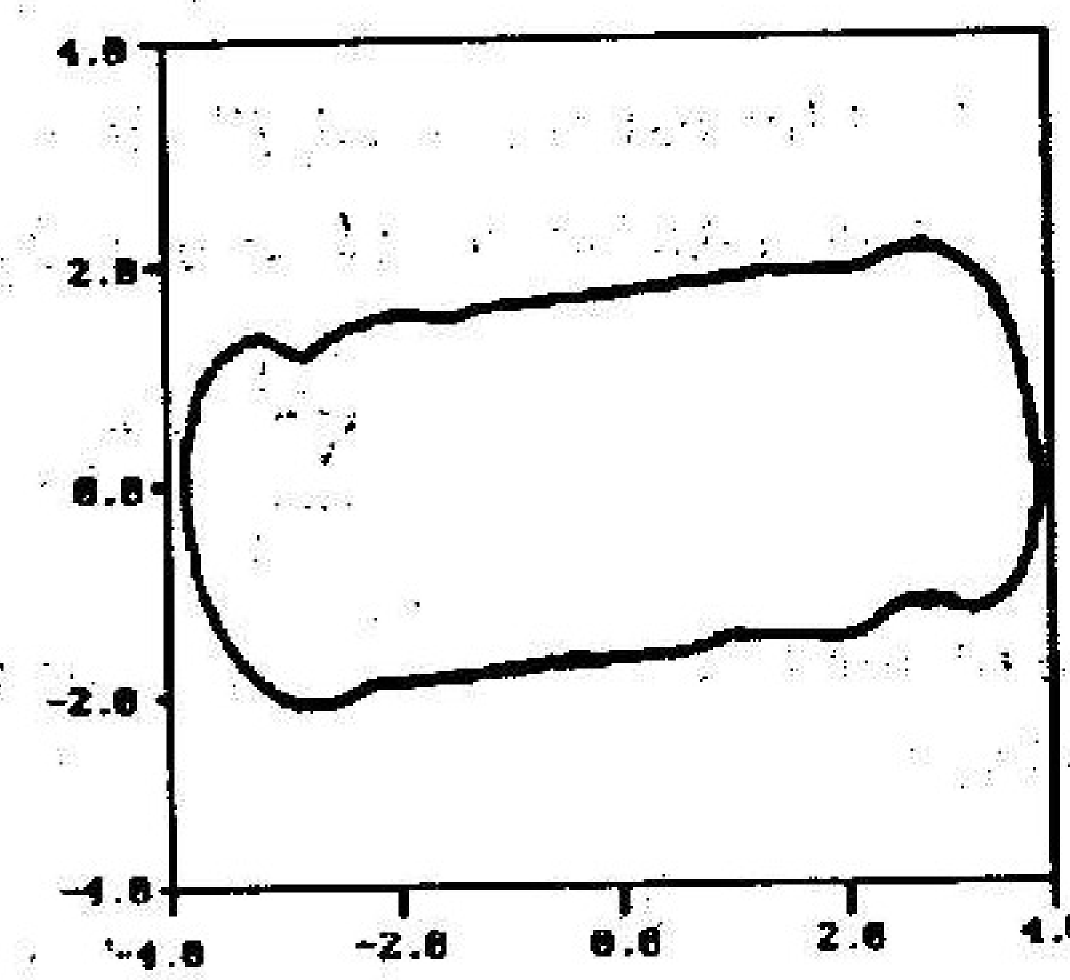
Example. Consider an autonomous system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x - \alpha(y^3/3 - y).$$

We use the spline collocation method to compute the periodic orbit and its period. Choose $N = 20$ and $p(x(0), y(0)) = -y(0)$. The diagrams shown below are the numerical periodic orbits for $\alpha = 2.5$ and $\alpha = 4.5$, where T_1 and T_2 are their periods.



$\alpha = 2.5,\ T_1 = 8.013$



$\alpha = 4.5,\ T_2 = 10.19$

References

[1] W.F. Langford, Numerical solution of bifurcation problems for ODEs, *Numer. Math.*, **28** (1977), 171–190.

[2] R. Seydel, Numerical computation of periodic orbits that bifurcate from stationary solutions of ODEs, *Applied Math. Comput.*, **9** (1981), 257–271.

[3] A. D. Jepson and H. P. Keller, Steady state and periodic solution paths: their bifurcations and computations, in "Numerical Method for Bifurcation Problem", ISNM 70, Ed: Küpper, Mittleman, Weber, Birkhauer Verlag, 1984.

[4] D. Deuflhard, Computation of periodic solutions of nonlinear ODEs, *BIT,* **24** (1984), 456–466.

[5] H. T. Doan, Invariant curves for numerical methods, *Quart. Appl. Math.*, **43** (1985), 385–393.

[6] L. Q. Zhang, Spline approximation to periodic solutions of ODEs, Ph. D. Dissertation of Zhongshan Univ., China, 1988.

[7] T. R. Lucas, Asymptotic expression for interpolating periodic splines, *SIAM. J. Numer. Anal.*, **19** (1982), 1051–1066.

[8] M. Urabe, Galerkin's procedure for nonlinear periodic systems, *Arch. Rat. Mech. Anal.*, **20** (1965), 120–152.