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ON THE NECESSARY CONDITIONS FOR THE SOLUBILITY OF ALGEBRAIC INVERSE EIGENVALUE PROBLEMS *1)

Xu Shu-fang
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

In this paper we give some necessary conditions for the solubility of additive inverse eigenvalue problems, multiplicative inverse eigenvalue problems and general inverse eigenvalue problems.

§1. Introduction

In this paper we shall consider the following inverse eigenvalue problems (see [1], [2]).

Problem A (Additive inverse eigenvalue problem). *Given an $n \times n$ Hermitian matrix $A = [a_{ij}]$, and n real numbers $\lambda_1, ..., \lambda_n$, find a real $n \times n$ diagonal matrix $D = \operatorname{diag}(c_1, ..., c_n)$ such that the matrix $A + D$ has eigenvalues $\lambda_1, ..., \lambda_n$.*

Problem M (Multiplicative inverse eigenvalue problem). Given an $n \times n$ positive definite Hermitian matrix $A = [a_{ij}]$, and n positive real numbers $\lambda_1, ..., \lambda_n$, find ân $n \times n$ positive definite diagonal matrix $D = (c_1, ..., c_n)$ such that the matrix DA has eigenvalues $\lambda_1, ..., \lambda_n$.

Problem G (General inverse eigenvalue problem). *Given $n + 1$ complex $n \times n$ Hermitian matrices $A_0, A_1, ..., A_n$ and n real numbers $\lambda_1, ..., \lambda_n$, find n real numbers $c_1, ..., c_n$, such that the matrix $A(c) = A_0 + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, ..., \lambda_n$.*

A number of sufficient conditions for these problems to have a solution have been discovered (see [1], [3]), but, to our knowledge, only one necessary condition is known and it applies only to Problem A. In the present note we shall give another necessary condition for the solubility of Problem A, which is equivalent to the condition in [4], but the form and the proof of this necessary condition is apparently simple and

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concise. Then we shall give some necessary conditions for the solubility of Problem M and Problem G.

Notation and Definitions. Throughout this paper we use the following notation. $\mathbb{C}^{m\times l}$ is the set of all $m\times l$ complex matrices. $\mathbb{C}^m$ is the set of all m-dimensional complex column vectors. The norm $\| \quad \|_F$ stands for Frobenious norm of a matrix. The superscripts T and H are for transpose and conjugate transpose, respectively. I is the $n \times n$ identity matrix, and e_i is the ith column of I. δ_{ij} is the Kronecker delta.

Let k and n be integers, $1 \leq k \leq n$. We use $G_{k,n}$ to denote the set of all increasing sequences of integers,

$$\pi = (j_1, j_2, \cdots, j_k) \text{ with } 1 \leq j_1 < j_2 < \cdots < j_k \leq n.$$

For arbitrary $\pi = (j_1, ..., j_k) \in G_{k,n}$ and $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, we use $A(\pi)$ to denote the $k \times k$ principal submatrix of A whose (i, l) entry is $a_{j_i j_l} (i, l = 1, 2, \cdots, n)$, tr$(A)$ to denote the trace of A, and $A^\dagger$ to denote the Moore-Penrose generalized inverse matrix of A. And we define

$$A^{(0)} = A - \operatorname{diag}(a_{11}, \cdots, a_{nn}).$$

Without loss of generality we can suppose that $a_{ii} = 0, i = 1, 2, \cdots, n$, in Problem A, $a_{ii} = 1, i = 1, 2, \cdots, n$, in Problem M, and $a_{ii}^{(k)} = \delta_{ik}, k, i = 1, 2, \cdots, n$, in Problem G, and suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ in the three problems.

§2. Main Results

Theorem 1. *The necessary conditions for the solubility of Problem M is*

$$\sum_{1\leq i<j\leq k} (\lambda_i - \lambda_{j+n-k})^2 \geq \lambda_n^2 k \max\left\{\|A^{(0)}(\pi)\|_F^2 \quad |\pi \in G_{k,n}\right\}, \quad 2 \leq k \leq n. \tag{2.1}$$

Theorem 2. *The necessary conditions for the solubility of Problem G is*

$$\sum_{1\leq i<j\leq k} (\lambda_i - \lambda_{j+n-k})^2 \geq k \max\left\{\|A^{(0)}(\pi) + \sum_{i=1}^{n} x_i(\pi) A_i^{(0)}(\pi)\|_F^2 \quad |\pi \in G_{k,n}\right\},$$
$$2 \leq k \leq n, \tag{2.2}$$

where

$$(x_1(\pi), \cdots, x_n(\pi))^T = [S(\pi)]^\dagger b(\pi),$$

$$S(\pi) = \begin{pmatrix} tr(A_1^{(0)}(\pi)A_1^{(0)}(\pi)) & \cdots & tr(A_1^{(0)}(\pi)A_n^{(0)}(\pi)) \\ \vdots & \ddots & \vdots \\ tr(A_n^{(0)}(\pi)A_1^{(0)}(\pi)) & \cdots & tr(A_n^{(0)}(\pi)A_n^{(0)}(\pi)) \end{pmatrix}, \tag{2.3}$$

$$b(\pi) = (-tr(A_1^{(0)}(\pi)A_0^{(0)}(\pi)), \cdots, -tr(A_n^{(0)}(\pi)A_0^{(0)}(\pi)))^T.$$

Letting $A_k = e_k e_k^T, k = 1, 2, \cdots, n$, in Theorem 2, we immediately obtain the necessary conditions for the solubility of Problem A as follows.

Corollary 1. The necessary conditions for the solubility of Problem A is

$$\sum_{1\le i<j\le k} (\lambda_i - \lambda_{j+n-k})^2 \ge k \max\left\{\|A(\pi)\|_F^2 \quad |\pi \in G_{k,n}\right\}, \quad 2 \le k \le n. \tag{2.4}$$

Remark. Corollary 1 is equivalent to Theorem 2 of [4].

§3. Proofs of Theorem 1 and Theorem 2

The proofs of Theorem 1 and Theorem 2 will be based on the following lemma.

Lemma 1. *Let $B = [b_{ij}]$ be a $k \times k$ Hermitian matrix, and its eigenvalues be $\mu_1 \ge \cdots \ge \mu_k$. Then*

$$\sum_{1\le i<j\le k} (\mu_i - \mu_j)^2 \ge k\|B^{(0)}\|_F^2. \tag{3.1}$$

Proof. Since

$$\sum_{i=1}^k \mu_i^2 = tr(B^2) = \|B^{(0)}\|_F^2 + \sum_{i=1}^k b_{ii}^2 \ge \|B^{(0)}\|_F^2 + \frac{1}{k}(\sum_{i=1}^k b_{ii})^2$$
$$= \|B^{(0)}\|_F^2 + \frac{1}{k}\Big(\sum_{i=1}^k \mu_i\Big)^2,$$

we have

$$\sum_{1\le i<j\le k} (\mu_i - \mu_j)^2 = k\sum_{i=1}^k \mu_i^2 - (\sum_{i=1}^k \mu_i)^2 \ge k\|B^{(0)}\|_F^2.$$

Now we prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Suppose that Problem M has a solution $D = (c_1, c_2, \cdots c_n)$, $c_i > 0$, $i = 1, 2, \cdots, n$, and $\pi = (j_1, \cdots, j_k) \in G_{k,n}$ is arbitrary. Let $\mu_1 \ge \cdots \ge \mu_k$ be eigenvalues of $[D(\pi)]^{\frac{1}{2}} A(\pi)[D(\pi)]^{\frac{1}{2}}$. Because $D^{\frac{1}{2}}AD^{\frac{1}{2}}$ is similar to DA and $[D(\pi)]^{\frac{1}{2}} A(\pi)[D(\pi)]^{\frac{1}{2}}$ is a $k \times k$ principal submatrix of $D^{\frac{1}{2}}AD^{\frac{1}{2}}$, by Lemma 1 and well-known interlacing inequalities for the eigenvalues of a Hermitian matrix, we obtain

$$\sum_{1\le i<j\le k} (\mu_i - \mu_j)^2 \ge k\|[D(\pi)]^{\frac{1}{2}} A^{(0)}(\pi)[D(\pi)]^{\frac{1}{2}}\|_F^2 \tag{3.2}$$

and

$$\lambda_i \ge \mu_i \ge \lambda_{i+n-k}, \qquad i = 1, 2, \cdots, k. \tag{3.3}$$

It follows from (3.3) that

$$\lambda_i - \lambda_{j+n-k} \ge \mu_i - \mu_j \ge 0, \qquad 1 \le i < j \le k. \tag{3.4}$$

Combining with (3.2) we get

$$\sum_{1\le i<j\le k}(\lambda_i-\lambda_{j+n-k})^2 \ge k\|[D(\pi)]^{\frac{1}{2}}A^{(0)}(\pi)[D(\pi)]^{\frac{1}{2}}\|_F^2. \tag{3.5}$$

By the min-max theorem of Hermitian matrices, we get

$$c_i \ge \lambda_n > 0 \qquad i=1,2,\cdots,n. \tag{3.6}$$

Let $A^{(0)}(\pi)=[b_{ij}]$. It follows from (3.6) that

$$\|[D(\pi)]^{\frac{1}{2}}A^{(0)}(\pi)[D(\pi)]^{\frac{1}{2}}\| = \sum_{i,l=1}^{k} c_{j_i}c_{j_l}|b_{li}|^2 \ge \lambda_n^2\sum_{i,l=1}^{k}|b_{li}|^2 = \lambda_n^2\|A^{(0)}(\pi)\|_F^2.$$

Combining with (3.5) we get

$$\sum_{1\le i<j\le k}(\lambda_i-\lambda_{j+n-k})^2 \ge \lambda_n^2\|A^{(0)}(\pi)\|_F^2. \tag{3.7}$$

Noting that $\pi\in G_{k,n}$ is arbitrary, we get (2.1) from (3.7) at once. The proof of Theorem 1 is completed.

Proof of Theorem 2. Suppose that $c_1,\cdots c_n$ is a solution of Problem G, and $\pi=(j_1,\cdots,j_k)\in G_{k,n}$ is arbitrary. As in the proof of (3.5), we have

$$\sum_{1\le i<j\le k}(\lambda_i-\lambda_{j+n-k})^2 \ge k\|A_0^{(0)}(\pi)+\sum_{i=1}^{n}c_iA_i^{(0)}(\pi)\|_F^2. \tag{3.8}$$

Now we define

$$\langle A,B\rangle = tr(A^HB) \qquad A,B\in \mathbb{C}^{k\times k}.$$

It is easy to prove that $\langle .,.\rangle$ is an inner product on $\mathbb{C}^{k\times k}$, and that norm $\| \quad \|_F$ is derived from this inner product.

We consider $A_i^{(0)}(\pi), i=1,2,\cdots,n$, as $n+1$ points in $(\mathbb{C}^{k\times k},\langle\cdot,\cdot\rangle)$. By the well-known property of the inner product space, we know that there exists $x^{(0)}=(x_1^{(0)},\cdots,x_n^{(0)})^T\in\mathbb{C}^n$ such that

$$\left\|A_0^{(0)}(\pi)+\sum_{i=1}^{n}x_i^{(0)}A_i^{(0)}(\pi)\right\|_F^2 = \min_{x=(x_i)\in\mathbb{C}^n}\left\|A_0^{(0)}(\pi)+\sum_{i=1}^{n}x_iA_i^{(0)}(\pi)\right\|_F^2 \tag{3.9}$$

and $x^{(0)}$ satisfies (3.9) if and only if

$$\langle A_j^{(0)}(\pi), A_0^{(0)}(\pi)+\sum_{i=1}^{n}x_i^{(0)}A_i^{(0)}(\pi)\rangle = 0 \text{ for } j=1,2,\cdots,n,$$

that is

$$\sum_{i=1}^{n}x_i^{(0)}tr(A_j^{(0)}(\pi)A_i^{(0)}(\pi)) = -tr(A_j^{(0)}(\pi)A_0^{(0)}(\pi)) \text{ for } j=1,2,\cdots,n. \tag{3.10}$$

(3.10) can be written as

$$S(\pi)x^{(0)} = b(\pi) \tag{3.11}$$

where $S(\pi)$ and $b(\pi)$ are defined by (2.3).

Since $S(\pi)$ is positive semidefinite, there exists a unitary matrix U such that

$$S(\pi) = U^H \Sigma U$$

where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_m, 0, \cdots, 0)$, $\sigma_i > 0$, $i = 1, 2, \cdots, m$. Let $y = (y_1, \cdots, y_n)^T = Ux^{(0)}$ and $d = (d_1, \cdots, d_n)^T = Ub(\pi)$. Then (3.11) is equivalent to

$$\Sigma y = d. \tag{3.12}$$

Since (3.12) must have a solution, it follows from (3.12) that

$$d_i = 0 \qquad i = m+1, \cdots, n.$$

Therefore, if we set

$$y_i^{(0)} = \begin{cases} \dfrac{d_i}{\sigma_i}, & 1 \le i \le m, \\ 0, & m+1 \le i \le n, \end{cases}$$

then $y^{(0)} = (y_1^{(0)}, \cdots, y_n^{(0)})^T$ is a solution of system (3.12).

Now let $x(\pi) = U^H y^{(0)}$, that is $x(\pi) = [S(\pi)]^\dagger b(\pi)$. Then $x(\pi)$ satisfies (3.11). Hence $x(\pi)$ satisfies (3.9). Note that $\pi \in G_{k,n}$ is arbitrary. Combining with (3.8), we obtain (2.2) at once. The proof of Theorem 2 is completed.

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