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A SIMPLIFIED VISCOSITY SPLITTING METHOD FOR SOLVING THE INITIAL BOUNDARY VALUE PROBLEMS OF NAVIER-STOKES EQUATION*1)

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Abstract

Based on the approximation of the linear operator semigroup, this paper proposes a simplified viscosity splitting method for solving the initial boundary value problems of the N-S equation. Some stability and convergence estimates of the method are proved. In particular, the mechanism of Chorin's method is explained and justified by the splitting method.

§1. Introduction

In this paper, the following initial boundary value problem of the Navier-Stokes (N-S) equation is considered

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\frac{1}{\rho}\nabla p=\nu\Delta u+f,\quad (x,t)\in\Omega\times[0,T), \tag{1.1}$$

$$\operatorname{div} u=\nabla\cdot u=0,\qquad (x,t)\in\Omega\times[0,T), \tag{1.2}$$

$$u(x,t)|_{x\in\partial\Omega}=0,\qquad t\in[0,T), \tag{1.3}$$

$$u(x,0)=u_o(x),\qquad x\in\Omega \tag{1.4}$$

where Ω is assumed to be a simply connected and bounded domain in $I\!R^2, u=(u^1(x,t),u^2(x,t))^T$ is the velocity, $p=p(x,t)$ the pressure and $f=(f^1(x,t),f^2(x,t))^T$ the body force, constants $\rho,\nu>0$ are the density and viscosity respectively. Re$=\frac{1}{\nu}$ represents Reynold's number.

Various viscosity splitting methods have been developed for solving the N–S equation. A remarkable one was proposed by Chorin in 1973 (see [3]) for calculating

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viscous incompressible flows with high Reynold's number. In terms of this method, the nonstationary N-S equation is solved by alternatively solving the Euler equation and the Stokes equation while the first equation by the characteristic vortex blob method and the next one by the random walk method, and in order to fulfil the no-slip condition $u \cdot \tau=0$ on the boundary $\partial\Omega$, vortex sheet is introduced to modify the solution obtained. A great deal of calculation has demonstrated that Chorin's method is effective for flows with high Reynold's number. However, the theoretical proof of convergence for Chorin's method has not been done yet, except for the case of pure initial value problem.

Recently, Ying Lungan has devoted several papers to the viscosity splitting method for the N–S equation in a bounded domain (see [4], [5]). He proposed a new viscosity splitting method of semidiscrete form with a special projection operator in it, and succeeded in proving the convergence and deriving an error estimate for his method under the assumption that the solution of the problem is sufficiently smooth. In [4], he gives a mathematical formulation for Chorin's method, and based on this formulation, he points out that "Chorin's scheme would cause divergence", i.e. the approximative solution obtained from Chorin's method does not converge to the solution of the original N-S equation. Since satisfactory results of Chorin's method have been shown in practice, a further discussion on the interpretation and the justice of Chorin's method might be valuable.

In this paper, making use of the approximation of the linear operator semigroup, we first present another viscosity splitting method with no projection operator like that in Ying's method. Then we prove the stability and convergence estimates for this method. Finally, based on these theoretical analyses, we give a new interpretation and justification to Chorin's method.

§2. A Simplified Viscosity Splitting Method

We first recall some concepts related to the semigroup theory of the N-S equation in two dimensions. Note that finding the projection in step 2 of the viscosity splitting method in [4] is equivalent to solving a boundary value problem of the biharmonic equation, so there is some computational complexity in doing it. In order to simplify the computation, we introduce a simplified viscosity splitting method of the N–S equation in a bounded domain.

2.1 Definitions and Concepts Related

Define subspaces

$$X = \text{ the closure of} \{u \in (C_0^\infty(\Omega))^2;\ \text{div } u = 0)\} \text{ in } (L^2(\Omega))^2,$$

$$G = \{\nabla q; q \in H^1(\Omega)\}.$$

Then there is a decomposition of form

$$(L^2(\Omega))^2 = X \oplus G.$$

Let P be the orthogonal projector: $(L^2(\Omega))^2 \to X$. Substituting P in equation (1.1), we have

$$\frac{\partial u}{\partial t} = -\nu A u - P((u \cdot \nabla)u) + Pf, \tag{2.1}$$

where $A = -P\Delta$, and $D(A) = X \cap \{u \in (H^2(\Omega))^2; u = 0 \text{ on } \partial\Omega\}$. It is known (see [2]) that the operator $-A$ generates a holomorphic semigroup in X, denoted by $e^{-At}, t \geq 0$.

For any given $0 < \alpha < 1$, as in [2], the power operator A^α can be defined, where

$$D(A^\alpha) = X \cap [(L^2(\Omega))^2, D(-\Delta)]_\alpha, \quad 0 \leq \alpha < 1.$$

Here $D(-\Delta) = X \cap (H^2(\Omega) \cap H_0^1(\Omega))^2$, and $[\ \cdot, \cdot\ ]_\alpha$ represents the interpolation of spaces. It is proved that

$$D(A^{(s+1)/2}) = D(A^{(s-1)/2}A) = D(A) \cap (H^{s+1}(\Omega))^2, \quad 1 \leq s < 3/2,$$

and $D(A^{1/2}) = X \cap (H_0^1(\Omega))^2$.

Lemma 1 (see [4]). *Let $u \in D(A^\alpha), 0 \leq \alpha < 5/4$. Then there exists a constant $C > 0$, such that*

$$C^{-1}\|A^\alpha u\|_0 \leq \|u\|_{2\alpha} \leq C\|A^\alpha u\|_0.$$

Lemma 2 (see [2]). *For any $\alpha > 0$, there exists a constant $C > 0$, such that*

$$\|A^\alpha e^{-tA}\|_0 \leq Ct^{-\alpha}, \quad \textit{for} \quad t > 0.$$

In particular,

$$\|e^{-tA}\|_0 \leq 1, \quad \textit{for} \quad t \geq 0.$$

It is well known that for any $v = (v^1, v^2)^T \in X$, there exists a unique function $\psi \in H_0^1(\Omega)$, called stream function, such that

$$v^1 = \frac{\partial \psi}{\partial y}, \quad v^2 = -\frac{\partial \psi}{\partial x}.$$

Let $\nabla\Lambda = (\partial/\partial y, -\partial/\partial x)$. Then $v = (\nabla\Lambda\psi)^T$, and the scalar function $\omega = -\nabla\Lambda v$ is called the vorticity of the given velocity field v. They satisfy

$$-\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0.$$

2.2 Construction of the method

Assume $u(t) : [0,T) \to D(A)$ is the solution of problem (1.1)–(1.4). Let $f_1 = P(f - (u \cdot \nabla)u)$. Then

$$\frac{\partial u}{\partial t} = -\nu A u + f_1.$$

Divide the interval $[0,T)$ into n subintervals of length $k = T/n$, and set $t_i = ik, i = 0, 1, \cdots, n$. By the Duhamel formula, we have

$$u(t) = e^{-\nu(t-ik)A}u(ik) + \int_{ik}^{t} e^{-\nu(t-s)A} f_1(s)ds \approx e^{-\nu(t-ik)A}[u(ik) + \int_{ik}^{t} f_1(s)ds],$$
$$ik \le t < (i+1)k,$$

where $e^{-\nu(t-s)A}$ is approximated by $e^{-\nu(t-ik)A}$. If we let

$$\tilde{u}_k((i+1)k - 0) = \tilde{u}_k(ik) + \int_{ik}^{(i+1)k} f_1(s)ds, \tag{2.3}$$

where $\tilde{u}_k(ik) = u_k(ik - 0)(= u(ik)$ now), and let

$$u_k^*((i+1)k - 0) = e^{-\nu k A}\tilde{u}_k((i+1)k - 0). \tag{2.4}$$

then $u_k^*((i+1)k-0)$ is an approximation of $u((i+1)k)$. Now replace the function f_1 in (2.3) and (2.4) by $\tilde{f}_1 = P(f - (\tilde{u}_k \cdot \nabla)\tilde{u}_k)$. Then from (2.3) and (2.4) we can obtain a new approximation $u_k((i+1)k-0)$ of $u((i+1)k)$, (2.3),(2.4) are equivalent to the following process:

Simplified Viscosity Splitting Method. For $i = 0, 1, \cdots, n$, on interval $[ik, (i+1)k)$

Step 1. Solve the Euler problem

$$\frac{\partial \tilde{u}_k}{\partial t} + (\tilde{u}_k \cdot \nabla)\tilde{u}_k + \frac{1}{\rho}\nabla \tilde{p}_k = f, \quad ik \le t < (i+1)k, \tag{2.5}$$

$$\nabla \cdot \tilde{u}_k = 0, \quad \tilde{u}_k \cdot n|_{\partial\Omega} = 0, \tag{2.6}$$

$$\tilde{u}_k(ik) = u_k(ik - 0), \quad u_k(-0) = u(0). \tag{2.7}$$

Step 2. Solve the Stokes problem

$$\frac{\partial u_k}{\partial t} + \frac{1}{\rho}\nabla p_k = \nu \Delta u_k, ik \le t < (i+1)k, \tag{2.8}$$

$$\nabla \cdot u_k = 0, u_k|_{\partial\Omega} = 0, \tag{2.9}$$

$$u_k(ik) = \tilde{u}_k((i+1)k - 0). \tag{2.10}$$

It is easy to see that for any $t \in [0,T)$,

$$u_k(t) = e^{-\nu tA}u_0 + \sum_{i=0}^{[t/k]} \int_{ik}^{(i+1)k} e^{-\nu(t-ik)A}\tilde{f}_1(s)ds. \tag{2.11}$$

2.3 Stability and error estimate

Consider the homogeneous Stokes equation

$$\frac{\partial u}{\partial t} = -\nu Au, \quad 0 \le t < T. \tag{2.12}$$

The solution $u(t) = e^{-\nu tA}v$ with the initial value $v \in X$ has singularity as $t \to 0$ in general. For the solution $u(t)$ to be smooth on $[0,T)$, it is necessary to demand, in addition to smoothness of v, that this initial value be compatible with the equation and the boundary condition at $\partial\Omega$ for $t = 0$. When $v \in D(A^{s/2})$, we say that v is a compatible initial value of order s of equation (2.12).

Lemma 3 (see [6]). *If $v \in (H^s(\Omega))^2$ is compatible of order s with equation* (2.12), *then*

$$\|e^{-\nu tA}v\|_{s+r} \le Ct^{-r/2}\|v\|_s, \quad r \ge 0, 0 < t < T,$$

where $C = C(\Omega, s, r, \nu, T)$.

The following theorem is our main result of this section.

Theorem. *Let u be the solution of problem* (1.1)–(1.4), *$\tilde{u}_k$ and u_k be the solution of* (2.5)–(2.10), *$u_0 \in D(A^{3/2})$, and u, f be sufficiently smooth in $\Omega \times [0,T], 1 \le s < 3/2$. Then for any $0 < \varepsilon < 1/4$, the following estimates hold for $0 \le t < T$,*

$$\|\tilde{u}(t)\|_{s+1} \le M, \tag{2.13}$$

$$\max(\|u(t) - u_k(t)\|_0, \|u(t) - \tilde{u}_k(t)\|_0) \le M'k^{3/4-\varepsilon}, \tag{2.14}$$

where constants M and M' depend only on $\Omega, \nu, s, T, u_0, u, f$ and ε.

To prove this theorem, we first consider the linear problem, i.e. assume $f_1 = P(f - (u \cdot \nabla)u)$ to be known, where u is the solution of (1.1)–(1.4) and u, f are sufficiently smooth. Then (2.5) becomes

$$\frac{\partial \tilde{u}_k}{\partial t} = f_1. \tag{2.15}$$

Lemma 4. *Assume $u_0 \in D(A^{3/2})$ and the solution u of* (1.1)–(1.4) *and f are sufficiently smooth. Let u^* and $\tilde{u}^*$ be the solution of* (2.15), (2.6)–(2.10). *Then*

$$\|u(t) - u^*(t)\|_0, \|u(t) - \tilde{u}^*(t)\|_0 \le C_1k, \quad 0 \le t < T, \tag{2.16}$$

and for any $0 < \varepsilon < 1/4$,

$$\|u(t) - \tilde{u}^*(t)\|_1 \leq C_2 k^{3/4-\varepsilon}, \quad 0 \leq t < T, \tag{2.17}$$

where C_1, C_2 *depend on* Ω, ν, T, u, f, *and* C_2 *depends on* ε *additionally.*

Proof. From the assumption, we have (with $I_t = [t/k]$)

$$\begin{aligned} u(t) - u^*(t) = & \sum_{i=0}^{I_t-1} \int_{ik}^{(i+1)k} (e^{-\nu(t-s)A} - e^{-\nu(t-ik)A}) f_1(s) ds \\ & + \int_{I_t k}^{t} (e^{-\nu(t-s)A} - e^{-\nu(t-I_t k)A}) f_1(s) ds \\ & - \int_{t}^{(I_t+1)k} e^{-\nu(t-I_t k)A} f_1(s) ds. \end{aligned} \tag{2.18}$$

Because

$$\begin{aligned} & \|(e^{-\nu(t-s)A} - e^{-\nu(t-ik)A}) f_1(s)\|_0 \\ & \leq C \|\nu A^{1-r/2} e^{-\nu(t-s)A} \int_0^{s-ik} e^{-\nu q A} dq\|_0 \|A^{r/2} f_1(s)\|_0 \\ & \leq C(t-s)^{-1+r/2} k \|f_1(s)\|_r, 0 < r < 1/2, ik \leq s < (i+1)k, \end{aligned}$$

and $\int_0^t (t-s)^{-1+r/2} ds \leq C$ and $\|e^{-\nu t A}\|_0 \leq 1$, we get for $0 \leq t < T$

$$\begin{aligned} \|u(t) - u^*(t)\|_0 & \leq \sum_{i=0}^{I_t-1} \int_{ik}^{(i+1)k} C(t-s)^{-1+r/2} k \|f_1(s)\|_r ds + Ck \sup_{0 \leq s \leq T} \|f_1(s)\|_0 \\ & \leq Ck \int_0^t (t-s)^{-1+r/2} ds \sup_{0 \leq s \leq t} \|f_1(s)\|_r + Ck \sup_{0 \leq s \leq T} \|f_1(s)\|_0 \\ & \leq Ck \sup_{0 \leq s \leq T} \|f_1(s)\|_{1/2}, \end{aligned}$$

which proves the first half of (2.6).

Since

$$u(t) = e^{-\nu(t-I_t k)A} u(I_t k) + \int_{I_t k}^{t} e^{-\nu(t-s)A} f_1(s) ds$$

and

$$\tilde{u}^*(t) = \tilde{u}^*(I_t k) + \int_{I_t k}^{t} f_1(s) ds,$$

then

$$\begin{aligned} u(t) - \tilde{u}^*(t) = & (e^{-\nu(t-I_t k)A} - I) u(I_t k) + \int_{I_t k}^{t} (e^{-\nu(t-s)A} - I) f_1(s) ds \\ & + u(I_t k) - \tilde{u}^*(I_t k). \end{aligned} \tag{2.19}$$

Noticing that $u \in D(A)$, we have

$$\|(e^{-\nu(t-I_t k)A} - I)u(I_t k)\|_0 = \Big\|\nu A \int_0^{t-I_t k} e^{-\nu q A} dq u(I_t k)\Big\|_0 \le \nu C \int_0^{t-I_t k} \|u(I_t k)\|_2 dq$$
$$\le \nu C k \sup_{0\le s\le T} \|u(s)\|_2.$$

By the first half of (2.16),

$$\|u(I_t k) - \tilde{u}^*(I_t k)\|_0 = \|u(I_t k) - u^*(I_t k - 0)\|_0 \le Ck \sup_{0\le s\le T} \|f_1(s)\|_{1/2}.$$

Further, from

$$\Big\| \int_{I_t k}^t (e^{-\nu(t-s)A} - I) f_1(s) ds \Big\|_0 \le Ck \sup_{0\le s\le T} \|f_1(s)\|_0,$$

we obtain

$$\|u(t) - \tilde{u}^*(t)\|_0 \le Ck \sup_{0\le s\le T} \|u(s)\|_2 + Ck \sup_{0\le s\le T} \|f_1(s)\|_{1/2}, \text{ for } 0 \le t < T.$$

Thus (2.16) is proved.

Now we turn to the proof of (2.17). By (2.18) and Lemma 1,

$$\|u(jk-0) - u^*(jk-0)\|_1 = \|\sum_{i=0}^{j-1} \int_{ik}^{(i+1)k} (e^{-\nu(jk-s)A} - e^{-\nu(jk-ik)A}) f_1(s) ds\|_1$$

$$\le C \sum_{i=0}^{j-1} \int_{ik}^{(i+1)k} \|A^{3/2} e^{-\nu(jk-s)A} \int_0^{s-ik} e^{-\nu q A} dq f_1(s)\|_0 ds$$

$$\le C \sum_{i=0}^{j-1} \int_{ik}^{(i+1)k} \|A^{1-r/2} e^{-\nu(jk-s)A} \int_0^{s-ik} A^{1/4+r} e^{-\nu q A} dq A^{1/4-r/2} f_1(s)\|_0 ds$$

$$\le C \sum_{i=0}^{j-1} \int_{ik}^{(i+1)k} (jk-s)^{-1+r/2} (s-ik)^{3/4-r} \|f_1(s)\|_{1/2-r} ds$$

$$\le C k^{3/4-r} \int_0^{jk} (jk-s)^{-1+r/2} ds \sup_{0\le s\le T} \|f_1(s)\|_{1/2}$$

$$\le C k^{3/4-r} \sup_{0\le s\le T} \|f_1(s)\|_{1/2}, \quad \text{for } 0 < r < 1/2, j = 0, 1, \cdots, n.$$

And by (2.19)

$$\|u(t) - \tilde{u}^*(t)\|_1 \le \|(e^{-\nu(t-jk)A} - I)u(jk)\|_1 + \|u(jk) - \tilde{u}^*(jk)\|_1$$
$$+ \Big\| \int_{ik}^t (e^{-\nu(t-jk)A} - I) f_1(s) ds \Big\|_1, \quad jk \le t < (j+1)k.$$

Since

$$\|(e^{-\nu(t-jk)A}-I)u(jk)\|_1 \le Ck \sup_{0\le s\le T}\|u(s)\|_3,$$

$$\|u(jk)-\tilde{u}^*(jk)\|_1 = \|u(jk)-u^*(jk-0)\|_1 \le Ck^{3/4-r}\sup_{0\le s\le T}\|f_1(s)\|_{1/2},$$

$$0<r<1/2,$$

and

$$\left\|\int_{jk}^t (e^{-\nu(t-jk)A}-I)f_1(s)ds\right\|_1 \le \int_{jk}^t (\|e^{-\nu(t-jk)A}f_1(s)\|_1 + \|f_1(s)\|_1)ds$$
$$\le C\int_{jk}^t (t-jk)^{-(1/4+\theta/2)}\|f_1(s)\|_{1/2-\theta}ds + Ck\sup_{0\le s\le T}\|f_1(s)\|_1$$
$$\le c(t-jk)^{3/4-\theta/2}\sup_{0\le s\le T}\|f_1(s)\|_{1/2} + Ck\sup_{0\le s\le T}\|f_1(s)\|_1,$$
$$\text{for } 0<\theta<1/2,\quad jk\le t<(j+1)k,\quad j=0,1,\cdots,n,$$

we have

$$\|u(t)-\tilde{u}^*(t)\|_1 \le Ck\sup_{0\le s\le T}\|u(s)\|_3 + Ck\sup_{0\le s\le T}\|f_1(s)\|_1 + Ck^{3/4-r}\sup_{0\le s\le T}\|f_1(s)\|_{1/2}$$
$$+Ck^{3/4-\theta/2}\sup_{0\le s\le T}\|f_1(s)\|_{1/2},\quad 0<r<1/2,\quad 0<\theta<1/2,$$

i.e.(2.17) is valid, and the proof of Lemma 4 is completed.

Consider the Euler equation problem as follows

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p = f,\quad 0\le t\le T,\quad x\in\Omega, \tag{2.20}$$

$$\operatorname{div} u = 0,\qquad 0\le t\le T,\quad x\in\Omega, \tag{2.21}$$

$$u\cdot n|_{\partial\Omega} = 0, \tag{2.22}$$

$$u(0) = u_0,\qquad x\in\Omega. \tag{2.23}$$

We are going to prove

Lemma 5. *Let $u_0\in D(A^{(s_1+1)/2})$, u be the solution of the above Euler equation problem, $1\le s<3/2, s_1=1+s/3$. If $\|u_0\|_{s_1+1}\le M_2$, then there exists a constant k_0 which depends only on Ω, s, T, M_2 and $\max_{0\le s\le T}\|f(s)\|_{s_1+1}$, such that*

$$\|u(t)\|_{s+1} \le C_3\|u_0\|_{s_1+1} + C_3 t,\quad 0\le t\le k_0, \tag{2.24}$$

where C_3 depends only on Ω, s, T and $\max\limits_{0\le s\le T}\|f(s)\|_{s_1+1}$.

Proof. Substiteting $-\nabla\Lambda$ in equation (2.20), we obtain

$$\frac{\partial\omega}{\partial t}+(u\cdot\nabla)\omega=F,\quad F=-\nabla\Lambda f,\quad u=(\nabla\Lambda\psi)^T,$$
$$-\Delta\psi=\omega,\psi|_{\partial\Omega}=0,\quad \omega(0)=\omega_0.$$

where $\omega=-\nabla\Lambda u$ is the vorticity corresponding to the velocity u, $\omega_0=-\nabla\Lambda u_0$.

Since $\|\omega_0\|_{s_1}=\|\nabla\Lambda u_0\|_{s_1}\le C\|u_0\|_{s_1+1}\le CM_2$, by Lemma 1 in [5] there exists a constant $k_0>0$ which depends only on Ω, s, T, M_2, C_0 and $\max\limits_{0\le s\le T}\|f(s)\|_{s_1+1}$, such that

$$\|\omega\|_s\le C_0\|\omega_0\|_{s_1}+C_0t\le C_0C\|u_0\|_{s_1+1}+C_0t,\quad 0\le t<k_0.$$

By the regularity of the elliptic operator,

$$\|u\|_{s+1}=\|\nabla\Lambda\psi\|_{s+1}\le C\|\psi\|_{s+2}\le c\|\omega\|_s\le C_3\|u_0\|_{s_1+1}+C_3t,\quad 0\le t\le k_0,$$

where C_0, C_3 depend only on Ω, s, T and $\max\limits_{0\le t\le T}\|f(s)\|_{s_1+1}$.

Lemma 6. *Let $u_0\in D(A^{(s_1+1)/2}), s_1=1+s/3, 1\le s<3/2$. If there exists a constant $M_0>0$, such that*

$$\|\tilde u_k(t)\|_1\le M_0,\quad 0\le t<T,$$

and there exist $C_3, k_0>0$, such that when $0<k<k_0$,

$$\|\tilde u_k(t)\|_{s+1}\le C_3\|\tilde u_k(ik)\|_{s_1+1}+C_3(t-ik), ik\le t<(i+1)k,\quad i=0,1,\cdots,\qquad(2.25)$$

then when $0<k<k_0$,

$$\sup_{0\le t<T}\|\tilde u_k(t)\|_{s+1}\le M_1,\qquad(2.26)$$

where M_1 depends only on $C_3, M_0, s, T, \nu, \Omega$, $\max\limits_{0\le s\le T}\|f(s)\|_{s_1+1}$ and $\|u_0\|_{s_1+1}$.

Proof. Let $\tilde f_1=P(f-(\tilde u_k\cdot\nabla)\tilde u_k)$. By (2.11),

$$\|u_k(t)\|_{s_1+1}\le C(\|e^{-\nu tA}u_0\|_{s_1+1}+\sum_{i=0}^{\left[\frac{t}{k}\right]}\int_{ik}^{(i+1)k}\|e^{-\nu(t-ik)A}\tilde f_1(s)\|_{s_1+1}ds)$$
$$\le C(\|u_0\|_{s_1+1}+\sum_{i=0}^{\left[\frac{t}{k}\right]}\int_{ik}^{(i+1)k}(t-ik)^{-(1+s_1-r)/2}\|\tilde f_1(s)\|_r ds),$$

for $s_1-1<r<1/2, (j-1)k\le t<jk, j=1,2,\cdots$. Since

$$\sup_{0\le s<jk}\|\tilde f_1(s)\|_r\le C\sup_{0\le s<jk}(\|f(s)\|_r+\|\tilde u_k(s)\|_{1+r'}^2)$$
$$\le C\sup_{0\le s<jk}(\|f(s)\|_r+\|\tilde u_k(s)\|_1^{2(1-r')}\|\tilde u_k(s)\|_2^{2r'})$$

and

$$\sum_{i=0}^{\left[\frac{t}{k}\right]}\int_{ik}^{(i+1)k}(jk-ik)^{-(1+s_1-r)/2}ds\le\int_0^{jk}(jk-s)^{-(1+s_1-r)/2}ds\le C,$$

where $(j-1)k\le t<jk, 0<r<r'<1/2$, we have

$$\|u_k(jk-0)\|_{s_1+1}\le C(\|u_0\|_{s_1+1}+\sup_{0\le s<jk}(\|f(s)\|_r+M_0^{2(1-r')}\|\tilde u_k(s)\|_2^{2r'}))$$
$$\le C(1+\sup_{0\le s<jk}\|\tilde u_k(s)\|_{s+1}^{2r'}).$$

By (2.25),

$$\|\tilde u_k(t)\|_{s+1}\le C_3C(1+\sup_{0\le s<jk}\|\tilde u_k(s)\|_{s+1}^{2r'}),\quad jk\le t<(j+1)k, j=0,1,\cdots.$$

And we have

$$\sup_{0\le t<T}\|\tilde u_k(t)\|_{s+1}\le CC_3(1+\sup_{0\le s<T}\|\tilde u_k(s)\|_{s+1}^{2r'}).$$

Then there exists a constant M_1 depending only on $C_3, M_0, T, s, \nu, \Omega, \sup_{0\le s<T}\|f(s)\|_{s_1+1}$ and $\|u_0\|_{s_1+1}$ such that (2.26) is fulfiled.

Lemma 7. *Let $u_0\in D(A^{(s_1+1)/2}), s_1=1+s/3, 1<s<3/2$, u and f be sufficiently smooth, $k\le 1$, $\max_{0\le t\le T}\|\tilde u_k(t)\|_{s+1}\le M_3$. Then for the solution $u_k,\tilde u_k$ of (2.5)–(2.10), we have*

$$\|u(t)-u_k(t)\|_0,\quad \|u(t)-\tilde u_k(t)\|_0\le C_4k^{3/4-\varepsilon},\quad\text{for}\quad 0<\varepsilon<1/4, 0\le t<T \tag{2.27}$$

where C_4 depends only on $\Omega, T, \nu, s, u, f, M_3$ and ε.

Proof. Let u^* and $\tilde u^*$ be the solution of (2.15), (2.6)–(2.10). Then,

$$\frac{\partial(\tilde u^*-\tilde u_k)}{\partial t}=P((\tilde u_k-u)\nabla u+(\tilde u_k\cdot\nabla)(\tilde u_k-u)),\quad ik\le t<(i+1)k,$$
$$\tilde u^*(ik)-\tilde u_k(ik)=u^*(ik-0)-u_k(ik-0),$$
$$\frac{\partial(u^*-u_k)}{\partial t}=-\nu A(u^*-u_k),\qquad ik\le t<(i+1)k,$$
$$u^*(ik)-u_k(ik)=\tilde u^*((i+1)k-0)-\tilde u_k((i+1)k-0).$$

Since $((\tilde{u}_k \cdot \nabla)(\tilde{u}^* - \tilde{u}_k), \tilde{u}^* - \tilde{u}_k) = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}^* - \tilde{u}_k\|_0^2 = ((\tilde{u}_k - u) \cdot \nabla u + (\tilde{u}_k \cdot \nabla)(\tilde{u}_k - u), (\tilde{u}^* - \tilde{u}_k)$$
$$= ((\tilde{u}_k - u) \cdot \nabla u + (\tilde{u}_k \cdot \nabla)(\tilde{u}^* - u), \tilde{u}^* - \tilde{u}_k).$$

Hence

$$\frac{d}{dt}\|\tilde{u}^* - \tilde{u}_k\|_0 \le C[\|u\|_{5/2}(\|\tilde{u}_k - \tilde{u}^*\|_0 + \|u^* - u\|_0) + \|\tilde{u}_k\|_2\|\tilde{u}^* - u\|_1]$$
$$\le C(\|\tilde{u}_k - \tilde{u}^*\|_0 + \|\tilde{u}^* - u\|_1)$$

and then

$$\|\tilde{u}^*(t) - \tilde{u}_k(t)\|_0 \le e^{C(t-ik)}(\|\tilde{u}^*(ik) - \tilde{u}_k(ik)\|_0 + k \max_{0\le t<T} \|\tilde{u}^*(t) - u(t)\|_1),$$
$$\|u^*(t) - u_k(t)\|_0 = \|e^{-\nu(t-ik)A}(\tilde{u}^*((i+1)k-0) - \tilde{u}_k((i+1)k-0)\|_0$$
$$\le \|\tilde{u}^*((i+1)k-0) - \tilde{u}_k((i+1)k-0)\|_0.$$

From the above two inequalities, we see

$$\|\tilde{u}^*((i+1)k-0) - \tilde{u}_k((i+1)k-0)\|_0 \le e^{Ck}(\|\tilde{u}^*(ik-0) - \tilde{u}_k(ik-0)\|_0$$
$$+k \max_{0\le t<T} \|\tilde{u}^*(t) - u(t)\|_1), \quad \text{for} \quad i = 0, 1, \cdots.$$

These together with (2.17) yield

$$\|\tilde{u}^*(t) - \tilde{u}_k(t)\|_0, \quad \|u^*(t) - u_k(t)\|_0 \le C_4' k^{3/4-\varepsilon}, \quad ik \le t < (i+1)k, i = 0, 1, \cdots.$$

Then Lemma 7 is proved by the triangle inequality and (2.16).

he proof of the theorem. Let $s_0 = s, s_l = 1 + s_{l-1}/3, l = 1, 2, \cdots$, and $H = \bigcap_{l=0}^{\infty}(H^{s_l}(\Omega))^2$. Since u is assumed to be sufficiently smooth in the hypothesis, we may assume $u \in H$ and set $M_0 = 2 \max_{0\le t\le T} \|u(t)\|_1$. Determine C_3 by Lemma 5, and then determine M_1 by Lemma 6. Considering M_1 as the constant M_2 in Lemma 5, we use Lemma 1 once more, and adjust k_0 such that (2.25) is valid for the above constant C_3. Then, determine M_3 according to Lemma 6 again. Set

$$M_4 = \max(M_1, M_3), \tag{2.28}$$

determine C_4 by Lemma 7, and reduce k_0 if necessary, such that

$$C(C_4 k_0^{3/4-\varepsilon})^{1/2}(\max_{0\le t\le T} \|u(t)\|_2 + M_4)^{1/2} \le M_0/2 \tag{2.29}$$

where C is a determined constant of the following inequality (2.30).

With these determined constants, we prove by induction that for any $0 < \varepsilon < 1/4$, when $0 < k < k_0$ and $0 \le t < T$,

$$\|\tilde{u}_k(t)\|_1 \le M_0, \quad \|\tilde{u}_k(t)\|_{s_l+1} \le M_1, \qquad l = 0, 1, \cdots,$$

$$\|u(t) - u_k(t)\|_0, \quad \|u(t) - \tilde{u}_k(t)\|_0 \le C_4 k^{3/4-\varepsilon}.$$

This is evident for $0 \le t < k$. Now we assume that for $0 \le t < jk, j > 0$, the above conclusions are true. Then by Lemma 5, Lemma 6 and (2.28),

$$\|\tilde{u}_k(t)\|_{s_{l-1}+1} \le M_3, \qquad l = 1, 2, \cdots, 0 \le t < (j+1)k.$$

By Lemma 7,

$$\|u(t) - u_k(t)\|_0, \|u(t) - \tilde{u}_k(t)\|_0 \le C_4 k^{3/4-\varepsilon}, \quad 0 \le t < (j+1)k.$$

From interpolation of norms,

$$\|u - u_k\|_1 \le C\|u - u_k\|_0^{1/2}\|u - u_k\|_2^{1/2}. \tag{2.30}$$

Noting (2.29), we have

$$\|u - \tilde{u}_k\|_1 \le M_0/2, \qquad 0 \le t < (j+1)k,$$

and therefore

$$\|\tilde{u}_k(t)\|_1 \le M_0, \qquad 0 \le t < (j+1)k.$$

By Lemmas 5 and 6,

$$\|\tilde{u}_k(t)\|_{s_{l-1}+1} \le M_1, \quad l = 1, 2, \cdots, 0 \le t < (j+1)k.$$

Then the argument of induction is completed.

If $k \ge k_0$, then there are T/k_0 steps at most, and the theorem is thus also valid.

2.4 The interpretation of Chorin's method

Substitute $-\nabla\Lambda$ in equation (1.1), and let $\omega = \nabla\Lambda u, F = -\nabla\Lambda f, \omega_0 = -\nabla\Lambda u_0$. We get the following form of the N–S equation

$$\frac{\partial\omega}{\partial t} + u \cdot \nabla\omega\nu\Delta\omega + F, \tag{2.31}$$

$$\Delta\psi\omega, \quad \psi|_{\partial\Omega} = \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0, \tag{2.32}$$

$$u = (\nabla\Lambda\psi)^T, \tag{2.33}$$

$$\omega(0) = \omega_0(x). \tag{2.34}$$

By a construction like that in Section 2.2, we get the viscosity splitting method for solving the N–S equation in vorticity-stream function form: for $i = 0, 1, \cdots$, on $[ik, (i+1)k)$.

Step 1. Solve the Euler problem

$$\frac{\partial \tilde{\omega}_k}{\partial t} + \tilde{u}_k \cdot \nabla \tilde{\omega}_k = F, \qquad ik \le t < (i+1)k, \tag{2.35}$$

$$-\Delta \tilde{u}_k = \tilde{\omega}_k, \qquad \tilde{\psi}_k|_{\partial\Omega} = 0, \tag{2.36}$$

$$\tilde{u}_k = (\nabla \Lambda \tilde{\psi}_k)^T, \tag{2.37}$$

$$\tilde{\omega}_k(ik) = \omega_k(ik - 0), \tag{2.38}$$

where $\omega_k(-0) = \omega_0$.

Step 2. Solve the Stokes problem

$$\frac{\partial \omega_k}{\partial t} = \nu \Delta \omega_k, \qquad ik \le t < (i+1)k, \tag{2.39}$$

$$\Delta \psi_k = \omega_k, \tag{2.40}$$

$$\psi_k|_{\partial\Omega} = \left.\frac{\partial \psi_k}{\partial n}\right|_{\partial\Omega} = 0, \tag{2.41}$$

$$u_k = (\nabla \Lambda \psi_k)^T, \tag{2.42}$$

$$\omega_k(ik) = \tilde{\omega}_k((i+1)k - 0). \tag{2.43}$$

The Euler problem in step 1 is relatively clear (see [1]) and can be solved effectively by a characteristic method or upwind type schemes. Step 2 is to solve a special diffusion problem of ω_k, where the boundary condition is implicit, so there are some difficulties in this step.

Chorin solved the N–S problem (2.31)–(2.34) in a small time interval $(ik, (i+1)k)$ in the following steps:

1° Solve the Euler problem (2.35)–(2.38) by characteristic vortex blob method;

2° Solve a pure initial value problem, i.e. (2.39) and (2.43), by the random walk method, and denote the solution obtained by $\bar{\omega}_k$ and define $\bar{\psi}_k$ by

$$-\Delta \bar{\psi}_k = \bar{\omega}_k, \qquad \bar{\psi}_k|_{\partial\Omega} = 0,$$

Then, get the corresponding velocity field $\bar{u}_k = (\nabla \Lambda \bar{\psi}_k)^T$ which satisfies condition $\bar{u}_k \cdot n|_{\partial\Omega} = 0$ automatically.

3° Fulfil another boundary condition

$$u_k \cdot \tau|_{\partial\Omega} = 0, \quad i.e. \quad \left.\frac{\partial \psi_k}{\partial n}\right|_{\partial\Omega} = 0,$$

and define

$$u_k = \begin{cases} \bar{u}_k, & \text{in}\,\Omega, \\ 0, & \text{in}\,\boldsymbol{R}^2 \setminus \Omega. \end{cases}$$

Due to the discontinuity of u_k on $\partial\Omega$, Chorin introduced a boundary vortex sheet with line density $\bar{u}_k \cdot \tau$ to modify the solution $\bar{\omega}_k$ of the diffusion problem (2.39) and (2.43); the modified solution can be written as

$$\omega_k(x) = \bar{\omega}_k(x)\chi(x)(dx) + \omega_k'(dx), \qquad \text{in}\,\overline{\Omega}$$

where

$$\chi(x) = \begin{cases} 1, & \text{in}\,\Omega, \\ 0, & \text{in}\,\boldsymbol{R}^2 \setminus \Omega. \end{cases}$$

and

$$\omega_k'(A) = \int_{A\cap\partial\Omega} \bar{u}_k \cdot dl, \qquad \text{for any open set } A \subset \boldsymbol{R}^2. \tag{2.44}$$

In a word, Chorin's method contains three steps in each time interval. Step 1, solve the Euler problem (2.35)–(2.38). Step 2: solve the pure initial value problem (2.39) and (2.43). Step 3, introduce the boundary vortex (2.44) to modify the solution obtained in step 2. The Step 2 combined with step 3 can be considered as a procedure of approximately solving the initial boundary value problem (2.31)–(2.34). Based on the above interpretation and the theoretical analyses in Section 2.3 for the simplified viscosity splitting method, we believe that Chorin's method is a reasonable method for solving the N–S flow with high Reynold's number.

§3. The Implementation of Simplified Splitting Method

It is identified in previous section that Chorin's method is a realization of the simplified viscosity splitting method. In this section, we provide another simple and convenient realization of this splitting method, which is also presented in correspondence with the vorticity-stream function formulation (2.35)–(2.43).

3.1 Method to solve the Euler problem (2.35)–(2.38)

Consider the Euler problem

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0, \qquad \text{in}\, \Omega \times [0,k)$$

$$u = (\nabla \Lambda \psi)^T, \quad \text{in}\, \Omega \times [0,k),$$

$$\omega|_{t=0} = \omega_0(x), \quad \text{in}\, \Omega, \tag{E}$$

where ψ is the solution of Dirichlet's problem

$$-\Delta\psi = \omega, \quad \text{in}\, \Omega, \quad \psi|_{\partial\Omega} = 0. \tag{D}$$

Let $\Omega_h = K \in \Omega$ be a quadrangular partition of domain Ω, K is the element, the boundary of K is denoted by ∂K. To define the approximation $(\psi_h, \omega_h)$of (ψ, ω), we make use of the following finite element spaces:

$H_h \subset H_0^1(\Omega)$ consists of piecewise bilinear functions on Ω_h,

$S_h \subset L_2(\Omega)$ consists of piecewise constants on Ω_h.

Given vorticity $\omega \in L_2(\Omega)$, the corresponding stream function ψ defined by (D) is approximated by $\psi_h \in H_h$ which satisfies

$$(\psi_h, \phi_h) = (\omega, \phi_h), \qquad \forall \phi_h \in H_h. \tag{D$_h$}$$

In terms of $u_h = (\nabla \Lambda \psi_h)^T$, we divide ∂K into two parts as follow

$$\partial K_- = \{x \in \partial K; u_h \cdot n(x) \le 0\}, \quad \text{inflow part}$$

$$\partial K_+ = \{x \in \partial K; u_h \cdot n(x) > 0\}, \quad \text{outflow part}$$

and for any given $\omega_h \in S_h$, we define on ∂K the "upwind value" of ω_h

$$\tilde{\omega}_h = \begin{cases} \text{the exterior limit value of}\, \omega_h, \text{ on}\, \partial K_-, \\ \text{the interior limit value of}\, \omega_h, \text{ on}\, \partial K_+, \end{cases}$$

and assume $\tilde{\omega}_h|_{\partial\Omega \cap \partial K} = 0$. Then as in [7], we define the semidiscrete finite element approximation of the Euler problem as

$$\begin{cases} \left(\frac{\partial \omega_h}{\partial t}, \eta_h\right)_K + \int_{\partial K} u_h \cdot n \tilde{\omega}_h \eta_h ds = 0, \quad \text{for any}\, \eta_h \in S_h, K \in \Omega_h, \\ \omega_h(0) = M_h \omega_0(x), \end{cases} \tag{E$_h$}$$

where

$$M_h \omega_0|_0 = \frac{1}{|K|}\int_0 \omega_0(x) dx, \quad |K| = \text{meas}\,(K).$$

Since div $u_h = 0$, we have $\int_{\partial K} u_h \cdot n ds = \int_K \operatorname{div} u_h dx = 0$, so that

$$\int_{\partial K} u_h \cdot n\tilde{\omega}_h \eta_h ds = \int_{\partial K} u_h \cdot n(\tilde{\omega}_h - \omega_h)\eta_h ds = \int_{\partial K_-} |u_h \cdot n|[\omega_h]\eta_h ds$$

where $[\omega_h] = \omega_h - \tilde{\omega}_h$, and then the first formula in (E_h) can be expressed as

$$\Big(\frac{\partial \omega_h}{\partial t}, \eta_h\Big)_K + \int_{\partial K_-} |u_h \cdot n|[\omega_h]\eta_h ds = 0$$

or

$$\frac{\partial \omega_h}{\partial t}|_K = -\frac{1}{|K|}\int_{\partial K_-} |u_h \cdot n|[\omega_h] ds, \qquad \text{for any } K \in \Omega_h.$$

Further, by using backward difference quotient $(\omega_h^{(1)} - \omega_h^{(0)})/k$ to approximate the derivative $\partial\omega_h/\partial t$ in the above formula, a completely discrete scheme for the Euler equation is set up, namely

$$\omega_h^{(1)}|_K = \omega_h^{(0)}|_0 - \frac{k}{|K|}\int_{\partial K_-} |u_h^{(0)} \cdot n|[\omega_h^{(0)}] ds, \qquad \text{for any } K \in \Omega_h, \tag{3.1}$$

where $\omega_h^{(0)}|_K = \omega_h(0)|_K, u_h^{(0)} = (\nabla\Lambda\psi_h^{(0)})^T$ and $\psi_h^{(0)} \in H_h$ is the solution of

$$(\nabla\psi_h^{(0)}, \nabla\phi_h) = (\psi_h^{(0)}, \phi_h), \qquad \text{for any } \phi_h \in H_h. \tag{3.2}$$

In addition, we have

$$u_h^{(0)} \cdot n = \partial_\tau \psi_h^{(0)}, \qquad \text{on } \partial K. \tag{3.3}$$

The formulae (3.1)–(3.3) derived for (E) can be easily changed into form that is suitable for the Euler problem (2.35)–(2.38).

3.2 Method to solve the Stokes problem (2.39)–(2.43)

Consider the Stokes problem of form

$$\frac{\partial \omega}{\partial t} = \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + f, \quad \text{in } \Omega \times (nk, (n+1)k), \tag{3.4}$$

$$-\Delta\psi = \omega, \qquad \text{in } \Omega \times (nk, (n+1)k), \tag{3.5}$$

$$\psi\Big|_{\partial\Omega} = \frac{\partial\psi}{\partial n}\Big|_{\partial\Omega} = 0, \qquad \text{for } t \in (nk, (n+1)k), \tag{3.6}$$

$$\omega(x, y, nk) = \omega_n(x, y), \tag{3.7}$$

where $\Omega = \{(x, y); 0 < x < 1, 0 < y < 1\}$ and k is the step size of time as before.

To discrete the problem (3.4)–(3.7) in space variable, we shall use the rectangular mesh Ω_h with nodes: (x_i, y_j), $i = 0, 1, \cdots, I, j = 0, 1, \cdots, J$ and cells:

$$K_{i,j} = [x_{i-1}, x_i; y_{j-1}, y_j], \quad i = 1, 2, \cdots, I, j = 1, 2, \cdots, J,$$

where $x_i = x_{i-1} + \Delta x_i, y_j = y_{j-1} + \Delta y_j$. Let $x_{i-1/2} = (x_{i-1} + x_i)/2, y_{j-1/2} = (y_{j-1} + y_j)/2$,then the center of cell $K_{i,j}$ is $(x_{i-1/2}, y_{j-1/2})$. The mesh with nodes: $(x_{i-1/2}, y_{j-1/2}).i = 1, 2, \cdots, I, j = 1, 2, \cdots, J$ is called the dual mesh of Ω_h and denoted by Ω_h^*. As an approximation of equation (3.4), we define on Ω_h^* the following finite difference equation

$$\omega^{(n+1)}_{i-1/2,j-1/2} = \omega^{(n)}_{i-1/2,j-1/2} + k(\delta_x^2 \omega^{(n)}_{i-1/2,j-1/2} + \delta_y^2 \omega^{(n)}_{i-1/2,j-1/2}) + k f^{(n)}_{i-1/2,j-1/2},$$
$$i = 2, 3, \cdots, I-1, j = 2, 3, \cdots, J-1 \qquad (3.8)$$

where δ_x^2 and δ_y^2 denote difference quotient operators of second order with respect to x and y on uniform mesh Ω_h^* respectively. As in Section 3.1, the equation (3.5) is approximated by finite element method: To find $\psi_h \in H_h$ such that

$$(\nabla \psi_h, \nabla \phi_h) = (\omega_h, \phi_h), \quad \text{for any } \phi_h \in H_h, \qquad (3.9)$$

where $H_h \subset H_0^1(\Omega)$ consists of piecewise bilinear functions on $K_{i,j}, i = 1, 2, \cdots, I, j = 1, 2, \cdots, J$. In boundary cells, for example in $K_{i,1} = [x_{i-1}, x_i; 0, y_1/2]$, we have

$$\psi_h(x, y) = \psi_h(x_{i-1}, y_1)(1 - (x - x_{i-1})/\Delta x_i) y/\Delta y_1$$
$$+ \psi_h(x_i, y_1)(x - x_{i-1}) y/\Delta x_i \Delta y_1,$$

then by the no-slip condition $u_h \cdot \tau = \partial \psi_h/\partial n = 0$ on the boundary, we get the following formula for calculating the average vorticity on boundary cell $K_{i,1}$

$$\omega_{i-1/2,1/2} = -\frac{1}{|K_{i,1}|} \int_{\partial K} u_h \cdot \tau ds = -\frac{1}{\Delta x_i \Delta y_1} \int_{\partial K} \frac{\partial \psi_h}{\partial n} ds$$
$$= -(\psi_h(x_{i-1}, y_1) + \psi_h(x_i, y_1))/2(\Delta y_1)^2. \qquad (3.10)$$

Similar formulae can be derived for other boundary cells.

The calculation of the method can be done in the following way:

First, solve problem (3.9), while ω_h in the right hand is taken as $\omega_h(x, y)$;

Second, calculate the boundary vortices by formulae as (3.10);

Third, let $\omega^{(n)}_{i-1/2,j-1/2} = \omega_n(x_{i-1/2}, y_{j-1/2})$, and use the boundary vortices obtained in the second step as the boundary values of mesh function $\omega^{(n-1)}_{i-1/2,j-1/2}$, then calculate by difference equation (3.8) the all values of $\omega_{i-1/2,j-1/2}^{(n+1)}$ at interior nodes of Ω_h^*.

It is easy to see that the approximate vorticity in S_h (Section 3.1), some piecewise constant function, can be identified with a mesh function defined on the dual mesh Ω_h^* (Section 3.2). And the approximate stream function $\psi_h \in H_h$, as piecewise bilinear function, can be identified with a mesh function defined on Ω_h. Therefore, the Euler problem (2.35)–(2.38) and the Stokes problem (2.39)–(2.43) are solved with the same mesh.

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