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A NEW KIND OF SCHEMES FOR THE OPERATOR EQUATIONS*

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Abstract

This paper provides a new kind of three-layer explicit schemes for solving the operator equation. It has good stability, and suits particularly the semidiscrete problems arising from solving multi-dimensional parabolic-type equations by the finite element method. The amount of its computation time is far less than that of any other algorithm of the finite element method and less than that of various economical schemes of the difference method. If the accuracy of the nonstandard finite element schemes (2.7) is not enough, it can be improved using extrapolations.

§1. Introduction

In numerical computation, for evolution equations the explicit schemes with good stability are more economical than the implicit schemes. In particular, when solving the evolution equation with the finite element method, we usually obtain a large system of ordinary differential equations about the time t. To discretie for time t, usually implicit schemes are used, but the amount of computation time is very large. In [1], the author has got a kind of explicit schemes for heat equations in the case of one and two dimensions under a special subdivision. In this paper, we remove the restriction of the number of dimensions and subdivision and consider the general operator equations. We get a kind of three-layer schemes with good stability. Of course, we are especially interested in the explicit forms of these schemes.

We consider discrete schemes for the operator equation

$$\bar{B}\frac{d\alpha(t)}{dt} + \bar{A}\alpha(t) = \bar{f}(t), \quad \alpha(0) = \alpha_0. \tag{1.1}$$

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For the concepts, definitions and notations in this paper, please refer to [2] (Chapters 1, 5, 6). For the operator equation (1.1), the standard form of three-layer schemes is

$$
\begin{aligned}
&By_{\overset{\circ}{t}} + \tau^2 R y_{\bar{t}t} + Ay = \varphi(t),\\
&y(0) = y_0, y(\tau) = y_1, 0 < t = n\tau < t, \\
&n = 1, 2, ..., \quad n_0 - 1, \quad t = n_0\tau.
\end{aligned} \tag{1.2}
$$

The solution of (1.2) may be presented in sum form $y = \bar{y} + \tilde{y}$, where $\bar{y}$ satisfies the homogeneous equation

$$By_{\overset{\circ}{t}} + \tau^2 R y_{\bar{t}t} + Ay = 0, \quad y(0) = y_0, \quad y(\tau) = y_1, \tag{1.2a}$$

and $\tilde{y}$ satisfies the inhomogeneous equation with homogeneous initial condition

$$By_{\overset{\circ}{t}} + \tau^2 R y_{\bar{t}t} + Ay = \varphi(t), \quad 0 < t = n\tau < t_0, \quad y(0) = y(\tau) = 0. \tag{1.2b}$$

§2. A New Kind of Schemes for the Operator Equations

We consider a kind of three-layer schemes of the finite-dimensional operator equation and their stability. In the finite-dimensional space, a linear operator is equivalent to a matrix. So, we let $\bar{A}$, $\bar{B}$ be matrices of order N and α_0 be a given vector of dimension N.

In (1.1), we make the following approximation: let

$$\bar{B} = \bar{B}_0 + \bar{B}_1 + \bar{B}_2, \tag{2.1}$$

where the diagonal elements of $\bar{B}_1$ are zero. Assume

$$\bar{B}_0 \frac{d\alpha(t)}{dt} \approx \bar{B}_0 \alpha_{\overset{\circ}{t}}(t), \tag{2.2}$$

$$\bar{B}_1 \frac{d\alpha(t)}{dt} \approx \bar{B}_1 \alpha_{\bar{t}}(t), \tag{2.3}$$

$$\bar{B}_2 \frac{d\alpha(t)}{dt} \approx \bar{B}_2 \alpha_t(t), \tag{2.4}$$

$$\bar{A}\alpha(t) \approx \bar{A}\alpha(t) + \tau^2 \frac{D_{\bar{A}}}{2} \alpha_{\bar{t}t}(t), \tag{2.5}$$

$$D_{\bar{A}} = \begin{pmatrix} \bar{a}_{11} & & & 0 \\ & \bar{a}_{22} & & \\ & & \ddots & \\ 0 & & & \bar{a}_{NN} \end{pmatrix}. \tag{2.6}$$

We substitute (2.1)—(2.5) into (1.1) and, noting that $\alpha_t(t) = \alpha_{\overset{\circ}{t}}(t) + \frac{\tau}{2}\alpha_{\bar{t}t}(t)$, $\alpha_{\bar{t}}(t) = \alpha_{\overset{\circ}{t}}(t) - \frac{\tau}{2}\alpha_{\bar{t}t}(t)$, use y to denote the discrete solution of (1.1) and rearrange. Then we have discrete schemes of (1.1):

$$\bar{B}y_{\overset{\circ}{t}}(t) + \tau^2\Big(\frac{D_{\bar{A}}}{2} + \frac{\bar{B}_2 - \bar{B}_1}{2}\Big)y_{\bar{t}t}(t) + \bar{A}y(t) = \varphi(t),$$

$$y(0) = y_0, \quad y(\tau) = y_1, \quad 0 < t = n\tau < t_0. \tag{2.7}$$

These are just of the standard form of three-layer schemes (1.2) where $B = \bar{B}$, $R = \frac{D_{\bar{A}}}{2} + \frac{\bar{B}_2 - \bar{B}_1}{2\tau}$, $A = \bar{A}$, and y_1 is to be given by an other method, such as two-layer schemes or the Taylor formula.

We point out that decomposition of $\bar{B}$ into $\bar{B}_0, \bar{B}_1$, and $\bar{B}_2$ is quite arbitrary. Different decompositions correspond to different schemes for solving (1.1).

According to A.A. Samarskys stability theory [2] we may immediately obtain the following results:

Theorem 2.1. *Assume $A = A^* \geq \delta E$, and $R = R^*$ are stationary operators. If*

$$B = B(t) \geq 0, \ \text{ for all } \ t \in \omega_\tau, \tag{2.8}$$

$$R > \frac{1}{4}A, \tag{2.9}$$

then scheme (1.2) is stable on the initial value and the right-hand side and we have the following estimate for $t > \tau$:

$$\|Y(t+\tau)\| \leq \|Y(\tau)\| + M_2 \max_{\tau < t' \leq t}(\|\varphi(t')\|_{A^{-1}} + \|\varphi_{\bar{t}}(t')\|_{A^{-1}}), \tag{2.10}$$

where M_2 is a positive constant, depending only on t_0.

Theorem 2.2. *Assume $A = A^*$ and $R = R^*$ are stationary nonnegative operators and $B(t)$ is a non-self-adjoint positive definite operator, i.e.,*

$$B(t) \geq \varepsilon E, \tag{2.11}$$

where ε is a positive constant independent of h and τ, and

$$R \geq \frac{1}{4}A. \tag{2.12}$$

Then the solution of problem (1.2b) has priori estimate

$$\|y(t+\tau)\| \leq \frac{2\sqrt{t}}{\varepsilon}\Big[\sum_{t'=t}^{t} \tau\|\varphi(t')\|^2\Big]^{1/2}. \tag{2.13}$$

Theorem 2.3. *Assume $A = A^* > 0$, and $R = R^*$ are stationary operators. When $B(t) \geq \varepsilon E$, and $R \geq \frac{1}{4}A$, scheme (1.2) is stable on the initial value and the right-hand side, and has an estimate*

$$\|Y(t+\tau)\| \leq \|Y(\tau)\| + \frac{1}{\sqrt{2\varepsilon}}\Big[\frac{t}{t'=\tau}\tau\|\varphi(t')\|^2\Big]^{1/2} \tag{2.14}$$

or

$$\|Y(t+\tau)\| \leq \|Y(\tau)\| + M_2 \max_{\tau\leq t'\leq t}\|\varphi(t')\|, \tag{2.14'}$$

where ε and M_2 are positive constans independent of h and τ.

If A and R depend on t, we must introduce the Lipschitz continuity for A and R about the time t.

Definition 2.1. *The operator $A(t)$ is Lipschitz continuous about the time t, if*

$$|((A(t) - A(t-\tau))x, x)| \leq C_3\tau(A(t-\tau)x, x) \tag{2.15}$$

for all $x \in H$ and $t = 2\tau, \cdots, (n-1)\tau$, where c_3 is a positive constant independent of h and τ.

For nonstationary operators A and R, the norm $\|Y(t+\tau)\| = \|Y(t+\tau)\|_{(t)}$ depends on t:

$$\begin{aligned}\|Y(t+\tau)\|^2_{(t)} = &\frac{1}{4}(A(t)(y(t+\tau)+y(t)), y(t+\tau)+y(t))\\ &+\tau^2((R(t) - \frac{1}{4}A(t))y_t(t), Y_t(t)),\end{aligned} \tag{2.16}$$

$$\begin{aligned}\|Y(t)\|^2_{(t-\tau)} = &\frac{1}{4}(A(t-\tau)(y(t)+y(t-\tau)), y(t)+y(t-\tau))\\ &+\tau^2((R(t-\tau) - \frac{1}{4}A(t-\tau))y_{\bar{t}}(t), y_{\bar{t}}(t)).\end{aligned} \tag{2.17}$$

Theorem 2.4. *Assume $A(t) = A^*(t) \geq \delta E$, and $R(t) = R^*(t)$ are nonstationary operators, and Lipschitz continuous about the time t, and*

$$R(t) \geq \frac{1+\bar{\varepsilon}}{4}A(t), \quad \text{for all} \quad 0 < t = n\tau < t_0, \tag{2.18}$$

where δ and $\bar{\varepsilon}$ are positive constants independent of h and τ, then scheme (1.2) is stable on the initial value and the right-hand side and has an estimate

$$\|Y(t+\tau)\|_{(t)} \leq M_1\|Y(\tau)\|_{(0)} + M_2 \max_{\tau<t'\leq t}[\|\varphi(t')\|_{A^{-1}(t)} + \|\varphi_{\bar{t}}(t')\|_{A^{-1}(t')}] \tag{2.19}$$

when $B(t) \geq 0, \tau < t = n\tau < t_0$;

$$\|Y(t+\tau)\|_{(t)} \leq M_1\|Y(\tau)\|_{(0)} + M_2\Big[\sum_{t'=\tau}^{t} \tau\|\varphi(t')\|^2\Big]^{1/2} \tag{2.20}$$

or

$$\|Y(t+\tau)\|_{(t)} \leq M_1\|Y(\tau)\|_{(0)} + M_2 \max_{\tau \leq t' \leq t} \|\varphi(t')\| \tag{2.20$'$}$$

when $B(t) \geq \varepsilon E$, *where* $\varepsilon . M_1, M_2$ *are positive constants independent of* h *and* τ.

Remark 1. Some conditions of Theorems 2.3 and 2.4 may be weakened. In Theorem 2.3, when $B(t) \geq 0$ there is stability on the initial value. In Theorem 2.4, when $A(t) = A^*(t) > 0$ (instead of $A(t) = A^*(t) \geq \delta E, \delta > 0$) there is stability on the initial value. In order to obtain estimate (2.20) or (2.20)$'$, it is sufficient that operator $A(t)$ is positive. The condition $B \geq 0$ may be replaced by

$$B \geq -c_4\tau^2 A \tag{2.21}$$

where c_4 is a positive constant independent of h and τ. If condition (2.21) is satisfied, then (2.19) holds, when $\tau \leq \tau_0, \tau_0 = \dfrac{1}{4c_4}$.

Remark 2. The semidiscrete standard finite element equations for several evolution equations are just equations of form (1.1). Hence, discrete schemes for (1.1) are also completely discrete schemes for the corresponding evolution equation. Therefore, if (1.1) is a semidiscrete standard finite element equation, then (2.7) are a kind of nonstandard finite element schemes. Especially, when $\bar{B}_0, \bar{B}_2$ are selected appropriately, i.e., $2\bar{B}_2 + \bar{B}_0$ is a diagonal matrix, then we obtain a kind of explicit schemes with good stability. For example, let us use the finite element method for the semidiscrete heat equation with the first kind initial-boundary condition.

For the one-dimensional case, $\bar{B}$ and $\bar{A}$ are three-diagonal symmetric positive-definite matrices. If the net of spatial variables is uniform, then

$$\bar{B} = \frac{1}{6}\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & 1 & 4 \end{pmatrix} \qquad \bar{A} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

For the case of two dimensions, $\bar{B}$ and $\bar{A}$ are relative to the subdivision of the region Ω. B and $\bar{A}$ are at least five-diagonal matrices. For instance, if Ω is divided according to Figure 3 in [1], then $\bar{B}$ is a seven-diagonal matrix. With the number of spatial dimensions increasing the diagonal band of matrices $\bar{B}$ and $\bar{A}$ widens rapidly.

In schemes (2.7), if $2\bar{B}_2+\bar{B}_0$ is a diagonal matrix, then we immediately obtain schemes (2.14) and (3.8) in [1] respectively.

In [1], we have proved that (2.14) and (3.8) have good stability.

For application, we provide the following example.

Example. Consider the heat equation

$$\begin{aligned}&\frac{\partial u}{\partial t}-\Delta u=f,\quad X\in\Omega\subset R^p,\quad 0<t\le t_0,\quad u=0,\\&(x,t)\in\partial\Omega\times[0,t_0],\quad u(x,0)=u_0(x),\quad X\in\Omega.\end{aligned}\tag{2.22}$$

Assume $u_h(X,t)=\sum_{j=1}^{N_h'}\alpha_j(t)\psi_j(X)$ is a semidiscrete standard finite element solution, where $\{\psi_j(x)\}$ are basis functions of the trial function space of finite elements, and $\{\alpha_j(t)\}$ are the solution for (1.1) (see [3]). Discretize (1.1) according to (2.7). For example, if $R\gtrsim\frac{1}{4}\bar{A}$ holds, then completely discrete schemes (2.7) for (2.22) are stable and have an error estimate

$$\|U^n-u(t_n)\|\le\|U^0-u_0\|+ch^r\Big\{\|u_0\|_r+\int_0^t\Big\|\frac{\partial u}{\partial t}\Big\|_r ds\Big\}+c_1(1+\tau/h^2)\tau\tag{2.23}$$

where U and u are respectively a completely discrete nonstandard finite element solution and a true solution for (2.22), and $U^n(X_i)=\sum_{j=1}^{N_h}y_j(t_n)\psi_j(x_i)$.

First, we consider stability. As the mass matrix $\bar{B}$ is a Gram matrix, it is positive definite. As the original differential equation is self-adjoint positive definite, $\bar{A}$ is symmetric positive definite. But the condition $R\ge\frac{1}{4}\bar{A}$ is generally not easy to test. In another paper [4], we provided a sufficient condition, which is easy to test. The result shows that $R\ge\frac{1}{4}\bar{A}$ holds for arbitrary $\bar{B}_1$ and $\bar{B}_2$. Therefore (2.7) are stable.

Secondly, we consider the error estimation. Note that

$$U^n-u(t_n)=U^n-u_n(t_n)+u_h(t_n)-u(t_n).\tag{2.24}$$

By means of [3], the semidiscrete standard finite element solution has the error estimate

$$\|u_h(t)-u(t)\|\le\|U^0-u_0\|+Ch^r\Big\{\|u_0\|_r+\int_0^t\Big\|\frac{\partial u}{\partial t}\Big\|_r ds\Big\},\quad\text{for } t\ge 0.\tag{2.25}$$

For the semidiscrete finite element equation (1.1) for (2.22), the truncation error of (2.7) is $O(\tau)+O(\tau^2)+O(\tau^2/h^2)$. Hence

$$\|U^n-u_h(t_n)\|\le C_1\tau(1+\tau/h^2),\tag{2.26}$$

thus

$$\|U^n - u(t_n)\| \le \|U^0 - u_0\| + ch^r\Big\{\|u_0\|_r + \int_0^t \Big\|\frac{\partial u}{\partial t}\Big\|_r ds\Big\}c_1\tau(1+\tau/h^2).$$

By means of the nonstandard finite element schemes (2.7) and their extrapolation, the accuracy of numerical solution for the parabolic equation achieves $O(\tau^2 + \tau^2/h^2)$ or $O(\tau^3)$. So the accuracy is good enough for general practical problems while the amount of computation time increases not too much.

§3. A Numerical Example

We consider a model problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < t \le 1, \quad 0 < x < \pi,$$

$$u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin 2x, \quad 0 < x < \pi.$$

Its analytic solution is

$$u(x,t) = e^{-4t}\sin 2x.$$

Let $h = \pi/2^{d+1}, d = 4, \tau = 1/4^d$. For the intervals $[0,\pi]$ and $[0,1]$ we make equidistant subdivision. On a computer M-150 we calculated this model problem by a standard linear finite element method, a difference method, and a nonstandard linear finite element method and its extrapolation. For the time variable, the standard linear finite element method used the Crank-Nicholson approximation. The difference method used a six-point symmetric difference scheme. The results of computation are shown in Table 1.

Table 1 indicates that the extrapolated solution of the nonstandard finite element method is very near to the standard finite element solution.

Table 1

t	x	t.s.	s.f.e.m.	d.m.	g.ext.	n.f.e.m.
	$\pi/8$	0.260130	0.259297	0.260972	0.259294	0.257619
0.25						
	$\pi/4$	0.367879	0.366701	0.369070	0.366697	0.364329
	$\pi/8$	0.012951	0.012785	0.013119	0.012781	0.012457
1						
	$\pi/4$	0.018316	0.018081	0.018553	0.018075	0.017617

t.s. — true solution, s.f.e.m. — standard finite element method,.

d.m. — difference method, g.ext. — global extrapolation
n.f.e.m. — nonstandard finite element method.

§4. Conclusions

(1) The amount of computation time is small. For the multi-dimensional case the amount of computation time of our schemes is far less than that of any other algorithm of finite element methods and also less than that of various economical schemes of difference methods, because our new schemes do only one explicit computation whatever dimensions the space variable has. Even if we extrapolate them, we need only to do three explicit computations. Other algorithms of finite element methods are implicit schemes, and so the amount of computation time is very big. For economical schemes of difference methods the increase in the amount of computation time follows the increase in the dimension. For example, in the two-dimensional case 2–4 one-dimensional implicit computations are needed, some schemes even need one more explicit computation. Thus the amount of their computation time approximately corresponds to 4–8 one-dimensional explicit computations. This amount of computation time is larger than that of the new schemes. Therefore the new schemes offer a way to reduce the amount of computation time needed in solving parabolic type equations using finite element methods. In addition, they are also effective for difference methods.

(2) The applicable range is vast. Our schemes apply not only to finite element methods and difference methods but also to other discrete methods for the space variable, and not only to the linear elements but also to high-order elements.

(3) The stability is good. For the parabolic type equations our schemes are almost absolutely stable.

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