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L^∞ CONVERGENCE OF CONFORMING FINITE ELEMENTS FOR THE BIHARMONIC EQUATION*[1)]

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Abstract

The paper considers the L^∞ convergence for conforming finite elements, such as Argyris element, Bell element and Bogner-Fox-Schmit element, solving the boundary value problem of the biharmonic equation. The nearly optimal order L^∞ estimates are given.

1. Introduction

The author has considered the L^∞ error estimates of the nonconforming finite elements for the biharmonic equation (see [3]). This paper will discuss the case of conforming finite elements.

Let Ω be a convex polygonal domain. The Dirichlet boundary value problem of the biharmonic equation is the following

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where $N = (N_x, N_y)$ is the unit normal of $\partial\Omega$.

For $p \in [1,\infty]$ and $m \geq 0$, let $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ be the usual Sobolev spaces, and $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ be the Sobolev norm and semi-norm respectively. When $p = 2$, denote them by $H^m(\Omega)$, $H_0^m(\Omega)$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. Let $H^{-m}(\Omega)$ be the dual space of $H_0^m(\Omega)$ with norm $\|\cdot\|_{-m,\Omega}$.

It is known that for $\forall f \in H^{-1}(\Omega)$, problem (1.1) has a unique solution $u \in H_0^2(\Omega) \cap H^3(\Omega)$, such that

$$\|u\|_{3,\Omega} \leq C\|f\|_{-1,\Omega}, \tag{1.2}$$

with C a positive constant.

Define, for $\forall u, v \in H^2(\Omega)$,

$$a(u,v) = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} \right) dxdy. \tag{1.3}$$

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Let $f \in L^2(\Omega)$. The variational form of problem (1.1) is to find $u \in H_0^2(\Omega)$, such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $(\cdot,\cdot)$ is the L^2 product.

For $h \in (0,h_0)$ with $h_0 \in (0,1)$, let $\mathcal{T}_h$ be a subdivision of Ω by triangles or rectangle. Let h_T =diam T and ρ_T the largest of the diameters of all circles contained in T. Assume that there exists a positive constant η, independent of h, such that $\eta h < \rho_T < h_T \le h$ for all $T \in \mathcal{T}_h$. Let $V_h \subset H_0^2(\Omega)$ be a finite element space associated with $\mathcal{T}_h$.

The finite element approximation to problem (1.1) is to find $u_h \in V_h$, such that

$$a(u_h,v) = (f,v), \qquad \forall v \in V_h. \tag{1.5}$$

This paper will show that the estimate of $|u-u_h|_{1,\infty,\Omega}$ is $\mathcal{O}(h^5|\ln h|)$ for Argyris element, $\mathcal{O}(h^4|\ln h|)$ for Bell element and $\mathcal{O}(h^3|\ln h|)$ for Bogner-Fox-Schmit element.

The remaining of the paper is arranged as follows. Section 2 will give the L^∞ estimates for Argyris element and its properties. Section 3 will give the proof of the L^∞ estimate for Argyris element. The last section will consider the case of Bell element and Bogner-Fox-Scĥmit element.

2. Argyris Element

From now on, let $\mathcal{T}_h$ be a subdivision of Ω by triangles and $V_h \subset H_0^2(\Omega)$ be Argyris finite element space associated with $\mathcal{T}_h$. Then $V_h = \{\, v \mid v \in H_0^2(\Omega), v|_T \in P_5(T), \forall T \in \mathcal{T}_h \,\}$. where $P_m(T)$ is the set of all polynomials with degree not greater than m for nonnegative integer m. Denote $Q_m(T)$ as the space consisting of all polynomials with degrees, with respect to x or y, not greater than m.

Let u be a solution of problem (1.1) and u_h that of problem (1.5). If $u \in H_0^2(\Omega) \cap H^6(\Omega)$, the following estimate is true:

$$\|u-u_h\|_{2,\Omega} \le Ch^4|u|_{6,\Omega}. \tag{2.1}$$

Throughout the paper, C always denotes the positive constant independent of h, with different values in different places. For L^∞ estimates, we have

Theorem 1. *Let V_h be Argyris finite element space, u the solution of problem* (1.1) *and u_h the solution of problem* (1.5). *Then*

$$|u-u_h|_{1,\infty,\Omega} \le Ch^5|\ln h||u|_{6,\infty,\Omega} \tag{2.2}$$

when $u \in W^{6,\infty}(\Omega)$, and

$$|u-u_h|_{0,\infty,\Omega} \le Ch^5|\ln h|^{1/2}|u|_{6,\Omega} \tag{2.3}$$

when $u \in H^6(\Omega) \cap H_0^2(\Omega)$.

The proof of Theorem 1 will be given in Section 3. Now we list some properties of Argyris element space.

For $T \in \mathcal{T}_h$, let Π_T be the interpolation operator of Argyris element. For $v \in H^6(\Omega)$, let $\Pi_h v|_T = \Pi_T v$ for all $T \in \mathcal{T}_h$. The following estimates are well known:

$$|v - \Pi_T v|_{m,T} \le Ch^{6-m}|v|_{6,T}, \quad 0 \le m \le 6, v \in H^6(T), T \in \mathcal{T}_h. \tag{2.4}$$

From [2,4], the following inequalities are true for $v \in V_h$:

$$\sum_{i=0}^{2} |v|_{i,\Omega} + |v|_{0,\infty,\Omega} \le C|v|_{2,\Omega}, \tag{2.5}$$

$$|v|_{0,\infty,\Omega} \le C|\ln h|^{1/2}|v|_{1,\Omega}, \tag{2.6}$$

$$|v|_{1,\infty,\Omega} \le C|\ln h|^{1/2}|v|_{2,\Omega}. \tag{2.7}$$

Let $P_h : L^2(\Omega) \to V_h$ be the $L^2(\Omega)$ orthogonal projection operator, i.e., for $\forall w \in L^2(\Omega)$,

$$(P_h w, v) = (w, v), \quad \forall v \in V_h. \tag{2.8}$$

The following is similar to Lemma 1 in [3]:

Lemma 1. *If* $0 \le m \le 2$, *then for* $w \in H_0^m(\Omega)$,

$$|w - P_h w|_{0,\Omega} \le Ch^m|w|_{m,\Omega}, \tag{2.9}$$

$$|P_h w|_{m,\Omega} \le C|w|_{m,\Omega}. \tag{2.10}$$

Now we want to show that the error bound of $u - u_h$ in norm $\|\cdot\|_{1,\Omega}$ is one order higher than that in $\|\cdot\|_{2,\Omega}$. To do it, we need to construct an interpolation operator $\tilde{\Pi}_h : H^3(\Omega) \cap H_0^2(\Omega) \to V_h$ with some interpolation properties.

For $T \in \mathcal{T}_h$, let $P_5'(T) = \{\ p|p \in P_5(T)$ and $\frac{\partial p}{\partial N} \in P_3(F)$ for each edge F of $T\ \}$. Let W_h be a finite element space associated with $\mathcal{T}_h$, which is determined as follows. For $w \in W_h$, $w|_T \in P_3(T)$ and $w, \frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are continuous at the vertices of $\mathcal{T}_h$ and vanish at the vertices along $\partial\Omega$.

For $T \in \mathcal{T}_h$, denote its vertices by A_T^1, A_T^2, A_T^3 and its center point by A_T^0. For $v \in H^3(T)$, let $\Pi_T^3 v$ be the interpolation polynomial of v, such that $\Pi_T^3 v \in P_3(T)$ and $\Pi_T^3 v$ equals v at $A_T^i, 0 \le i \le 3$, and $\frac{\partial}{\partial x}\Pi_T^3 v, \frac{\partial}{\partial y}\Pi_T^3 v$ equal $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at vertices A_T^1 to A_T^3 respectively. For $v \in H^3(\Omega)$, let $\Pi_h^3 v$ be determined by $\Pi_h^3 v|_T = \Pi_T^3(v|_T), \forall T \in \mathcal{T}_h$.

For an arbitrary vertex A, let N_A be the number of elements containing point A. Obviously, N_A is bounded. For $v \in H^3(\Omega) \cap H_0^2(\Omega)$, $\tilde{\Pi}_h v \in V_h$ is determined as follows:

1) For each $T \in \mathcal{T}_h$, $\tilde{\Pi}_h v|_T \in P_5'(T)$.

2) $\tilde{\Pi}_h v, \frac{\partial}{\partial x}\tilde{\Pi}_h v$ and $\frac{\partial}{\partial y}\tilde{\Pi}_h v$ equal $v, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at the vertices respectively.

3) Let A be an arbitrary vertex. If $A \in \Omega$, then

$$\frac{\partial^2}{\partial x^i \partial y^j}\tilde{\Pi}_h v(A) = \frac{1}{N_A}\sum_{\substack{T \in \mathcal{T}_h \\ A \in T}} \frac{\partial^2}{\partial x^i \partial y^j}\Pi_T^3 v(A). \quad i + j = 2. \tag{2.11}$$

If A is also a vertex of Ω, then

$$\frac{\partial^2}{\partial x^i \partial y^j}\tilde{\Pi}_h v(A) = 0, \quad i+j=2. \tag{2.12}$$

If A is on an edge of $\partial\Omega$ and is not the vertex of Ω, then

$$\begin{cases} \dfrac{\partial^2}{\partial s^2}\tilde{\Pi}_h v(A) = 0, \\ \dfrac{\partial^2}{\partial s \partial N}\tilde{\Pi}_h v(A) = 0, \\ \dfrac{\partial^2}{\partial N^2}\tilde{\Pi}_h v(A) = \dfrac{1}{N_A}\displaystyle\sum_{\substack{T\in\mathcal{T}_h \\ A\in T}} \dfrac{\partial^2}{\partial N^2}\Pi_T^3 v(A), \end{cases} \tag{2.13}$$

where s is the tangent vector of $\partial\Omega$.

Lemma 2. *Let T_1 and T_2 be two triangles with common edge F, and let A_1 and A_2 be the endpoints of F. Then there exists a constant C independent of T_1 and T_2, such that*

$$\sum_{k=1}^{2}\sum_{i+j=2}\left|\frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_1}^3 v(A_k) - \frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_2}^3 v(A_k)\right| \le C|v|_{3,T_1\cup T_2} \tag{2.14}$$

for all $v \in H^3(T_1 \cup T_2)$.

Proof. If $p \in P_2(T_1 \cup T_2)$, then $\Pi_{T_1}^3(p|_{T_1}) = p|_{T_1}, \Pi_{T_2}^3(p|_{T_2}) = p|_{T_2}$. Hence

$$\frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_1}^3 p(A_k) = \frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_2}^3 p(A_k), \quad i+j=2,\ k=1,2.$$

By Bramble-Hilbert lemma and the affine technique, we get (2.14).

Lemma 3. *Let T be a triangle, and F be an edge of T with endpoints A_1 and A_2. Denote the tangent vector of F by s. Then there exists a constant C independent of T, such that*

$$\sum_{k=1}^{2}\left(\left|\frac{\partial^2}{\partial s \partial x}\Pi_T^3 v(A_k)\right| + \left|\frac{\partial^2}{\partial s \partial y}\Pi_T^3 v(A_k)\right|\right) \le C|v|_{3,T} \tag{2.15}$$

for all $v \in H^3(T)$ with $\frac{\partial v}{\partial x}(A_k) = 0, \frac{\partial v}{\partial y}(A_k) = 0$ for $k = 1, 2$.

Proof. From the definition of Π_T^3,

$$\frac{\partial}{\partial x}\Pi_T^3 v(A_k) = 0, \quad \frac{\partial}{\partial y}\Pi_T^3 v(A_k) = 0, \quad k=1,2.$$

Hence, by the affine technique, we can get

$$\sum_{k=1}^{2}\left(\left|\frac{\partial^2}{\partial s \partial x}\Pi_T^3 v(A_k)\right| + \left|\frac{\partial^2}{\partial s \partial y}\Pi_T^3 v(A_k)\right|\right) \le C|\Pi_T^3 v|_{3,T}.$$

(2.15) follows from $|\Pi_T^3 v|_{3,T} \le C|v|_{3,T}$.

Lemma 4. *For $\forall v \in H^3(\Omega) \cap H_0^2(\Omega)$,*

$$|v - \tilde{\Pi}_h v|_{m,\Omega} \le Ch^{3-m}|v|_{3,\Omega}, \quad 0 \le m \le 2. \tag{2.16}$$

Proof. From the interpolation theory (see [1]), we have

$$|v - \Pi_h^3 v|_{m,\Omega} \le Ch^{3-m}|v|_{3,\Omega}, \quad 0 \le m \le 2. \tag{2.17}$$

For each $T \in \mathcal{T}_h$, we can show that the following inequality:

$$\|p\|_{0,T}^2 \le Ch^2 \sum_{k=1}^{3} [|p(A_T^k)|^2 + h^2|\frac{\partial}{\partial x}p(A_T^k)|^2 + h^2|\frac{\partial}{\partial y}p(A_T^k)|^2 + h^4 \sum_{i+j=2} |\frac{\partial^2}{\partial x^i \partial y^j}p(A_T^k)|^2]$$

for all $p \in P_5'(T)$. Using $\Pi_h^3 v - \tilde{\Pi}_h v|_T \in P_5'(T)$ and the definitions of Π_h^3 and $\tilde{\Pi}_h$, we have

$$\begin{aligned}\|\Pi_h^3 v - \tilde{\Pi}_h v\|_{0,\Omega}^2 &= \sum_{T\in\mathcal{T}_h} \|\Pi_h^3 v - \tilde{\Pi}_h v\|_{0,T}^2 \\ &\le Ch^6 \sum_{T\in\mathcal{T}_h} \sum_{k=1}^{3} \sum_{i+j=2} \left|\frac{\partial^2}{\partial x^i \partial y^j}\Pi_h^3 v(A_T^k) - \frac{\partial^2}{\partial x^i \partial y^j}\tilde{\Pi}_h v(A_T^k)\right|^2.\end{aligned} \tag{2.18}$$

For $T \in \mathcal{T}_h$ and A_T^k, we consider three cases.

Firstly, A_T^k is in Ω. Let $T_1, \cdots, T_l$ be all elements containing A_T^k, such that $T_1 = T$, $T_j \cap T_{j+1} (1 \le j \le l-1)$ is the common edge of T_j and T_{j+1}. From Lemma 2, we have

$$\begin{aligned}&\sum_{i+j=2} \left|\frac{\partial^2}{\partial x^i \partial y^j}\Pi_h^3 v(A_T^k) - \frac{\partial^2}{\partial x^i \partial y^j}\tilde{\Pi}_h v(A_T^k)\right|^2 \\ &\le C \sum_{i+j=2} \sum_{n=1}^{l-1} \left|\frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_n}^3 v(A_T^k) - \frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_{n+1}}^3 v(A_T^k)\right|^2 (\frac{l-n}{l})^2 \\ &\le C \sum_{n=1}^{l} |v|_{3,T_n}^2.\end{aligned} \tag{2.19}$$

Secondly, A_T^k is also a vertex of $\partial\Omega$. Let the number l of elements containing A_T^k be greater than 1. There must be two different elements T' and T'', such that an edge F' of T' and an edge F'' of T'' are along $\partial\Omega$ and A_T^k is an endpoint of both F' and F''. Let N', N'' and s', s'' be the unit normals and the unit tangents of F', F'', respectively. Let $T_1, \cdots, T_l$ be the elements containing A_T^k and $T_1 = T, T_l = T'$ and $T_j \cap T_{j+1} (1 \le j \le l-1)$ be a common edge of T_j and T_{j+1}. Then, from Lemma 2, we get

$$\begin{aligned}\sum_{i+j=2} \left|\frac{\partial^2}{\partial x^i \partial y^j}\Pi_h^3 v(A_T^k) - \frac{\partial^2}{\partial x^i \partial y^j}\tilde{\Pi}_h v(A_T^k)\right|^2 &= \sum_{i+j=2} |\frac{\partial^2}{\partial x^i \partial y^j}\Pi_h^3 v(A_T^k)|^2 \\ &\le C \sum_{i+j=2} \sum_{n=1}^{l-1} |\frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_n}^3 v(A_T^k) - \frac{\partial^2}{\partial x^i \partial y^j}\Pi_{T_{n+1}}^3 v(A_T^k)|^2\end{aligned}$$

$$+C\sum_{i+j=2}|\frac{\partial^2}{\partial x^i\partial y^j}\Pi^3_{T'}v(A^k_T)|^2$$

$$\leq C\left(\sum_{n=1}^{l}|v|^2_{3,T_n}+\sum_{i+j=2}|\frac{\partial^2}{\partial x^i\partial y^j}\Pi^3_{T'}v(A^k_T)|^2\right). \tag{2.20}$$

On the other hand, we have

$$\sum_{i+j=2}\left|\frac{\partial^2}{\partial x^i\partial y^j}\Pi^3_{T'}v(A^k_T)\right|^2\leq C\sum_{i+j=2}\left|\frac{\partial^2}{\partial s'^i\partial s''^j}\Pi^3_{T'}v(A^k_T)\right|^2,$$

where constant C is dependent on the angles at all vertices of Ω. From Lemmas 2 and 3, we have

$$\sum_{i+j=2}\left|\frac{\partial^2}{\partial s'^i\partial s''^j}\Pi^3_{T'}v(A^k_T)\right|^2\leq C\Big(|v|^2_{3,T'}+\left|\frac{\partial^2}{\partial s''^2}\Pi^3_{T'}v(A^k_T)\right|^2\Big)$$

$$\leq C\Big(|v|^2_{3,T'}+\left|\frac{\partial^2}{\partial s''^2}\Pi^3_{T''}v(A^k_T)\right|^2+\left|\frac{\partial^2}{\partial s''^2}\Pi^3_{T'}v(A^k_T)-\frac{\partial^2}{\partial s''^2}\Pi^3_{T''}v(A^k_T)\right|^2\Big)$$

$$\leq C\sum_{\substack{\tilde{T}\in\mathcal{T}_h\\ A^k_T\in\tilde{T}}}|v|^2_{3,\tilde{T}}.$$

Hence, from (2.20), we have

$$\sum_{i+j=2}\left|\frac{\partial^2}{\partial x^i\partial y^j}\Pi^3_h v(A^k_T)-\frac{\partial^2}{\partial x^i\partial y^j}\tilde{\Pi}_h v(A^k_T)\right|^2\leq C\sum_{\substack{\tilde{T}\in\mathcal{T}_h\\ A^k_T\in\tilde{T}}}|v|^2_{3,\tilde{T}}. \tag{2.21}$$

If A^k_T is a vertex of $\partial\Omega$ and only element T contains A^k_T, then the three vertices of T are all on $\partial\Omega$. Hence $\Pi^3_h v|_T\equiv 0$ and (2.21) is also true.

Thirdly, A^k_T is on an edge of $\partial\Omega$ and is not the vertex of Ω. Denote the unit normal of the edge by N and the unit tangent of the edge by s. From the definitions of $\tilde{\Pi}_h v$, we have

$$\sum_{i+j=2}\left|\frac{\partial^2}{\partial x^i\partial y^j}\Pi^3_h v(A^k_T)-\frac{\partial^2}{\partial x^i\partial y^j}\tilde{\Pi}_h v(A^k_T)\right|^2$$

$$\leq C\sum_{i+j=2}\left|\frac{\partial^2}{\partial N^i\partial s^j}\Pi^3_h v(A^k_T)-\frac{\partial^2}{\partial N^i\partial s^j}\tilde{\Pi}_h v(A^k_T)\right|^2$$

$$=C\Big(\left|\frac{\partial^2}{\partial s^2}\Pi^3_h v(A^k_T)\right|^2+\left|\frac{\partial^2}{\partial N\partial s}\Pi^3_h v(A^k_T)\right|^2$$

$$+\left|\frac{\partial^2}{\partial N^2}\Pi^3_h v(A^k_T)-\frac{\partial^2}{\partial N^2}\tilde{\Pi}_h v(A^k_T)\right|^2\Big).$$

As in the second case, we can show that

$$|\frac{\partial^2}{\partial s^2}\Pi^3_h v(A^k_T)\Big|^2+\left|\frac{\partial^2}{\partial N\partial s}\Pi^3_h v(A^k_T)\right|^2\leq C\sum_{\substack{\tilde{T}\in\mathcal{T}_h\\ A^k_T\in\tilde{T}}}|v|^2_{3,\tilde{T}},$$

and in the way for the first case, we can prove that

$$\Big|\frac{\partial^2}{\partial N^2}\Pi_h^3 v(A_T^k) - \frac{\partial^2}{\partial N^2}\tilde{\Pi}_h v(A_T^k)\Big|^2 \leq C \sum_{\substack{\tilde{T}\in\mathcal{T}_h\\ A_T^k\in\tilde{T}}} |v|_{3,\tilde{T}}^2.$$

It follows that (2.21) is true for the third case as well.

Combining the discussion of three cases and (2.18), we get

$$|\Pi_h^3 v - \tilde{\Pi}_h v|_{0,\Omega} \leq Ch^3|v|_{3,\Omega}. \tag{2.22}$$

Lemma 4 follows from (2.17), (2.22) and the inverse inequality for polynomials.

From Lemma 4 and the Aubin-Nitshe technique, we can prove the following lemma.

Lemma 5. *Let u be the solution of (1.1) and u_h the solution of (1.5) with V_h the Argyris element space. Then*

$$\|u-u_h\|_{m,\Omega} \leq Ch^{3-m}|u|_{3,\Omega}, \quad m=1,2 \tag{2.23}$$

when $f\in H^{-1}(\Omega)$, and

$$\|u-u_h\|_{1,\Omega} \leq Ch^5|u|_{6,\Omega} \tag{2.24}$$

when $u\in H^6(\Omega)\cap H_0^2(\Omega)$.

3. The proof of Theorem 1

In this section, we will prove Theorem 1. Firstly, let the solution u of (1.1) be in $H^6(\Omega)$. From (2.6), Lemma 5 and the interpolation theory, we have

$$\begin{aligned}\|u-u_h\|_{0,\infty,\Omega} \leq& \|u-\Pi_h u\|_{0,\infty,\Omega} + \|\Pi_h u - u_h\|_{0,\infty,\Omega}\\ \leq& C\Big(h^5|v|_{6,\Omega} + |\ln h|^{1/2}|\Pi_h u - u_h|_{1,\Omega}\Big)\\ \leq& Ch^5|\ln h|^{1/2}|u|_{6,\Omega},\end{aligned}$$

i.e., (2.3) is true. What remains is to prove (2.2). Assume $u\in W^{6,\infty}(\Omega)$. By the interpolation result (see [1]), we have

$$\begin{aligned}|u-u_h|_{1,\infty,\Omega} \leq& |u-\Pi_h u|_{1,\infty,\Omega} + |\Pi_h u - u_h|_{1,\infty,\Omega}\\ \leq& Ch^5|u|_{6,\infty,\Omega} + |\Pi_h u - u_h|_{1,\infty,\Omega}.\end{aligned} \tag{3.1}$$

So we must estimate $|\Pi_h u - u_h|_{1,\infty,\Omega}$. Let $T'\in\mathcal{T}_h$ be the element such that $|\Pi_h u - u_h|_{1,\infty,\Omega} = |\Pi_h u - u_h|_{1,\infty,T'}$. Without loosing generality, suppose that

$$|\Pi_h u - u_h|_{1,\infty,T'} = \Big|\frac{\partial(\Pi_h u - u_h)}{\partial x}\Big|_{0,\infty,T'}.$$

Let $(x_0,y_0)\in T'$ be the point such that

$$\Big|\frac{\partial(\Pi_h u - u_h)}{\partial x}\Big|_{0,\infty,T'} = \Big|\frac{\partial(\Pi_h u - u_h)}{\partial x}(x_0,y_0)\Big|.$$

To prove (2.2), we need some results about the weight function and the regular Green function. For (x_0, y_0), define the weight function ρ as

$$\rho(x,y) = (x-x_0)^2 + (y-y_0)^2 + \lambda^2 h^2,$$

with λ a fixed positive number. For integer α and a bounded domain $G \in R^2$, define

$$|v|_{m,(\alpha),G} = \left(\sum_{i+j=m} \int_G \rho^{-\alpha} \left| \frac{\partial^m v}{\partial x^i \partial y^j} \right|^2 dxdy \right)^{1/2} \tag{3.2}$$

when $v \in H^m(G)$. When $G = \Omega$, $|\cdot|_{m,(\alpha),G}$ is replaced by $|\cdot|_{m,(\alpha)}$. For the weight function, the following inequalities are true:

$$|v|_{m,(\gamma)} \le (\lambda h)^{-(\gamma-\alpha)} |v|_{m,(\alpha)}, \quad \gamma > \alpha, v \in H^m(\Omega), \tag{3.3}$$

$$|v|_{0,(1)} \le C|\ln h|^{1/2} \|v\|_{0,\infty,\Omega}, \qquad \forall v \in L^\infty(\Omega), \tag{3.4}$$

$$\left| \int_\Omega vw\, dxdy \right| \le |v|_{0,(\alpha)}\, |v|_{0,(-\alpha)}, \quad v, w \in L^2(\Omega), \tag{3.5}$$

$$|v - \Pi_T v|_{k,(\alpha),T} \le Ch^{6-k} |v|_{6,(\alpha),T}, \quad 0 \le k \le 6, v \in H^6(T), T \in \mathcal{T}_h. \tag{3.6}$$

Lemma 6. *There exists a constant C such that, for $v \in H_0^2(\Omega) \cap H^3(\Omega)$, the following inequalities are true:*

$$|v - \tilde{\Pi}_h v|_{m,(\alpha)} \le Ch^{3-m} |v|_{3,(\alpha)}, \quad 0 \le m \le 2. \tag{3.7}$$

Proof. Let $0 \le m \le 2$. Then

$$|v - \tilde{\Pi}_h v|^2_{m,(\alpha)} = \sum_{T \in \mathcal{T}_h} |v - \tilde{\Pi}_h v|^2_{m,(\alpha),T} \le \sum_{T\in\mathcal{T}_h} \|\rho^{-\alpha}\|_{0,\infty,T} |v - \tilde{\Pi}_h v|^2_{m,T}. \tag{3.8}$$

For $T \in \mathcal{T}_h$, let $S_h(T) = \{\tilde{T} \in \mathcal{T}_h; T \cap \tilde{T} \ne \emptyset\}$. From the proof of Lemma 4, we see

$$|v - \tilde{\Pi}_h v|^2_{m,T} \le Ch^{2(3-m)} \sum_{\tilde{T} \in S_h(T)} |v|^2_{3,\tilde{T}}. \tag{3.9}$$

For $\forall T \in \mathcal{T}_h$, we have

$$\max_{\substack{\tilde{T}\in S_h(T)\\(x,y)\in\tilde{T}}} \rho(x,y) \le C \min_{\substack{\tilde{T}\in S_h(T)\\(x,y)\in\tilde{T}}} \rho(x,y).$$

Then from (3.8) and (3.9), we get

$$|v - \tilde{\Pi}_h v|^2_{m,(\alpha)} =\le Ch^{2(3-m)} \sum_{T\in\mathcal{T}_h} \sum_{\tilde{T}\in S_h(T)} |v|^2_{3,\tilde{T}}.$$

Because the number of the elements in $S_h(T)$ is bounded, Lemma 6 is proved.

Now we turn to the regular Green function. Let $q \in P_5(T')$ satisfy

$$\int_{T'} qp\, dxdy = \frac{\partial}{\partial x} p(x_0, y_0), \qquad \forall p \in P_5(T').$$

Define $\delta_h \in L^2(\Omega)$ such that

$$\delta_h(x,y) = \begin{cases} q(x,y), & (x,y) \in T', \\ 0, & \text{otherwise}. \end{cases} \tag{3.10}$$

Let g be the regular Green function determined by

$$\begin{cases} \Delta^2 g = \delta_h, & \text{in } \Omega, \\ g|_{\partial\Omega} = \dfrac{\partial g}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{3.11}$$

and g_h be its finite element solution by Argyris element, i.e.,

$$a(g_h, v_h) = (\delta_h, v_h), \qquad \forall v_h \in V_h. \tag{3.12}$$

For δ_h, g and g_h, we have the following estimates:

$$\|\delta_h\|_{0,\Omega} \le Ch^{-2}, \tag{3.13}$$

$$\|\delta_h\|_{-1,\Omega} \le Ch^{-1}, \tag{3.14}$$

$$\|g - g_h\|_{1,\Omega} + h|g - g_h|_{2,\Omega} + h^2|g|_{3,\Omega} \le Ch, \tag{3.15}$$

$$\|g\|_{2,\Omega} \le C|\ln h|^{1/2}, \tag{3.16}$$

$$|g|_{3,(-1)} \le C|\ln h|^{1/2}, \tag{3.17}$$

which are similar to those in [3].

Lemma 7.

$$|g - g_h|_{2,(-1)} \le Ch|\ln h|^{1/2}. \tag{3.18}$$

Proof. By simple computation, we have

$$|g-g_h|^2_{2,(-1)} \le \int_\Omega \rho \left[\left(\frac{\partial^2}{\partial x^2}(g-g_h)\right)^2 + 2\left(\frac{\partial^2}{\partial x \partial y}(g-g_h)\right)^2 + \left(\frac{\partial^2}{\partial y^2}(g-g_h)\right)^2 \right] dxdy$$

$$\le |a(g-g_h, \rho(g-g_h))| + \left|\left(\frac{\partial^2(g-g_h)}{\partial x^2}, g-g_h\right)\right| + \left|\left(\frac{\partial^2(g-g_h)}{\partial y^2}, g-g_h\right)\right|$$

$$+C\sum_{i+j=2}\int_\Omega \left|\frac{\partial^2}{\partial x^i \partial y^j}(g-g_h)\right|\rho^{1/2}\left|\frac{\partial(g-g_h)}{\partial x}\right| dxdy$$

$$+C\sum_{i+j=2}\int_\Omega \left|\frac{\partial^2}{\partial x^i \partial y^j}(g-g_h)\right|\rho^{1/2}\left|\frac{\partial(g-g_h)}{\partial y}\right| dxdy.$$

From (3.5) and Green formula, we get

$$|g-g_h|^2_{2,(-1)} \le |a(g-g_h, \rho(g-g_h))| + C|g-g_h|_{2,(-1)}|g-g_h|_{1,\Omega}$$

$$+ \left|\left(\frac{\partial^2(g-g_h)}{\partial x^2}, g-g_h\right)\right| + \left|\left(\frac{\partial^2(g-g_h)}{\partial y^2}, g-g_h\right)\right|$$

$$\leq |a(g-g_h,\rho(g-g_h))| + C|g-g_h|_{2,(-1)}|g-g_h|_{1,\Omega}$$
$$+ |\frac{\partial}{\partial x}(g-g_h)|^2_{0,\Omega} + |\frac{\partial}{\partial y}(g-g_h)|^2_{0,\Omega}.$$

By (3.15), we have

$$|g-g_h|^2_{2,(-1)} \leq |a(g-g_h,\rho(g-g_h))| + Ch|g-g_h|_{2,(-1)} + Ch^2. \tag{3.19}$$

From (3.3), (3.5) and (3.7), we have

$$|a(g-g_h,\rho(g-\tilde{\Pi}_h g))| \leq C|g-g_h|_{2,(-1)}|\rho(g-\tilde{\Pi}_h g)|_{2,(1)}$$
$$\leq C|g-g_h|_{2,(-1)}(|g-\tilde{\Pi}_h g|_{2,(-1)} + |g-\tilde{\Pi}_h g|_{1,(0)} + |g-\tilde{\Pi}_h g|_{0,(1)})$$
$$\leq Ch|g-g_h|_{2,(-1)}(|g|_{3,(-1)} + h|g|_{3,\Omega} + h^2|g|_{3,(1)})$$
$$\leq Ch|g-g_h|_{2,(-1)}(|g|_{3,(-1)} + h|g|_{3,\Omega}).$$

From (3.17), we get

$$|a(g-g_h,\rho(g-\tilde{\Pi}_h g))| \leq Ch|\ln h|^{1/2}|g-g_h|_{2,(-1)}. \tag{3.20}$$

For $v \in L^2(T)$, let $P_T^0 v \in L^2(T)$ be the orthogonal projection of v in $P_0(T)$. From (1.4), (1.5), (3.5) and (3.6), we have

$$|a(g-g_h,\rho(\tilde{\Pi}_h g-g_h))| = |a(g-g_h,\rho(\tilde{\Pi}_h g-g_h) - \Pi_h(\rho(\tilde{\Pi}_h g-g_h)))|$$
$$\leq C|g-g_h|_{2,(-1)}\Big(\sum_{T\in\mathcal{T}_h}\Big|\rho(\tilde{\Pi}_h g-g_h) - \Pi_h(\rho(\tilde{\Pi}_h g-g_h))\Big|^2_{2,(1),T}\Big)^{1/2}$$
$$= C|g-g_h|_{2,(-1)}\Big(\sum_{T\in\mathcal{T}_h}|(\rho-P_T^0\rho)(\tilde{\Pi}_h g-g_h) - \Pi_h(\rho-P_T^0\rho)(\tilde{\Pi}_h g-g_h)|^2_{2,(1),T}\Big)^{1/2}$$
$$\leq Ch^4|g-g_h|_{2,(-1)}\Big(\sum_{T\in\mathcal{T}_h}|(\rho-P_T^0\rho)(\tilde{\Pi}_h g-g_h)|^2_{6,(1),T}\Big)^{1/2}.$$

On the other hand,

$$\sum_{T\in\mathcal{T}_h}|(\rho-P_T^0\rho)(\tilde{\Pi}_h g-g_h)|^2_{6,(1),T} \leq C\sum_{T\in\mathcal{T}_h}\Big(h^2|\tilde{\Pi}_h g-g_h|^2_{6,T}$$
$$+ |\tilde{\Pi}_h g-g_h|^2_{5,T} + |\tilde{\Pi}_h g-g_h|^2_{4,(1),T}\Big) \leq Ch^{-6}\sum_{T\in\mathcal{T}_h}|\tilde{\Pi}_h g-g_h|^2_{2,T}$$
$$= Ch^{-6}|\tilde{\Pi}_h g-g_h|^2_{2,\Omega} \leq Ch^{-6}$$

by (2.16), (3.3), (3.15), the inverse inequality, and the inequality

$$\Big|\frac{\partial\rho}{\partial x}\Big|^2 + \Big|\frac{\partial\rho}{\partial y}\Big|^2 \leq C\rho.$$

Hence

$$|a(g-g_h,\rho(\tilde{\Pi}_h g-g_h))| \leq Ch|g-g_h|_{2,(-1)}. \tag{3.21}$$

Combining (3.19) to (3.21), we get

$$|g-g_h|_{2,(-1)}^2 \leq Ch|\ln h|^{1/2}(|g-g_h|_{2,(-1)}+h).$$

Inequality (3.18) follows.

From (1.4), (1.5), (3.5) and (3.10) and (3.12), we have

$$\begin{aligned}|\Pi_h u-u_h|_{1,\infty,\Omega} &= |(\delta_h, \Pi_h u-u_h)| = |a(g_h, \Pi_h u-u_h)|\\ &=|a(g-g_h, u-\Pi_h u)+(\Delta^2 g, \Pi_h u-u)|\\ &\leq C\Big(|g-g_h|_{2,(-1)}|u-\Pi_h u|_{2,(1)}+|\delta_h|_{0,(-1)}|u-\Pi_h u|_{0,(1)}\Big).\end{aligned}$$

By (3.4), (3.6), (3.10) and (3.18), we get

$$|\Pi_h u-u_h|_{1,\infty,\Omega} \leq Ch^5|\ln h|^{1/2}|u|_{6,(1)} \leq Ch^5|\ln h||u|_{6,\infty,\Omega}. \tag{3.22}$$

Inequalities (3.1) and (3.22) imply (2.2). Theorem 1 is proved.

4. Bell Element and Bogner-Fox-Schmit Element

First, let $\mathcal{T}_h$ be a subdivision of Ω by triangles and $V_h \subset H_0^2(\Omega)$ be Bell finite element space associated with $\mathcal{T}_h$. Then $V_h = \{\, v \mid v \in H_0^2(\Omega), v|_T \in P_5'(T), \forall T \in \mathcal{T}_h \,\}$.

Similarly to the analysis for Argyris element, we can prove the following.

Theorem 2. *Let V_h be Bell finite element space, u the solution of problem* (1.1) *and u_h the solution of problem* (1.5). *Then*

$$|u-u_h|_{1,\infty,\Omega} \leq Ch^4|\ln h||u|_{5,\infty,\Omega} \tag{4.1}$$

when $u \in W^{5,\infty}(\Omega)$, and

$$|u-u_h|_{0,\infty,\Omega} \leq Ch^4|\ln h|^{1/2}|u|_{5,\Omega} \tag{4.2}$$

when $u \in H^5(\Omega) \cap H_0^2(\Omega)$.

From now on, let Ω be a rectangle with its edges parallel to x or y axis respectively. Let $\mathcal{T}_h$ be a subdivision of Ω by rectangles with their edges parallel to x or y axis, and $V_h \subset H_0^2(\Omega)$ be Bogner-Fox-Schmit finite element space associated with $\mathcal{T}_h$. Then $v \in V_h$ if and only if the following are true:

1) $v|_T$ is in $Q_3(T)$ for all $T \in \mathcal{T}_h$.

2) v, $\frac{\partial}{\partial x}v$, $\frac{\partial}{\partial y}v$ and $\frac{\partial^2}{\partial x\partial y}v$ are continuous at the vertices and vanish at the vertices along $\partial\Omega$.

For this element, we have

Theorem 3. *Let V_h be Bogner-Fox-Schmit finite element space, u the solution of problem* (1.1) *and u_h the solution of problem* (1.5). *Then*

$$|u-u_h|_{1,\infty,\Omega} \leq Ch^3|\ln h||u|_{4,\infty,\Omega} \tag{4.3}$$

when $u \in W^{4,\infty}(\Omega)$, *and*

$$|u-u_h|_{0,\infty,\Omega} \le Ch^3|\ln h|^{1/2}|u|_{4,\Omega} \tag{4.4}$$

when $u \in H^4(\Omega) \cap H_0^2(\Omega)$.

The proof of Theorem 3 is similar to that of Theorem 1. The orders of h and the Sobolev norms should be changed relevantly, and the operator $\tilde{\Pi}_h$ should be replaced by the following.

For $T \in \mathcal{T}_h$, denote its vertices by $A_T^1, A_T^2, A_T^3, A_T^4$. For $v \in H^3(T)$, let $P_T^a v$ be the interpolation polynomial of v by Adini element, i.e., $P_T^a v \in P_3(T)+$ spann $\{x^3y, xy^3\}$, and let $\Pi_T^a v$, $\frac{\partial}{\partial x}\Pi_T^a v$ and $\frac{\partial}{\partial y}\Pi_T^a v$ equal v, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at vertices A_T^1 to A_T^4 respectively. For $v \in H^3(\Omega)$, let $\Pi_h^a v$ be determined by $\Pi_h^a v|_T = \Pi_T^a(v|_T), \forall T \in \mathcal{T}_h$.

For an arbitrary vertex A, let N_A be the number of elements containing point A. Obviously, N_A is bounded. For $v \in H^3(\Omega) \cap H_0^2(\Omega)$, $\tilde{\Pi}_h v \in V_h$ is determined by the following:

i) For each $T \in \mathcal{T}_h$, $\tilde{\Pi}_h v|_T \in Q_3(T)$.

2) $\tilde{\Pi}_h v$, $\frac{\partial}{\partial x}\tilde{\Pi}_h v$ and $\frac{\partial}{\partial y}\tilde{\Pi}_h v$ equal v, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at the vertices respectively.

3) Let A be an arbitrary vertex. If $A \in \Omega$, then

$$\frac{\partial^2}{\partial x \partial y}\tilde{\Pi}_h v(A) = \frac{1}{N_A}\sum_{\substack{T\in\mathcal{T}_h \\ A\in T}} \frac{\partial^2}{\partial x \partial y}\Pi_T^a v(A). \tag{4.5}$$

If A is on $\partial\Omega$, then

$$\frac{\partial^2}{\partial x \partial y}\tilde{\Pi}_h v(A) = 0. \tag{4.6}$$

For the operator $\tilde{\Pi}_h$ defined this way, Lemmas 4 and 6 are also true.

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