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APPROXIMATE METHODS FOR GENERALIZED INVERSES OF OPERATORS IN BANACH SPACES*

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Abstract

In this paper, we present the necessary and sufficient condition of convergence of several iterative methods for computing the generalized inverses of operators in Banach spaces. It is proved that the iterative methods converge to the generalized inverse of an Operator in Banach spaces if and only if these conditions are satisfied.

1. Introduction

In this paper, we will present the necessary and sufficient condition of convergence of the iterative methods for computing the generalized inverses of operators. Let X and Y be two Banach spaces, and $B[X, Y]$ be the Banach space consisting of all bounded linear operators from X into Y. $D(T)$, $R(T)$ and $N(T)$ denote the domain, range and null of T respectively. We assume that the closed subspace $N(T)$ of x has a topological complement $N(T)^c$ and the closed subspace $\overline{R(T)}$ of Y has a topological complement $\overline{R(T)}^c$. Thus,

$$X = N(T) \bigoplus N(T)^c, \quad Y = \overline{R(T)} \bigoplus \overline{R(T)}^c.$$

The above decomposition exists if and only if there exist projectors P and Q such that

$$PX = N(T), \quad QY = \overline{R(T)}.$$

In this case, Nashed [1] pointed out that the operator T has unique generalized inverses $T^+ \equiv T^+_{P,Q}$ ($T^+_{P,Q}$ implies that the operator T^+ depends on the projectors P and Q) such that

$$\begin{cases} D(T^+) = R(T) \bigoplus \overline{R(T)}^c, \quad N(T^+) = \overline{R(T)}^c, \\ R(T^+) = N(T)^c, \quad TT^+T = T, \quad T^+TT^+ = T^+, \quad \text{on} \quad D(T^+), \\ T^+T = I - P, \quad TT^+ = Q|_{D(T^+)}, \end{cases} \tag{1}$$

where $Q|_{D(T+)}$ is the restriction of Q on $D(T^+)$. In addition, T^+ is bounded if and only if $R(T)$ is closed in Y. In this paper, $R(T)$ is assumed to be closed. Then, we

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obviously have

$$\begin{cases} X = N(T) \bigoplus N(T)^c, \quad Y = R(T) \bigoplus R(T)^c, \\ D(T^+) = Y, \quad N(T^+) = R(T^c, \quad R(T^+) = N(T)^c \end{cases} \tag{2}$$

and

$$\begin{cases} TT^+T = T, \quad T^+TT^+ = T^+, \\ T^+T = P_{N(T)^C}, \quad TT^+ = P_{R(T)}. \end{cases} \tag{3}$$

From (3) we can easily obtain

$$\begin{cases} T^+P_{R(T)} = T^+, \quad P_{N(T)^C}T^+ = T^+, \\ TP_{N(T)^C} = T, \quad P_{R(T)}T = T. \end{cases} \tag{4}$$

2. The Newton-Raphson Iterative Methods

Theorem 1. *Let*

$$R_k = P_{R(T)} - TT_k, \quad T_{k+1} = T_k [I + R_k], \quad k \geq 0. \tag{5}$$

The sequence $\{T_k\}$ converges quadratically to T^+ if and only if

$$P_{N(T)^C}T_0 = T_0 \quad \text{and} \quad r_\sigma(R_0) < 1, \tag{6}$$

where I is the identity operator in Y, and $r_\sigma(R_0)$ is the spectral radius of R_0.

Proof. Suppose condition (6) holds. Then we can show that $P_{R(T)}R_k = R_k$ and $P_{N(T)^C}T_k = T_k$, for any $K = 0, 1, 2, \dots$ [3]. Thus,

$$\begin{aligned} R_k &= P_{R(T)} - TT_k = P_{R(T)} - TT_{k-1}[I + R_{k-1}] \\ &= R_{k-1} - TT_{k-1}R_{k-1} = P_{R(T)}R_{k-1} - TT_{k-1}R_{k-1} \\ &= R_{k-1}^2 = \cdots = R_0^{2^k} \end{aligned}$$

and

$$T^+R_k = T^+P_{R(T)} - T^+TT_k = T^+ - T_k. \tag{7}$$

Since $r_\sigma(R_0) < 1$, we have

$$\lim_k \|T^+ - T_k\| \leq \lim_k \|T^+\| \cdot \|R_0^{2^k}\| = 0.$$

Inversely, suppose $\|T_k - T^+\| \to 0$ as $K \to \infty$. Then,

$$\|R_0^{2^k}\| = \|TT^+ - TT_k\| \leq \|T\| \cdot \|T^+ - T_k\| \to 0$$

or

$$r_\sigma(R_0)^{2^k} = r_\sigma\left(R_0^{2^k}\right) \leq \|R_0^{2^k}\| \to 0.$$

Thus,

$$r_\sigma(R_0) < 1. \tag{8}$$

From (5) and (8), it follows that

$$T_k = T_0, \Pi_{i=0}^{k-1}\left[I + R_0^{2^i}\right] = T_0\left[I + R_0 + R_0^2 + \cdots + R_0^{2^k-1}\right],$$

and

$$\lim_k T_k = \lim_k T_0\left[I + R_0 + R_0^2 + \cdots + R_0^{2^k-1}\right],$$

namely

$$T^+ = T_0(I - R_0)^{-1},$$

or

$$T_0 = T^+(I - R_0) = T^+ - T^+P_{R(T)} + T^+TT_0 = P_{N(T)}cT_0,$$

and this completes the proof of Theorem 1.

Corollary 1. Let $X = H_1$, $Y = H_2$ be two Hilbert spaces. Suppose $T_0 = \alpha T^*$ and the scalar α satisfies $0 < \alpha < 2\|T\|^{-2}$. Then the iterative sequence

$$T_{k+1} = T_k\left[2I_2 - TT_k\right], \qquad k \geq 0, \tag{9}$$

converges unifo**r**mly to the Moore-Penrose inverses T^+. Here T^* is the conjugate operator of T, and I_2 is the identity operator in H_2.

Proof. To prove Corollary 1, we need only to show that T_0 satisfies condition (6) and (9) is equivalent to (5). First of all, we notice

$$P_{N(T)^\perp}T_0 = P_{R(T^*)}\alpha T^* = \alpha T^* = T_0.$$

Hence, the first condition of (6) is satisfied. We will prove by induction that $T_kP_{R(T)} = T_k$ for any integer $k \geq 0$.

Since

$$T_0P_{R(T)} = \alpha T^*P_{N(T^*)^\perp} = \alpha T^*,$$

we have

$$T_0P_{R(T)} = T_0.$$

Suppose

$$T_kP_{R(T)} = T_k.$$

Then,

$$T_{k+1}P_{R(T)} = T_k\left[2I_2 - TT_k\right]P_{R(T)} = 2T_kI_2 - T_kTT_k = T_k\left[2I_2 - TT_k\right] = T_{k+1},$$

and this completes the proof of induction. Thus (10) becomes (5):

$$T_{k+1} = T_k\left[2I_2 - TT_k\right] = T_k\left[I_2 + P_{R(T)} - TT_k\right] = T_k\left[I_2 + R_k\right].$$

To examine the second condition of (6), we notice that the operator $P_{R(T)} - \alpha TT^*$ is self-conjugate in H_2. Then,

$$r_\sigma(P_{R(T)} - \alpha TT^*) = \|P_{R(T)} - \alpha TT^*\| = \sup_{\|x\|=1} |\langle P_{R(T)}x, x\rangle - \alpha\langle TT^*x, x\rangle|.$$

Let $x \in H_2$,

$$x = x_1 + x_2 \in R(T) \overset{\perp}{\bigoplus} N(T^*).$$

Thus

$$\|P_{R(T)} - \alpha TT^*\| = \sup_{\substack{\|x\|=1 \\ x=x_1+x_0}} |\langle P_{R(T)}x_1, x_1\rangle - \alpha\langle TT^*x_1, x_1\rangle|$$

$$= \sup_{\substack{\|x_1\|\le 1 \\ x_1\in N(T^*)^\perp}} |\langle x_1, x_1\rangle - \alpha\langle TT^*x_1, x_1\rangle|.$$

Consider the Hilbert space $\tilde{H} = N(T^*)^\perp$. Then the operator $\tilde{T} = TT^*|_{\tilde{H}}$ is self-conjugate in $\tilde{H}$. By (10), we have

$$\|P_{R(T)} - \alpha TT^*\| = \|I - \alpha\tilde{T}\| = r_\sigma\left(I - \alpha\tilde{T}\right).$$

Using the spectral mapping theorem, we obtain

$$r_\sigma\left(I - \alpha\tilde{T}\right) = \sup\left\{|\beta| : \beta = 1 - \alpha\lambda, \lambda \in \sigma\left(\tilde{T}\right)\right\}.$$

But $\sigma(\tilde{T}) \subseteq (0, \|T\|^2]$[4] and $0 < \alpha < 2\|T\|^{-2}$. We conclude that

$$r_\sigma(I - \alpha\tilde{T}) < 1$$

or

$$\|P_{R(T)} - TT_0\| < 1.$$

Thus

$$r_\sigma(R_0) \le \|P_{R(T)} - TT_0\| < 1,$$

and this completes the proof of Corollary 1.

3. Hypepower Methods and Euler-Knopp Iteration

Theorem 2. *Let*

$$R_k = P_{R(T)} - TT_k, \quad k \ge 0, \tag{11}$$

$$T_{k+1} = T_k[I + R_k + R_k^2 + \cdots + R_k^{P-1}] \quad k \ge 0, \quad p \ge 2. \tag{12}$$

The sequence $\{T_k\}$ *converges to* T^+ *with order* P *if and only if* $P_{N(T)}cT_0 = T_0$ *and* $r_\sigma(R_0) < 1$.

Proof. The proof is analogous to the proof of Theorem 1.

Corollary 2. Let $X = H_1$, $Y = H_2$ be two Hilbert spaces, $T_0 = \alpha T^*$, and let the scalar α satisfy $0 < \alpha < 2\|T\|^{-2}$. Then the iterative sequence

$$T_{k+1} = T_k \sum_{i=0}^{p-1} [2I_2 - TT_k]^i \quad k \ge 0, \quad p \ge 2 \tag{13}$$

converges uniformly to the Moore-Penrose inverses T^+ with order P.

Theorem 3. *The Euler-Knopp iterative sequence*

$$T_{n+1} = T_n \left[P_{R(T)} - TT_0\right] + T_0, \quad n \geq 0 \tag{14}$$

converges uniformly to

$$T^+ = T_0(I - P_{R(T)} + TT_0)^{-1}$$

if and only if $P_{N(T)}cT_0 = T_0$ *and* $r_\sigma(R_0) < 1$, *where* $R_0 = P_{R(T)} - TT_0$.

Proof. From (14), we can show

$$T_n = T_0 \sum_{k=0}^{n} \left(P_{R(T)} - TT_0\right)^k, \quad n \geq 0. \tag{15}$$

Noticing (4) and $P_{N(T)}cT_0 = T_0$, we have

$$\begin{aligned}
T^+ - T_n &= T^+ - T_0 \sum_{k=0}^{n} \left(P_{R(T)} - TT_0\right)^k \\
&= P_{N(T)}cT^+ - P_{N(T)}cT_0 \sum_{k=0}^{n} \left(P_{R(T)} - TT_0\right)^k \\
&= T^+TT^+ - T^+TT_0 \sum_{k=0}^{n} \left(P_{R(T)} - TT_0\right)^k \\
&= T^+ \left\{P_{R(T)} - TT_0 \sum_{k=0}^{n} \left(P_{R(T)} - TT_0\right)^k\right\} \\
&= T_+ \left\{P_{R(T)} - TT_0 - TT_0 \sum_{k=1}^{n} \left(P_{R(T)} - TT_0\right)^k\right\} \\
&= T^+ \left\{(P_{R(T)} - TT_0 \left(P_{R(T)} - TT_0\right) - TT_0 \sum_{k=2}^{n} \left(P_{R(T)} - TT_0\right)^k\right\}.
\end{aligned}$$

Since $P_{R(T)}TT_0 = TT_0$ and $P^2_{R(T)} = P_{R(T)}$, we have

$$\begin{aligned}
T^+ - T_n &= T^+ \left\{(P_{R(T)} - TT_0)^2 - TT_0 \sum_{k=2}^{n} \left(P_{R(T)} - TT_0\right)^k\right\} \\
&\vdots \\
&= T^+ \left\{\left(P_{R(T)} - TT_0\right)^n\right\}.
\end{aligned}$$

Noticing the condition $r_\sigma(R_0) < 1$, we have

$$\|T^+ - T_n\| \leq \|T^+\| \cdot \|R_0^n\| \to 0, \quad n \to \infty.$$

In addition, the Neumann series

$$\sum_{k=0}^{\infty} \left(P_{R(T)} - TT_0\right)^k$$

converges uniformly to

$$\left(I - P_{R(T)} + TT_0\right)^{-1}$$

if and only if $r_\sigma(R_0) < 1$. Inversely, suppose

$$\lim_n T_n = \lim_n T_0 \quad \sum_{k=0}^{n} \left(P_{R(T)} - TT_0\right)^k,$$

namely

$$T^+ = T_0\left(I - P_{R(T)} - TT_0\right)^{-1}.$$

This implies $r_\sigma(R_0) < 1$ and

$$T_0 = T^+\left(I - P_{R(T)} + TT_0\right) = T^+ - T^+P_{R(T)} + T^+TT_0 = P_{N(T)^c}T_0.$$

This completes the proof of Theorem 3.

Corollary 3. Let $X = H_1$, $Y = H_2$ be two Hilber spaces. Suppose $T_0 = \alpha T^*$, and the scalar α satisfies $0 < \alpha < 2\|T\|^{-2}$. Then the Euler-Knopp iterative sequence

$$T_0 = \alpha T^*,$$

$$T_{n+1} = T_n\left[I_2 - \alpha TT^*\right] + \alpha TT^*, \quad n \geq 0,$$

converges uniformly to the Moore-Penrose inverses T^+.

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