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INVERSE SPECTRUM PROBLEMS FOR BLOCK JACOBI MATRIX*

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Abstract

By establishing the spectrum (matrix) function for the block Jacobi matrix, theorems of existence and uniqueness for the inverse problem and algorithms for its solution are obtained. The study takes into account all possible multiple-eigenvalue cases that are very difficult to deal with by other means.

1. Introduction

There is an extensive study [1–5] on inverse eigenvalue problems for Jacobi matrices, but only a few papers [4] deal with block or banded matrices which arise more often in practice. In these papers most work is restricted to the case of simple eigenvalue only, which is often not the case in practice. Further study of such problems involving multiple eigenvalues is urgently needed.

In this paper, we study the Jacobi matrix with entries as $r\times r$ matrices of the form:

$$A=\begin{bmatrix} a_1 & b_1 & & & \\ b_1^* & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & b_{n-1} \\ & & & b_{n-1}^* & a_n \end{bmatrix},$$

$a_j,\ b_j\ \ r\times r$ matrices on C, $a_j=a_j^*$, all b_j invertible.

We denote the set of all $r\times r$ matrices on C as a ring F. The main points we must bear in mind are

(1) for $a,b\in F$, then $ab\neq ba$ in general,

(2) for $a\in F$ and $a\neq 0, a^{-1}$ may not exist in general.

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First, we need to establish operations in F, and some definitions. Then is Section 2 we will apply the Fourier theory to the matrix A. This fresh approach will turn out to be very useful. In Section 3. a general inverse spectrum problem and the main theorems of the problem will be given. Two algorithms based on G-L theory are formulated, and another algorithm based on orthogonalization is given in Section 4. Finally, a brief description of the applications and numerical testing will be presented. The main advantage of our treatment is that; our conclusion takes into account all possible multiple-eigenvalue cases that are difficult to deal with by other means.

Beside the well known operations such as multiplication, scalar multiplication and addition, we recall the conjugate operation '$*$' in F:

(1) for $a \in F, a^* = \bar{a}^T, (a^*)^* = a,$ (2)

(2) for $a, b \in F, (ab)^* = b^*a^*,$

(3) for $a \in F, a^*a$ and aa^* are semi-positive and self-adjoint,

(4) $a = 0 \leftrightarrow a^*a = aa^* = 0.$

Let H denote a linear space of n-dimensional column vectors on F, that is

$$H = \{f = (f_1, f_2, \cdots, f_n)^T, \text{ all } f_K \in F\}.$$

And denote H^* as its conjugate space, that is

$$H^* = \{f^* = (f_1^*, f_2^*, \cdots, f_n^*), \text{ all } f_K \in F\}.$$

Then we define the inner product of $f, g \in H$ as follows:

(1) $(f, g) = f^*g = f_1^*g_1 + f_2^*g_2 + \cdots + f_n^*g_n \in F,$ (3)

(2) $(f, g) = (g, f)^*,$

(3) $(f, f) = f^*f \geq 0,$

(4) $f = 0 \leftrightarrow (f, f) = 0.$

Denote by 0 and I the zero element and identity element respectively of either F or H or some matrices without causing confusion.

Since (f, g) is a matrix in F, then it is natural to regard $a^*a(aa^*)$ in (2) and (f, f) in (3) as certain matrix measurement instead of a usual norm. This is something interesting beyond our problem.

2. Fourier Theory for a Block Jacobi Matrix

First, we introduce an eigenfunction $\phi(\lambda), \lambda \in (-\infty, +\infty)$, for (1) as follows:

$$\phi(\lambda) = (\phi_1(\lambda), \phi_2(\lambda), \cdots \phi_n(\lambda))^T, \phi(\lambda) = I,$$

$$A\phi(\lambda) = \lambda\phi(\lambda) + R(\lambda), \quad R(\lambda) = (0, 0, \cdots, \Phi_n(\lambda))^T \tag{4}$$

where $\Phi_n(\lambda) = b_{n-1}^*\phi_{n-1}(\lambda) + (a_n - \lambda)\phi_n(\lambda)$. In general $\Phi_n(\lambda) \neq 0$, unless when λ is an eigenvalue, such that det $|\Phi_n(\lambda)| = 0$ and the homogeneous system $\Phi_n(\lambda)\gamma = 0$ has nontrival solutions $\gamma = \gamma(\lambda)$.

Let λ_k be an eigenvalue with multiplicity s_k. Then the homogeneous system possesses s_k linearly independent solutions $\gamma^l(\lambda_k), l = 1, 2, \cdots, s_k \le r$. Furthermore, we normalize $\phi(\lambda_k)\gamma^l(\lambda_k)$ as follows:

$$(\phi(\gamma^l)^*(\phi\gamma^k)_{\lambda=\lambda_k} = \gamma^{l*}\phi^*\phi\gamma^k = \delta_{kl}I, \quad \delta_{kk} = 1; \quad \delta_{kl} = 0 \text{ for } k \ne l. \tag{5}$$

If we take into account the multiplicity, then we may omit the superscript of γ, and simply write $\gamma(\lambda_k)$.

Definition. *The spectrum function $\rho(\lambda)$ of the block matrix (1) is defined by*

$$\rho(\lambda) = \sum_{\lambda_k<\lambda} \gamma(\lambda_k)\gamma^*(\lambda_k), \quad \lambda \in (-\infty, +\infty).$$

By the definition, $\rho(\lambda)$ is a nondecreasing and semi-positive function:

(1) *for $\lambda, \mu \in (-\infty, +\infty)$ and $\lambda < \mu$, $\rho(\lambda) \le \rho(\mu)$;*

(2) $\rho(\lambda) = \rho^*(\lambda) \ge 0$.

Remark 1. It is easy to check that for matrix (1) the spectrum function is uniquely determined and assumption (A2) in **3** must be valid.

Denote by $\hat{F}\rho$ the set of all functions of λ in F with measure $d\rho(\lambda)$. Then we define

$$(1)\ a(\lambda) \overset{\cdot\rho}{=} b(\lambda) \Leftrightarrow \text{ for all } c(\lambda) \in \hat{F}_\rho, \int a(\lambda)d\rho(\lambda)c(\lambda) = \int b(\lambda)d\rho(\lambda)c(\lambda), \tag{6}$$

$$(2)\ a(\lambda) \overset{\rho\cdot}{=} b(\lambda) \Leftrightarrow \text{ for all } c(\lambda) \in \hat{F}_\rho, \int c(\lambda)d\rho(\lambda)a(\lambda) = \int c(\lambda)d\rho(\lambda)b(\lambda),$$

(3) $a(\lambda) \overset{\cdot\rho}{=} b(\lambda) \Leftrightarrow a^*(\lambda) \overset{\rho\cdot}{=} b^*(\lambda)$.

The same definitions are used for vectors. If $\hat{H}\rho$ denotes the space of vector functions of λ, then $\phi(\lambda) \in \hat{H}\rho$ and in the above sense we have

$$A\phi(\lambda) \overset{\cdot\rho}{=} \lambda\phi(\lambda). \tag{7}$$

For each $f \in H$, we define its Fourier transform with respect to $\phi(\lambda)$ by

$$\hat{f}(\lambda) = \phi^*(\lambda)f \in \hat{F}_\rho \tag{8}$$

and the inverse Fourier transform by

$$f = \int_{-\infty}^{+\infty} \phi(\lambda)d\rho(\lambda)\hat{f}(\lambda). \tag{8'}$$

By direct calculating it is easy to derive (8′) from (8). Since

$$f = \int_{-\infty}^{+\infty} \phi(\lambda)d\rho(\lambda)\hat{f}(\lambda) = \int_{-\infty}^{+\infty} \phi d\rho\phi^* f, \quad \text{for all } f \in H, \tag{9}$$

we have

$$I = \int_{-\infty}^{+\infty} \phi(\lambda)d\rho\phi^*(\lambda). \tag{10}$$

Notice that if all b_j's are invertible, then for $\phi(\lambda) = (\phi_1, \phi_2, \cdots, \phi_n), \phi_k(\lambda)$ is just a polynomial in λ of order $k-1$, i.e.

$$\phi_k(\lambda) = b_{k-1}^{-1} \cdots b_1^{-1}\lambda^{k-1} + \cdots = (b_1 b_2 \cdots b_{k-1})^{-1}\lambda^{k-1} + \cdots, I = \phi_1(\lambda) \tag{11}$$

and it is easy to see that the transforms defined by (8) and by (8') are in one to one correspondence. Furthermore,

$$\hat{f}(\lambda) = \int_{-\infty}^{+\infty} \phi^*(\lambda)\phi(\mu)d\rho(\mu)\hat{f}(\mu) = \int_{-\infty}^{+\infty} \delta_\rho(\lambda,\mu)d\rho(\mu)\hat{f}(\mu), \qquad \text{for all } \hat{f} \in \hat{H}_\rho. \quad (12)$$

So we denote $\phi^*(\lambda)\phi(\mu) = \delta_\rho(\lambda,\mu)$-the Dirac delta function with respect to $\rho(\lambda)$. Now we reach a generalized form of Parsaval equality:

$$f^*g = \int_{-\infty}^{+\infty} \hat{f}^*(\lambda)d\rho(\lambda)\hat{g}(\lambda), \qquad \text{for all } f,g \in H,$$

or in particular

$$f^*f = \int_{-\infty}^{+\infty} \hat{f}^*(\lambda)d\rho(\lambda)\hat{f}(\lambda), \qquad \text{for all } f \in H. \quad (13)$$

3. The Inverse Eigenvalue Problem

In this section we will formulate the inverse eigenvalue problem and present the main result for a block Jacobi matrix. First we need the following two assumptions (A1) and (A2):

(A1) all $b_k, k = 1, 2, \cdots, n-1$, are invertible, and are fully determined by $a_1, \cdots, a_k$ and $b_1, \cdots b_{k-1}$.

In practice b_k may be given, or only depend on a_k. For the completeness of the spectrum function $\rho(\lambda)$ we also assume that

(A2) rank $\{\rho(\lambda_k + 0) - \rho(\lambda_k - 0)\} = r_k \le r, r_k$-multiplicity of λ_k; $\sum_k r_k = r \cdot n$.

It is easy to see that for a given block Jacobi matrix (1) assumption (A2) must be fulfilled.

Theorem 1. *Under* (A1) *and* (A2) *matrix* A *of form* (1) *is uniquely determined by the spectrum function* $\rho(\lambda)$.

Proof. Our proof of this theorem is constructive. First from (7) we have

$$\phi^* A = (A\phi)^* = \lambda\phi^* + R^*(\lambda),$$

i.e.

$$(\phi_1^*(\lambda), \cdots, \phi_n^*(\lambda))A = (\lambda\phi_1^*, \lambda, \phi_2^*, \cdots, \lambda\phi_n^* + \Phi^*(\lambda)).$$

This impplies

$$\phi_1^*(\lambda)a_1 + \phi_2^*(\lambda)b_1^* = \lambda\phi_1^*(\lambda),$$

$$\phi_1^*(\lambda)b_1 + \phi_2^*(\lambda)a_2^* + \phi^*(\lambda)b_2^* = \lambda\phi_2^*(\lambda),$$

$$\cdots\cdots$$

$$\phi_{n-1}^*(\lambda)b_{n-1} + \phi_n^*(\lambda)a_n^* = \lambda\phi_n^*(\lambda) + \Phi_n^*(\lambda) \overset{\rho}{=} \lambda\phi_n^*. \quad (14)$$

Moreover, relation (10) implies

$$\int_{-\infty}^{+\infty} \phi_k(\lambda)d\rho(\lambda)\phi_j^*(\lambda) = I\delta_{kj}, \delta_{kk} = 1, \delta_{kj} = 0 \text{ for } k \neq j.$$

Therefore, premultiplying the first euqation of (14) by $\phi_1(\lambda)$ and using (14) gives

$$a_1 = \int \lambda\phi_1(\lambda)d\rho\phi_1^*(\lambda), \phi_1(\lambda) = \phi_1^*(\lambda) = I.$$

Now, by (A1), b_1 can be found and $\phi_1(\lambda) = b_1^{-1}(\lambda - a_1), \cdots$. If $a_1, a_2, \cdots, a_{k-1}$; $b_1, b_2, \cdots, b_{k-1}$ and $\phi_1, \phi_2, \cdots, \phi_k$ are known, then from the k-th equality of (14), we can find

$$a_k = \int_{-\infty}^{+\infty} \lambda\phi_k(\lambda)d\rho\phi_k^*(\lambda), b_k, \text{ and } \phi_{k+1}(\lambda), \quad k = 2, 3, \cdots.$$

By induction, the proof is completed.

The proof of the theorem is constructive, but it does not provide a practical algorithm for numerical computation, because the construction is numerically unstable for large n (see de Boor and Golub [2] for $r = 1$). Interestingly we have the following theorem:

Theorem 2. *If A is a block Jacobi matrix of form* (1), *where all b_k's are nonsingular, then under* (A2) *all a_k and $b_k b_k^*$ are uniquely determined by the spectrum function $\rho(\lambda)$.*

Proof. There is only a little difference in the proof. First we have

$$a_1 = \int \lambda\phi_1 d\rho\phi_1^* \quad \text{and} \quad b_1 = \int \lambda\phi_1 d\rho\phi_2^*, \phi_1 = I.$$

Then we have

$$b_1 b_1^* = \int \lambda\phi d\rho(b_1\phi_2)^* = \int \lambda\phi_1 d\rho((\lambda - a_1)\phi_1)^*.$$

Similarly,

$$a_k = \int \lambda\phi_k d\rho\phi_k^*,$$

$$b_k b_k^* = \int \lambda\phi_k d\rho\Phi_k^*(\lambda), \quad \Phi_k(\lambda) = b_{k-1}^*\phi_{k-1} + (\lambda - a_k)\phi_k,$$

$$k = 1, 2, \cdots, n-1, \quad a_n = \int \lambda\phi_n d\rho\phi_n^*.$$

The proof is completed.

Remark 2. Since $b_k b_k^*$ is unique, the by QL factorization method we may suppose b_k are lower triangular matrices, and A becomes a band matrix.

Denote

$$(a(\lambda), b(\lambda)) = \int_{-\infty}^{+\infty} a(\lambda)d\rho b^*(\lambda), \quad a, b \in \hat{F}_\rho.$$

Let $P_k(\lambda) = \lambda^{k-1} + \cdots$ be orthogonal monic polynomials, then we have $P_k(\lambda) = b_1 b_2 \cdots b_{k-1}\phi_k(\lambda)$ and $(P_k, P_k) = b_1 b_2 \cdots b_{k-1}(b_1 b_2 \cdots b_{k-1})^*$.

Theorem 3. *Assumption* (A2) *insures all (P_k, P_k) and (b_k) invertible.*

Proof. If $(P_1,P_1),\cdots,(P_{k-1},P_{k-1})>0$, but (P_k,P_k) is singular, i.e. $\det|(P_k,P_k)|=0$, then for the following block $k\times k$ matrix

$$\Lambda=\begin{pmatrix}(1,1) & (1,\lambda) & & (1,\lambda^{k-1})\\ (\lambda,1)) & (\lambda,\lambda) & \cdots & (\lambda,\lambda^{k-1})\\ \cdots & \cdots & \cdots & \cdots\\ (\lambda^{k-1},1) & (\lambda^{k-1},\lambda) & \cdots & (\lambda^{k-1},\lambda^{k-1})\end{pmatrix},\quad \Lambda Q=0$$

the homogeneous system has a nontrivial solution $Q=(q_1,q_2,\cdots,q_k)^T$, which implies

$$(Q(\lambda),Q(\lambda))=0,\quad Q(\lambda)=q_k+q_{k-1}+\cdots q_1\lambda^{k-1},$$

i.e. $\det|Q(\lambda_j)|=0,\ j=1,2,\cdots,nr$. This means $\det Q(\lambda)$ must be a polynomial of order nr, which is impossible for $k\le n$.

Remark 3. Theorem 3 also shows the solvability of the inverse problem under assumption (A2).

4. Some Algorithms for Solving the Problem

Our original algorithm is based on the recurrence

$$b_{i-1}^*\phi_{i-1}(\lambda)+a_i\phi_i(\lambda)+b_i\phi_{i+1}(\lambda)=\lambda\phi_i(\lambda) \tag{1}$$

from Theorems 1–2, where $\phi_1(\lambda)=I$. Since $\{\phi_i(\lambda)\}$ form an orthonormal set with respect to the inner product associated with matrix A, it follows immediately that $a_i=(\lambda\phi_i(\lambda),\phi_i(\lambda))$ and $b_i=(\lambda\phi_i(\lambda),\phi_{i+1}(\lambda))$ whence

$$\begin{aligned}b_ib_i^*&=(\lambda\phi_i(\lambda),\phi_{i+1}(\lambda))b_i^*=(\lambda\phi_i(\lambda),b_i\phi_{i+1}(\lambda))\\&=(\lambda\phi_i(\lambda),\lambda\phi_i(\lambda)-a_i\phi_i(\lambda)-b_{i-1}^*\phi_{i-1}(\lambda))\\&=(\lambda\phi_i(\lambda),\lambda\phi_i(\lambda))-(\lambda\phi_i(\lambda),\phi_i(\lambda))a_i^*-(\lambda\phi_i(\lambda),\phi_{i-1}(\lambda))b_{i-1}.\end{aligned}$$

We use the recurrence, the equation for a_i and the last equation for $b_ib_i^*$ as the basis for our original algorithm. We take b_i to be the Cholesky factor of $b_ib_i^*$. Below we call this scheme Algorithm 1*a*.

We can improve this scheme by using

$$b_ib_i^*=(\lambda\phi_i(\lambda),\lambda\phi_i(\lambda))-a_ia_i^*-b_{i-1}^*b_{i-1}$$

in place of the equation for $b_ib_i^*$. To derive this last equation, start with the equation for $b_ib_i^*$ and substitute a_i for $(\lambda\phi_i(\lambda),\phi_i(\lambda))$ amd b_{i-1} for $(\lambda\phi_i(\lambda),\phi_{i-1}(\lambda))$, where the last substitution is based on the equality $b_{i-1}=(\lambda\phi_i(\lambda),\phi_{i-1}(\lambda))$ and the orthonormality of ϕ. Again we take b_i to be the Cholesky factor of $b_ib_i^*$. Below we call this revised scheme Algorithm 1b.

The new equation for $b_ib_i^*$ can be derived more easilty by noting

$$(\lambda\phi_i(\lambda), \lambda\phi_i(\lambda)) = (b_{i-1}^*\phi_{i-1}(\lambda) + a_i\phi_i(\lambda) + b_i\phi_{i+1}(\lambda), b_{i-1}^*\phi_{i-1}(\lambda) + a_i\phi_i(\lambda) + b_i\phi_{i+1}(\lambda)) = b_{i-1}^*b_{i-1} + a_ia_i^* + b_ib_i^*.$$

The third algorithm is based on the unique set of monic matrix polynomials satisfying $p_i(\lambda) = 0, p_1(\lambda) = I$, the recurrence

$$p_{i+1}(\lambda) = (\lambda - \alpha_i)p_i(\lambda) - \beta_i p_{i-1}(\lambda)$$

and orthogonality with respect to the inner product associated with A. Given p_i and P_{i-1}, orthogonality requires that

$$\alpha_i = (\lambda p_i(\lambda), p_i(\lambda))(p_i(\lambda), p_i(\lambda))^{-1}$$

and

$$\beta_i = (p_i(\lambda), p_i(\lambda))(p_{i-1}(\lambda), p_{i-1}(\lambda))^{-1}$$

since

$$(\lambda p_i(\lambda), p_{i-1}(\lambda)) = (p_i(\lambda), p_i(\lambda)).$$

Hence we can compute $p_{i+1}(\lambda)$ (Note β_i is not defined for $i = 1$, but this just reflects the fact that $p_2(\lambda) = (\lambda - \alpha_1)$). These equations for α_i and β_i can be used to calculate all α and β recursively. In this paper we have shown

$$a_i = b_{i-1}^{-1} \cdots b_1^{-1}\alpha_i b_1 \cdots b_{i-1}$$

and

$$b_ib_i^* = b_{i-1}^{-1} \cdots b_1^{-1}\beta_{i+1}b_1 \cdots b_{i-1}.$$

These last two equations can be used to calculate a and b from α and β, again using the Cholesky factor of $b_ib_i^*$ for b_i.

In our algorithm implementing this shceme, we start with α_i and multiply on the left and right by b_1^{-1} and b_1 respectively. We then multiply on the left and right by b_2^{-1} and b_2 respectively and so on. We use a similar approach to calculate $b_ib_i^*$. Below we call this scheme Algorithm 2. (It would be more efficient to form and update the products $b_{i-1}^{-1} \cdots b_1^{-1}$ and $b_1 \cdots b_{i-1}$, but we think this could lead to a loss of precision due to roundoff. We'll implement a scheme that uses this alternate approach later to check if this is the case.)

Problem 1.

$$a_i = \begin{bmatrix} 1.8480 & -0.3767 & -0.2032 \\ -0.3767 & 0.6053 & 0.0566 \\ -0.2032 & 0.0566 & 1.0467 \end{bmatrix} \quad \text{and } b_i = I \quad \text{for all } i.$$

	Algorithm 1a				
	$n=3$	$n=6$	$n=9$	$n=12$	$n=15$
$\|\Lambda-\tilde{\Lambda}\|_2$	9.2149D-15	1.8652D-14	7.9936D-15	1.2795D-14	1.1303D-14
$\|A-\tilde{A}\|_2$	1.0958D-14	4.5079D-14	3.2878D-13	3.3642D-11	4.9396D-10
$\|A\tilde{V}-\tilde{V}\tilde{\Lambda}\|_2$	1.2921D-14	4.4815D-14	3.2939D-13	3.3641D-11	4.9396D-10
$\|\tilde{A}V-V\Lambda\|_2$	1.1944D-14	4.4247D-14	3.2802D-13	3.3642D-11	4.9396D-10

	Algorithm 1b				
	$n=3$	$n=6$	$n=9$	$n=12$	$n=15$
$\|\Lambda-\tilde{\Lambda}\|_2$	7.7716D-15	1.8652D-14	6.2172D-15	1.8971D-14	1.5987D-14
$\|A-\tilde{A}\|_2$	1.3296D-14	4.4522D-14	3.4571D-13	3.4134D-11	5.0618D-10
$\|A\tilde{V}-\tilde{V}\tilde{\Lambda}\|_2$	1.3857D-14	4.3468D-14	3.4365D-13	3.4133D-11	5.0618D-10
$\|\tilde{A}V-V\Lambda\|_2$	1.39[illegible]1D-14	4.3663D-14	3.4480D-13	3.4134D-11	5.0618D-10

	Algorithm 2				
	$n=3$	$n=6$	$n=9$	$n=12$	$n=15$
$\|\Lambda-\tilde{\Lambda}\|_2$	3.1086D-15	4.4409D-15	4.4409D-15	3.9968D-15	5.9952D-15
$\|A-\tilde{A}\|_2$	3.1693D-15	2.0006D-14	3.4738D-13	3.3915D-11	5.0056D-10
$\|A\tilde{V}-\tilde{V}\tilde{\Lambda}\|_2$	4.4976D-15	2.1418D-14	3.4637D-13	3.3916D-11	5.0056D-10
$\|\tilde{A}V-V\Lambda\|_2$	3.5966D-15	1.9280D-14	3.4658D-13	3.3915D-11	5.0056D-10

Problem 2.

$$a_i = \begin{bmatrix} 1.8480 & -0.3767 & -0.2032 \\ -0.3767 & 0.6053 & 0.0566 \\ -0.2032 & 0.0566 & 1.0467 \end{bmatrix} \quad \text{and } b_i = \begin{bmatrix} .5 & 0 & 0 \\ 1 & 1 & 0 \\ .3 & .5 & 2 \end{bmatrix} \quad \text{for all } i.$$

	Algorithm 1a				
	$n=3$	$n=6$	$n=9$	$n=12$	$n=15$
$\|\Lambda-\tilde{\Lambda}\|_2$	2.3848D-13	1.0083D-09	9.6689D-06	0.5910	failed
$\|A-\tilde{A}\|_2$	6.0747D-13	4.8896D-09	6.1796D-05	2.6999	failed
$\|A\tilde{V}-\tilde{V}\tilde{\Lambda}\|_2$	6.0809D-13	4.8896D-09	6.1796D-05	2.6999	failed
$\|\tilde{A}V-V\Lambda\|_2$	6.0661D-13	4.8896D-09	6.1796D-05	2.6999	failed

	Algorithm 1b				
	$n = 3$	$n = 6$	$n = 9$	$n = 12$	$n = 15$
$\|\Lambda - \tilde{\Lambda}\|_2$	3.0309D-13	1.4447D-09	8.0614D-06	failed	failed
$\|A - \tilde{A}\|_2$	7.7805D-13	7.0050D-09	5.1522D-05	failed	failed
$\|A\tilde{V} - \tilde{V}\tilde{\Lambda}\|_2$	7.7690D-13	7.0050D-09	5.1522D-05	failed	failed
$\|\tilde{A}V - V\Lambda\|_2$	7.7706D-13	7.0050D-09	5.1522D-05	failed	failed

	Algorithm 2				
	$n = 3$	$n = 6$	$n = 9$	$n = 12$	$n = 15$
$\|\Lambda - \tilde{\Lambda}\|_2$	2.2204D-15	5.3291D-15	1.2879D-14	6.6169D-13	9.8049D-08
$\|A - \tilde{A}\|_2$	8.3176D-15	1.8641D-12	8.6729D-10	1.3233D-06	0.0013
$\|A\tilde{V} - \tilde{V}\tilde{\Lambda}\|_2$	7.8671D-15	1.8635D-12	8.6729D-10	1.3233D-06	0.0013
$\|\tilde{A}V - V\Lambda\|_2$	7.6669D-15	1.8665D-12	8.6729D-10	1.3233D-06	0.0013

In all tables of this paper "failed" means the program terminated with an error message indicating that $b_i b_i^*$ was not positive definite so it could not calculate its Cholesky factorization. For $n = 15$, we got several warnings from Algorithm 2 that the condition number of some matrices was large than 10^{17}, but the program did not crash and in fact managed to construct a matrix having eigenvalues relatively close to the required set.

Problem 3.

$$a_i = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \qquad \text{and} \qquad b_i = I \qquad \text{for all } i.$$

	Algorithm 1a				
	$n = 3$	$n = 6$	$n = 9$	$n = 12$	$n = 15$
$\|\Lambda - \tilde{\Lambda}\|_2$	1.3944D-13	1.1129D-08	1.2069D-05	failed	failed
$\|A - \tilde{A}\|_2$	3.4874D-13	3.8038D-08	6.7913D-05	failed	failed
$\|A\tilde{V} - \tilde{V}\tilde{\Lambda}\|_2$	3.4775D-13	3.8038D-08	6.7913D-05	failed	failed
$\|\tilde{A}V - V\Lambda\|_2$	3.4752D-13	3.8038D-08	6.7913D-05	failed	failed

	Algorithm 1b				
	$n = 3$	$n = 6$	$n = 9$	$n = 12$	$n = 15$
$\|\Lambda - \tilde{\Lambda}\|_2$	2.3448D-13	1.0357D-08	4.1061D-06	failed	failed
$\|A - \tilde{A}\|_2$	5.9899D-13	3.5400D-08	8.6591D-05	failed	failed
$\|A\tilde{V} - \tilde{V}\tilde{\Lambda}\|_2$	5.9729D-13	3.5400D-08	8.6591D-05	failed	failed
$\|\tilde{A}V - V\Lambda\|_2$	5.9889D-13	3.5400D-08	8.6591D-05	failed	failed

	Algorithm 2				
	$n=3$	$n=6$	$n=9$	$n=12$	$n=15$
$\|\Lambda-\tilde{\Lambda}\|_2$	3.5527D-15	5.3291D-15	9.7700D-15	6.2172D-15	9.7700D-15
$\|A-\tilde{A}\|_2$	5.4888D-15	1.0273D-13	1.1780D-11	3.4780D-10	1.8456D-08
$\|A\tilde{V}-\tilde{V}\tilde{\Lambda}\|_2$	7.2797D-15	1.0324D-13	1.1782D-11	3.4781D-10	1.8416D-08
$\|\tilde{A}V-V\Lambda\|_2$	6.6025D-15	1.0338D-13	1.1783D-11	3.4781D-10	1.8456D-08

For each of the test problems outlined above and the yack algorithm, the test procedure was as follows.

1. We generated the spectral data $\{(\lambda_i, v_i, \gamma_i)\}$ from A, where λ_i is an eigenvalue, v_i is the corresponding eigenvector and where γ_i is the first r components of v_i. We sorted $\{\lambda_i\}$ from the smallest to the largest and re-ordered the spectral data based on this. We formed and saved $\Lambda = \text{diag }(\lambda_i), V = [v_1, \cdots, v_{r,n}]]$ and $\Gamma = \{\gamma_1, \cdots \gamma_{r.n}]$.

2. We input the spectral data $\{(\lambda_i, \gamma_i)\}$ to the algorithm and generated a block Jacobi matrix $\tilde{A}$ baving $\tilde{b}_i$ lower triangular with positive diagonal elements. (This restricition makes the reconstruction unique.)

3. We computed the spectral data $\{(\tilde{\lambda}_i, \tilde{v}_i, \tilde{\gamma}_i)\}$ from $\tilde{A}$. We sorted $\{\tilde{\lambda}_i\}$ from the smallest to the largest and re-ordered the spectral data based on this. We formed and saved $\tilde{\Lambda} = \text{diag }(\tilde{\lambda}), \tilde{V} = \{\tilde{v}_1, \cdots, \tilde{v}_{r.n}]$ and $\tilde{\Gamma} = [\tilde{\gamma}_1, \cdots, \tilde{\gamma}_{r\cdot n}]$.

4. We printed the error measures $\|\Lambda-\tilde{\Lambda}\|_2, \|A-\tilde{A}\|_2, \|A\tilde{V}-\tilde{V}\tilde{\Lambda}\|_2$, and $\|\tilde{A}V-V\Lambda\|_2$.

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