Journal of Computational Mathematics, Vol.11, No.4, 1993, 301–312.

ON A CLASS OF ELLIPTIC PROBLEMS AND ITS APPLICATION TO HEAT TRANSFER IN NONCONVEX BODIES*

Rogério Martins Saldanha da Gama
(Laboratório Nacional de Comuptação Cientifica, Brazil)

Abstract

This work presents a procedure for constructing the solution to a class of problems with application to heat transfer processes in which the energy reemission is not negligible. Such problems are characterized by a Poisson equation subjected to certain nonlinear boundary conditions. The solution is constructed from a sequence whose elements may be obtained from a minimum principle. Some practical situations are presented.

1. Introduction

There exist many situations, in engineering design, in which the temperature distribution is an important consideration in the determination of the geometry, the dimensions, the material, etc., $\cdots$ of a given part of a body.

The most common mathematical model, for describing the energy transfer phenomenon and obtaining the temperature field in a body, is the linear one represented by a Poisson equation subjected to Dirichlet/Neumann boundary conditions[1,2]. This well known mathematical problem is present in almost all books on partial differential equations as, for instance, in [3].

In spite of its "popularity", the above mentioned model is not adequate for some important and complex phenomena such as the ones in which the temperature levels are so high and/or the ones in atmosphere-free space. Examples of such phenomena can be found in [4, 5, 6].

When some subset of the boundary of a body is at high temperature, the radiative energy transfer from the body to the environment can not be neglected. In addition, when the body boundary is not convex, the reemission phenomenon (a radiant emission from the body to itself) must also be taken into account.

The radiation heat transfer is, due to Stefan-Boltzmann law, an inherently nonlinear phenomenon[7].

Besides the radiative transfer, if there exists an atmosphere surrounding the body, the convective transfer from/to the body must be taken into account[2,6,8].

* Received May 7, 1991.

Combining the above mentioned energy transfer mechanisms from/to a body with the conduction heat transfer, which takes place inside the body, we have a large and interesting class of nonlinear mathematical problems.

The main objective of this work is to study the above mentioned class of problems, providing a way for constructing the solutions and presenting some of the most important physical applications.

2. Governing Equations

Let us consider a rigid and opaque body $\mathcal{B}$ represented by the bounded open set $\Omega(\Omega \subset I\!R^3$ - with the cone properties[9]) with regular boundary $\partial\Omega$. The steady-state energy transfer process, inside $\mathcal{B}$, is mathematically described by[2]

$$\Delta u = -r \quad \text{in } \Omega \tag{1}$$

in which r represents an energy source. When (1) represents a real heat transfer process we have that r is a piecewise continuous and bounded function that may depend on the position X and on the temperature $u(u = \tilde{u}(X), X \in \bar{\Omega})$. Here, we shall suppose that

$$\begin{cases} r = \hat{r}(\tilde{u}(X), X) \quad \text{for all } X \in \Omega, \\ \hat{r}(b, X) \leq \hat{r}(a, X) \text{ if } b > a \text{ for any } X \in \Omega, \\ \lim\limits_{a \to -\infty} \hat{r}(a, X) = \hat{r}^*(X), \quad r^* = \hat{r}^*(X), \quad r^* \in L^2(\Omega). \end{cases} \tag{2}$$

The main objective of this work is to construct the solution to (1) subjected to the following boundary condition:

$$-\frac{\partial u}{\partial n} = f - \mathcal{L}[g] - h \quad \text{on } \partial\Omega \tag{3}$$

in which $\partial/\partial n$ indicates differentiation in the direction of the exterior normal to $\partial\Omega, h$ is a known piecewise continuous and bounded function, f is a function such that

$$\begin{cases} f = \hat{f}(\tilde{u}(X), X) \text{ for all } X \in \partial\Omega, \\ \hat{f}(b, X) > \hat{f}(a, X) \text{ if } b > a \text{ for any } X \in \partial\Omega^* \subseteq \partial\Omega, \\ \hat{f}(b, X) \geq \hat{f}(a, X) \text{ if } b > a \text{ for any } X \in \partial\Omega, \\ \lim\limits_{|a| \to \infty} \left[\frac{a}{|a|}\hat{f}(a, X)\right] = \infty \text{ for any } X \in \partial\Omega^* \subseteq \partial\Omega, \\ -\infty < a < \infty \leftrightarrow -\infty < \hat{f}(a, X) < \infty \text{ for any } x \in \partial\Omega, \end{cases} \tag{4}$$

g is a function such that

$$\begin{cases} g = \hat{g}(\widetilde{u}(X), X) \text{ for all } X \in \partial\Omega, \\ \hat{g}(b, X) \geq \hat{g}(a, X) \text{ if } b > a \text{ for all } X \in \partial\Omega, \\ \hat{f}(b, X) - \hat{f}(a, X) \geq \hat{g}(b, X) - \hat{g}(a, X) \text{ if } b > a \text{ for all } X \in \partial\Omega, \\ 0 \leq \hat{g}(a, X) \leq [\hat{f}(a, X)]_+ \text{ for any } X \in \partial\Omega \end{cases} \tag{5}$$

in which $[\ \ ]_+$ denotes the "positive part of" and the linear operator $\mathcal{L}$ is defined as follows:

$$\mathcal{L}[g] = \int_{Y \in \partial\Omega} \hat{g}(\widetilde{u}(Y), Y) K(X, Y) dS, \ x \in \partial\Omega; \quad g = \hat{g}(\widetilde{u}(X), X), \ X \in \partial\Omega \tag{6}$$

where the kernel $K(X, Y)$ satisfies the following conditons:

$$0 \leq K(X, Y) \equiv K(Y, X) < \infty \text{ for all } X \in \partial\Omega \text{ and } Y \in \partial\Omega, \tag{7}$$

$$\int_{Y \in \partial\Omega} K(X, Y) dS \leq \delta < 1 \text{ for all } X \in \partial\Omega. \tag{8}$$

The boundary condition (3) is a generalization of the equations which represent the conduction/convection/radiation heat exchange on the body's boundary. The operator $\mathcal{L}$ represents, in practical problems, the thermal radiant reemission[7].

Some practical applications of model (1) + (3) will be discussed later. Now, the objective is to construct the solution to (1) + (3). This solution will be reached from a sequence of problems in which the reemission term is known (in other words, $\mathcal{L}[g]$ is known).

3. An Auxiliary Problem

Here we shall consider the following auxiliary problem:

$$\Delta u = -r \quad \text{in } \Omega,$$

$$-\frac{\partial u}{\partial n} = f - \bar{h} \quad \text{on } \partial\Omega \tag{9}$$

in which $\bar{h}$ is a (known) piecewise continuous and bounded field.

The solution to (9) is the field which minimizes the following functional:

$$I[w] = \frac{1}{2} \int_\Omega \nabla W \circ \nabla w dV - \int_\Omega \rho dV + \int_{\partial\Omega} \phi dS \tag{10}$$

in which

$$\phi = \hat{\phi}(\widetilde{w}(X), X) = \int_0^{\widetilde{w}(X)} \hat{f}(\xi, X) d\xi + \widetilde{h}(X)\widetilde{w}(X), \quad w = \widetilde{w}(X), \quad \bar{h} = \widetilde{h}(X) \tag{11}$$

and

$$\rho = \hat{\rho}(\widetilde{w}(X), X) = \int_0^{\widetilde{w}(X)} \hat{r}(\xi, X) d\xi, \quad w = \widetilde{w}(X). \tag{12}$$

Taking the first variation of I, employing Green's identity and imposing that $\delta I = 0$ we obtain

$$\delta I = \int_\Omega \{\Delta u + r\}\eta dV + \int_{\partial\Omega} \{f - \bar{h} + \frac{\partial u}{\partial n}\}\eta dS = 0 \tag{13}$$

in which η is any admissible variation. Equations (13) and (9) are, in a weak sense, equivalent.

Taking into account the properties of f and r we have that $I[w]$ is a strictly convex functional. Therefore, $\delta I = 0$ corresponds to a minimum and the field u, which minimizes $I[w]$, is unique.

In addition, defining $\| \ \|$ as the usual norm of $H^1(\Omega)$, we conclude that (coerciveness)

$$\lim_{\|w\|\to\infty} \frac{I[w]}{\|w\|} = +\infty \tag{14}$$

and, hence, the minimum exists and belongs to $H^1(\Omega)$ (existence of the solution u). A detailed discussion on coerciveness can be found in [10, 11].

Since $r \in L^2(\Omega)$, we have that the field u belongs to $H^2(\Omega)$ and is a strong solution to (9)[12]. In addition, since Ω has the cone properties, u is continuous in Ω[9].

4. Constructing the Solution to (1) + (3)

The field u, solution to (1) + (3), is the limit of the sequence $[v_0, v_1, v_2, v_4, v_5, \cdots]$ in which $v_0 \equiv 0$ and $v_i (v_i = \tilde{v}_i(X), i = 1, 2, 3, \cdots)$ is the unique solution to

$$\Delta v_i = -r_i \text{ in } \Omega, \quad r_i = \hat{r}(\tilde{v}_i(X), X),$$
$$-\frac{\partial v_i}{\partial n} = f_i - \bar{h}_{i-1} \text{ on } \partial\Omega, \quad f_i = \hat{f}(\tilde{v}_i(X), X) \tag{15}$$

in which $\bar{h}_{i-1}$ is, for each i, the known field given by

$$\bar{h}_{i-1} = \tilde{h}_{i-1}(X), \quad \bar{h}_{i-1} = \mathcal{L}[g_{i-1}] + h \text{ on } \partial\Omega, \quad g_{i-1} = \hat{g}(\tilde{v}_{i-1}(X), X). \tag{16}$$

Problem (15) is, for each $i \geq 1$, a problem like (9). Hence, the fields v_i are continuous and the fields $\bar{h}_i$ are bounded.

From (15), we may write the following

$$\int_\Omega \{r_i - r_{i-1}\} dV = \int_{\partial\Omega} \{f_i - f_{i-1} - \mathcal{L}[g_{i-1} - g_{i-2}]\} dS. \tag{17}$$

Since (2) holds and (see Appendix I)

$$0 \equiv v_0 \leq v_1 \leq v_2 \leq v_3 \leq \cdots \leq v_{i-1} \leq v_i \leq \cdots \quad \text{for all } X \in \bar{\Omega}, \tag{18}$$

we may conclude the following

$$\int_{\partial\Omega} |f_i - f_{i-1}| dS \leq \int_{\partial\Omega} |\mathcal{L}[g_{i-1} - g_{i-2}]| dS. \tag{19}$$

Since, from (5),

$$f_i - f_{i-1} \geq g_i - g_{i-1} \quad \text{for all } X \in \partial\Omega \tag{20}$$

we have that, from (6), (7) and (8),

$$\int_{\partial\Omega} |f_i - f_{i-1}| dS \le \delta \int_{\partial\Omega} |f_{i-1} - f_{i-2}| dS, \quad 0 \le \delta < 1. \tag{21}$$

Since $f_i \in L^1(\partial\Omega)$, the sequence $[f_1, f_2, f_3, \cdots]$ converges in $L^1(\partial\Omega)$[13]. Therefore, taking into account (7), we may define the bounded field $\bar{h}_\infty$ as

$$\bar{h}_\infty = \widetilde{h}_\infty(X), \quad \bar{h}_\infty \equiv \lim_{i\to\infty} \bar{h}_i \equiv \lim_{i\to\infty} \mathcal{L}[g_i] + h, \quad X \in \partial\Omega. \tag{22}$$

Hence, the field v defined by

$$v = \widetilde{v}(X), \quad v \equiv \lim_{i\to\infty} v_i \quad \text{in } \bar{\Omega} \tag{23}$$

is the solution to

$$\begin{cases} \Delta v = -r \text{ in } \Omega, \quad r = \hat{r}(\widetilde{v}(X), X), \\ -\dfrac{\partial v}{\partial n} = f - \bar{h}_\infty \text{ on } \partial\Omega, \quad f = \hat{f}(\widetilde{v}(X), X). \end{cases} \tag{24}$$

Since, from (21),

$$\hat{h}_\infty = \mathcal{L}[g] + h, \quad g = \hat{g}(\widetilde{v}(X), X) \text{ for all } X \in \partial\Omega \tag{25}$$

we have that v is a solution to (1) + (3), being a continuous field as well as v_i's.

5. Uniqueness

In order to prove that v is the unique solution to (1) + (3), let us assume that u, solution to (1) + (3), is different from v. Since u is a solution to (1) + (3), we have

$$\begin{cases} \Delta(u - v_i) = -(\hat{r}(\widetilde{u}(X), X) - \hat{r}(\widetilde{v}_i(X), X)) \quad \text{in } \Omega, \\ -\dfrac{\partial}{\partial n}(u - v_i) = \hat{f}(\widetilde{u}(X), X) - \hat{f}(\widetilde{v}_i(X), X) \\ \qquad - \displaystyle\int_{Y\in\partial\Omega} \{\hat{g}(\widetilde{u}(Y), Y) - \hat{g}(\widetilde{v}_{i-1}(Y), Y)\} K(X, Y) dS \quad \text{on } \partial\Omega \end{cases} \tag{26}$$

in which $u = \widetilde{u}(X)$ and $v_i = \widetilde{v}_i(X)$.

Employing a procedure analogous to the one presented in Appendix I, we conclude that

$$u \ge v_i \quad \text{in } \bar{\Omega} \quad \text{for any } i \tag{27}$$

and, therefore,

$$u \ge v \quad \text{in } \bar{\Omega}. \tag{28}$$

Since $u \ge v$ in $\bar{\Omega}$, we have that $\hat{r}(\widetilde{u}(X), X) \le \hat{r}(\widetilde{v}(X), X)$ and, once u and v satisfy (1) + (3), we may write

$$\int_{X\in\partial\Omega} \{\hat{f}(\widetilde{u}(X), X) - \hat{f}(\widetilde{v}(X), X)\} dS$$
$$\le \int_{X\in\partial\Omega} \Big[\int_{Y\in\partial\Omega} \{\hat{g}(\widetilde{u}(Y), Y) - \hat{g}(\widetilde{v}(Y), Y)\} K(X, Y) dS \Big] dS. \tag{29}$$

Therefore, from the properties of $K(X,Y)$, f and g, we have that

$$\int_{X\in\partial\Omega}\{\hat{f}(\tilde{u}(X),X)-\hat{f}(\tilde{v}(X),X)\}dS \leq \delta\int_{X\in\partial\Omega}\{\hat{g}(\tilde{u}(X),X)-\hat{g}(\tilde{v}(X),X)\}dS$$

$$\leq \delta\int_{X\in\partial\Omega}\{\hat{f}(\tilde{u}(X),X)-\hat{f}(\tilde{v}(X),X)\}dS. \tag{30}$$

Since (28) holds, f satisfies (4) and $0 \leq \delta < 1$, we conclude that

$$\int_{X\in\partial\Omega}\{\hat{g}(\tilde{u}(X),X)-\hat{g}(\tilde{v}(X),X)\}dS = \int_{X\in\partial\Omega}\{\hat{f}(\tilde{u}(X),X)-\hat{f}(\tilde{v}(X),X)\}dS = 0. \tag{31}$$

Consequently, combining (4) with (31), we have

$$u \equiv v \quad \text{on } \partial\Omega^*. \tag{32}$$

Taking into account that u and v are solutions to (1) + (3) and that (28), (31) and (32) hold, we have that u and v must satisfy the following conditions:

$$\begin{cases} \Delta(u-v) \geq 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial n}(u-v) = 0 & \text{on } \partial\Omega, \\ u - v = 0 & \text{on } \partial\Omega^* \subseteq \partial\Omega. \end{cases} \tag{33}$$

Hence, employing the result of Appendix II, we must have (uniqueness)

$$u \equiv v \quad \text{in } \bar{\Omega}. \tag{34}$$

6. An Application–Energy Transfer in Black Bodies Surrounded by Vacuum

In order to present a physical application, let us consider a rigid and block body with unitary thermal conductivity.

In such a case, the steady-state energy transfer process is described by (see Appendix III)

$$\begin{cases} \Delta u = -r & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \sigma u^4 - \displaystyle\int_{\partial\Omega}\sigma\tilde{u}(Y)^4K(X,Y)dS & \text{on } \partial\Omega \end{cases} \tag{35}$$

in which σ is the Stefan-Boltzmann constant[14] and $K(X,Y)$ depends only on the shape of the body[7]. The kernel $K(X,Y)$ is given by

$$K(X,Y) = \begin{cases} \dfrac{[(X-Y)\circ n_X][(Y-X)\circ n_Y]}{\pi[(X-Y)\circ(X-Y)]^2} & \text{if } X \text{ and } Y \text{ can be connected by a straight line which does not intercept the body,} \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

in which n_X is the unit outward normal at $X \in \partial\Omega$ and n_Y is the unit outward normal at $Y \in \partial\Omega$.

Actually, (35) is not a problem like (1) + (3). But, consider, instead of (35), the following problem[5]:

$$\begin{cases} \Delta u = -r \quad \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \sigma |u|^3 u - \displaystyle\int_{Y\in\partial\Omega} \sigma [\tilde{u}(Y)]_+^4 K(X,Y) dS \text{ on } \partial\Omega. \end{cases} \tag{37}$$

The above problem, if r satisfies (2), is a problem like (1) + (3). However, (35) and (37) are not, mathematically, the same problem. Although mathematically different, (35) and (37) are physically equivalent, since u makes physical sense only when it is nonnegative for all $X \in \bar{\Omega}$ (because u represents an absolute temperature [15]). Hence, from a practical viewpoint, (35) can be substituted by (37).

Since (37) has one, and only one, solution, we conclude that this solution is the one, with physical sense, associated to (35), provided (35) represents a physically admissible process.

It is to be noticed that there exists infinitely many bodies for which (36) satisfies (7) and (8).

Condition (7), for instance, is satisfied for any body with regular boundary.

Condition (8) is satisfied for any body such that any point $X \in \partial\Omega$ can exchange, directly, thermal radiant energy with the environment.

For instance, if Ω is given by

$$\Omega \equiv \{(x,y,z) \in I\!R^3 \text{ such that } 1 < x^2 + y^2 + z^2 < 2 \text{ and } z > 0\},$$

conditions (7) and (8) are satisfied. The kernel $K(X,Y)$ is, in this case, given as[7]:

$$K(X,Y) = \begin{cases} \dfrac{1}{4\pi} & \text{for } x^2 + y^2 + z^2 = 1 \text{ and } z > 0, \\ 0 & \text{for } x^2 + y^2 + z^2 = 2 \text{ and } z > 0, \\ 0 & \text{for } 1 < x^2 + y^2 + z^2 < 2 \text{ and } z = 0. \end{cases} \tag{38}$$

7. An Application–Energy Transfer in Black Bodies Surrounded by Atmosphere

When there exists atmosphere there must be considered, besides the radiative losses, the losses due to convective heat transfer[2,8].

Considering the losses by convection, the energy transfer phenomenon is described by (see Appendix III)

$$\begin{cases} \Delta u = -r \quad \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \sigma u^4 - \displaystyle\int_{Y\in\partial\Omega} \sigma \tilde{u}(Y)^4 K(X,Y) dS + \beta(u - u_{\text{ref}}) \text{ on } \partial\Omega, \end{cases} \tag{39}$$

in which β is a positive valued field (called "convection heat transfer coefficient") and u_{ref} is a temperature of reference (usually the temperature of the environment).

Problem (39) is physically equivalent to[6]

$$\begin{cases} \Delta u = -r \quad \text{in } \Omega, \\ -\dfrac{\partial u}{\partial n} = \sigma |u|^3 u - \displaystyle\int_{Y \in \partial\Omega} \sigma [\tilde{u}(Y)]_+^4 K(X,Y) dS + \beta(u - u_{\text{ref}}) \text{ on } \partial\Omega \end{cases} \tag{40}$$

once u represents an absolute temperature.

Problem (40) is a problem like (1) + (3), provided (2) is satisfied.

8. Final Remarks

This work presents an interdisciplinary contribution which involves a solution procedure for a given class of problems and its applications to some practical heat transfer processes.

The procedure employed for solution's construction consists of an efficient algorithm for simulating problems like (1) + (3). In particular, when employing a finite element method[16] or a finite difference method[17], this algorithm allows effective storage schemes since, at each step of solution's construction, $\mathcal{L}[g]$ is known.

Appendix I. Proving That $v_{i+1} \geq v_i$ in $\bar{\Omega}$

Let us consider the following problem:

$$\begin{cases} \Delta V = -R \quad \text{in } \Omega, \\ -\dfrac{\partial}{\partial n} V = F - G \quad \text{on } \partial\Omega \end{cases} \tag{AI.1}$$

in which the fields V, R, F and G are such that

$$\begin{cases} RV \leq 0 \quad \text{for all } X \in \Omega, \\ FV \geq 0 \quad \text{for all } X \in \partial\Omega, \\ G \geq 0 \quad \text{for all } X \in \partial\Omega. \end{cases} \tag{AI.2}$$

We shall demonstrate that V, solution to $(AI - 1)$, is a field such that

$$V \geq 0 \quad \text{in } \bar{\Omega}. \tag{AI.3}$$

In order to demonstrate that (AI.3) holds, let us assume that

$$\inf_{\bar{\Omega}} V = \inf_{\Omega} V < \inf_{\partial\Omega} V. \tag{AI.4}$$

The above inequality holds if, and only if, there exists a subset $\Omega^* \subseteq \Omega$ such that

$$R < 0 \quad \text{in } \Omega^* \quad \text{and} \quad \inf_{\Omega^*} V = \inf_{\Omega} V. \tag{AI.5}$$

Since, from (AI.2), $RV \leq 0$ in Ω, we conclude that

$$V \geq 0 \quad \text{in } \Omega^* \tag{AI.6}$$

and, therefore,

$$\inf_{\Omega^*} V = \inf_{\Omega} V = \inf_{\bar{\Omega}} V \geq 0. \tag{AI.7}$$

Now, let us suppose that

$$\inf_{\bar{\Omega}} V = \inf_{\partial\Omega} V \leq \inf_{\Omega} V. \tag{AI.8}$$

The above inequality assures that there exists a subset $\partial\Omega^* \subseteq \partial\Omega$ such that

$$\frac{\partial V}{\partial n} \leq 0 \quad \text{on } \partial\Omega^* \quad \text{and} \quad \inf_{\partial\Omega^*} V = \inf_{\partial\Omega} V. \tag{AI.9}$$

If (AI.9) holds, we have, from (AI.1), that

$$F - G \geq 0 \quad \text{on } \partial\Omega^*. \tag{AI.10}$$

Since $G \geq 0$ on $\partial\Omega$, we conclude that $F \geq 0$ on $\partial\Omega^*$. Hence, from (AI.2), we may write

$$V \geq 0 \quad \text{on } \partial\Omega^*. \tag{AI.11}$$

Therefore,

$$\inf_{\partial\Omega^*} V = \inf_{\partial\Omega} V = \inf_{\bar{\Omega}} V \geq 0. \tag{AI.12}$$

Inequalities (AI.7) and (AI.12) allow us to conclude that

$$\inf_{\bar{\Omega}} V \geq 0 \leftrightarrow V \geq 0 \quad \text{in } \bar{\Omega}. \tag{AI.13}$$

Since (15) holds, the difference $v_{i+1} - v_i$ will satisfy

$$\begin{aligned} \Delta(v_{i+1} - v_i) &= -(r_{i+1} - r_i) \quad \text{in } \Omega, \\ -\frac{\partial}{\partial n}(v_{i+1} - v_i) &= f_{i+1} - f_i - \mathcal{L}[g_i - g_{i-1}] \quad \text{on } \partial\Omega. \end{aligned} \tag{AI.14}$$

Begining with $i = 1$ we have

$$\begin{aligned} \Delta(v_2 - v_i) &= -(r_2 - r_i) \quad \text{in } \Omega, \\ -\frac{\partial}{\partial n}(v_2 - v_i) &= f_2 - f_i - \mathcal{L}[g_1] \quad \text{on } \partial\Omega. \end{aligned} \tag{AI.15}$$

Since $\mathcal{L}[g_1] \geq 0$, Problem (AI.15) is like problem (AI.1). Thus from (AI.13)

$$v_2 \geq v_1 \quad \bar{\Omega}. \tag{AI.16}$$

From (AI.16) we have that

$$\mathcal{L}[g_2 - g_1] \geq 0. \tag{AI.17}$$

Therefore, taking $i = 2$ in (AI.14), we have

$$v_3 \geq v_2 \quad \bar{\Omega}. \tag{AI.18}$$

The above procedure may be successively employed for proving that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_i \leq v_{i+1} \quad \text{in } \bar{\Omega}. \tag{AI.19}$$

Appendix II. On the Solution of (33)

Here we shall consider the following problem:

$$\begin{cases} \Delta Z \geq 0 & \text{in } \Omega, \\ \dfrac{\partial Z}{\partial n} = 0 & \text{on } \partial\Omega, \\ Z = 0 & \text{on } \partial\Omega^* \subseteq \partial\Omega. \end{cases} \tag{AII.1}$$

From Green's identity we have that

$$\int_\Omega \Delta Z dV = \int_{\partial\Omega} \frac{\partial Z}{\partial n} dS \tag{AII.2}$$

and, hence, from (AII.1) and (AII.2), we may write

$$\int_{\partial\Omega} \frac{\partial Z}{\partial n} dS = 0 \to \int_\Omega \Delta z dV = 0 \to \Delta Z = 0 \text{ in } \Omega. \tag{AII.3}$$

Since $\Delta Z = 0$ in Ω and $\partial Z/\partial n = 0$ on $\partial\Omega$, the field Z must be constant in $\bar{\Omega}$. Hence, once $Z = 0$ on $\partial\Omega^*$, we have

$$Z \equiv 0 \quad \text{in } \bar{\Omega}. \tag{AII.4}$$

Appendix III. Physical Meaning of the Boundary Conditions of (35) and (39)

In this Appendix we shall present a brief discussion concerning the physical meaning of the terms appearing in the boundary conditions of (35) and (39).

Begining with (35), we have that σu^4 represents that amount of thermal radiant energy leaving, per unit time and unit area, the point $X \in \partial\Omega$. This term arises directly from Stefan-Boltzmann law. The integral over $\partial\Omega$ represents the amount of energy that, coming from $\partial\Omega$, reaches, per unit time and unit area, the point $X \in \partial\Omega$. A detailed discussion on radiative transfer can be found in [7, 14].

In problem (39) we have, besides the radiative loss, a convective loss. The convective loss is usually assumed to be proportional to the difference between the local temperature and a temperature of reference (generally the temperature of the environment). The energy loss, by convective transfer (per unit time and unit area) is repesented by $\beta(u - u_{\text{ref}})$ ar each $X \in \partial\Omega$ [8].

The exterior normal derivative $\partial u/\partial n$, with the minus sign, represents, for each $X \in \partial\Omega$, the amount of energy (per unit time and unit area) that reaches the boundary $\partial\Omega$ by conduction [2].

The boundary conditions of (35) and (39) arise from the following energy balance:

$$\text{losses by conduction} \equiv \text{losses by radiation} + \text{losses by convection} \tag{AIII.1}$$

in which

$$\text{losses (per unit time and unit area) by conduction} \equiv -\frac{\partial u}{\partial n}, \tag{AIII.2}$$

$$\text{losses (per unit time and unit area) by convection} \equiv \beta(u - u_{\text{ref}}), \tag{AIII.3}$$

losses (per unit time and unit area) by radiation

$$\equiv \sigma u^4 - \int_{Y \in \partial\Omega} \sigma \tilde{u}(y)^4 K(X,Y) dS. \tag{AIII.4}$$

References

[1] R.B. Bird, W.E. Stewart and E.N. Lightfoot, Transport Phenomena, John Wiley & Sons, 1960.

[2] J.P. Holman, Heat Transfer, McGraw-Hill, Kogakusha, 1976.

[3] F. John, Partial Differential Equaiotns, Springer-Verlag, New York, 1982.

[4] R.M. Saldanha da Gama, On the physical solutions to the heat equation subjected to nonlinear boundary conditions, *Journal of Computational Mathematics,* accepted for publication.

[5] R.M. Saldanha da Gama, Existence, uniqueness and construction of the solution to the energy transfer problem in a rigid and nonconvex black body, *Journal of Applied Mathematics and Physics* (ZAMP), accepted for publication.

[6] R.M. Saldanha da Gama, On the solution of the energy transfer problem in a nonconvex black body with convective/radiative boundary conditions, *International Journal of Non-linear Mechanics,* accepted for publication.

[7] E.M. Sparrow and R.D. Cess, Radiation Heat Transfer, McGraw-Hill, Washington, 1976.

[8] F. Kreith, Principles of Heat Transfer, Intext Educational Publishers, New York, 1973.

[9] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[10] M.S. Berger, Nonlinearity and Functional Analysis, Academic Press, New York, 1977.

[11] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[12] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1977.

[13] K. Yosida, Functional Analysis, Springer-Verlag, 1966.

[14] R. Siegel and J.R. Howell, Thermal Radiative Heat Transfer, McGraw-Hill, New York, 1981.

[15] H.B. Callen, Thermodynamics, John Wiley & Sons, New York, 1960.

[16] K.J. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.

[17] A.A. Samarski and V.B. Andreiev, Methods en Diferencias para las Ecuaciones Elipticas, Editorial MIR, Moscow, 1976.