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# ITERATIVE CORRECTIONS AND A POSTERIORI ERROR ESTIMATE FOR INTEGRAL EQUATIONS*

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**Abstract**

Starting from a well known operator identity we obtain a recurrence formula, i.e., an iterative correction scheme, for the integral equations with computable kernel. From this we can increase the order of convergence step by step, say, from 4th to 8th to 12th. What is more interesting in this scheme, besides its fast acceleration, is its weak requirement on the integral kernel: the regularity of the kernel will not be strengthened during the correction procedure.

## §1. Operator Framework

Suppose that the linear operator equation

$$Lu = f \tag{1}$$

is approximated by another equation

$$L_0 u_0 = f, \tag{2}$$

where $L_0$ is an approximation of $L$ in the sense of operator norm:

$$\varepsilon_0 \equiv \|L_0 - L\| \ll 1. \tag{3}$$

Thus, if $L^{-1}$ exists, then $L_0^{-1}$ exists:

$$\|L_0^{-1}\| \le (1 - \varepsilon_0 \|L^{-1}\|)^{-1} \|L^{-1}\|. \tag{4}$$

For simplicity we will use the notation:

$$A_0 = L_0^{-1}(L_0 - L).$$

A well known identity

$$u - u_0 = A_0 u$$

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leads to a recurrence formula:

$$u - u_0 = A_0 u_0 + A_0^2 u = A_0 u_0 + A_0^2 u_0 + A_0^3 u = \cdots.$$

Setting

$$u_1 = A_0 u_0, \quad \ldots, \quad u_i = A_0^i u_0,$$

we obtain an iterative correction scheme:

$$u - \sum_{i=0}^{\gamma-1} u_i = A_0^\gamma u, \tag{5}$$

where the remainder is of high order:

$$\|A_0^\gamma u\| \le \|A_0\|^\gamma \|u\| \le O(\varepsilon_0^\gamma), \tag{6}$$

and $u_i$ are nothing but the solutions of the same approximating equation (2) with different right hand sides:

$$L_0 u_i = (L_0 - L) u_{i-1}, \quad i = 1, 2, \cdots.$$

These $u_i$ provide, as a by-product, an a posteriori error estimate.

**Remark 1.** The scheme (5) is nothing but a variant of formula (21) in [2]. If we set

$$L = I - K, \; L_0 = I - KP, \; \overline{u}^\gamma = \sum_{i=0}^{\gamma} u_i,$$

then formula (21) in [2] can be read as (5).

## 2. Iterated Galerkin Method

Let (1) be an integral equation of second kind:

$$Lu(s) \equiv u(s) - \int_0^1 K(s,t) u(t)\, dt = f(s) \tag{7}$$

and (2) the iterated Galerkin method:

$$L_0 u_0(s) \equiv u_0(s) - \int_0^1 K(s,t) P u(t)\, dt = f(s),$$

where $P$ is the $L_2$-orthogonal projection onto a subspace with the standard approximation property:

$$\|u - Pu\| \le O(n^{-R}) \|\partial^R u\|,$$

where the lower order of derivatives on the right hand side are omitted.

Only one thing has to be done in applying the scheme (5), that is the estimate for (3). For this, we need the regularity condition on $K$:

$$\partial_s^R \partial_t^R K \in L_2 \tag{8}$$

and let $f \in H^R$ the Sobolev space. Then, the behavior of the integral operator will help us to compensate the loss of derivatives: for $v \in L_2$,

$$\begin{aligned}(\partial^R(L_0 - L)u, v) &= (\partial^R K(I-P)u, v) = ((I-P)u, K^*v) \\ &= ((I-P)u, (I-P)K^*v) \le O(n^{-2R})\|\partial^R u\|\|\partial^R K^* v\|,\end{aligned}$$

where

$$K^*v(t) \equiv \int_0^1 \partial_s^R K(s,t)v(s)\,ds,$$

$$\|\partial^R K^* v\| = \|\int_0^1 \partial_t^R \partial_s^R K(s,t)v(s)ds\| \le O(1)\|v\|,$$

and hence we obtain a full estimate in $H^R$-norm:

$$\|\partial^R(L_0 - L)u\| \le c \cdot n^{-2R}\|\partial^R u\|,$$

i.e. (3) holds in $H^R$-space.

Once (3) is obtained, (5) and (6) give us a high accuracy:

$$\|\partial^R(u - \sum_{i=0}^{\gamma-1} u_i)\| \le (c\|L_0^{-1}\|n^{-2R})^\gamma \|\partial^R u\|.$$

Note that the regularity requirment (8) (with parameter $R$) remains the same durig the correction procedure. So, in the case $K$ is of weak regularity: $R = 1$, or $\|\partial^2 K v\| \le c\|v\|$ and $K$ is computable, we can still use the $\gamma$-1-th correction to increase the accuracy of $u_0$ from 2nd to $2\gamma$-th order, though the other high order methods do not work. Even for the case where $\|\partial K v\| \le c\|v\|$, we can have $\gamma$-th order by $\gamma$-1-th correction when $K$ is computable.

## 3. Nystrom Method

Equation (7) may be approximated by the Nystrom method:

$$L_0 u_0(s) \equiv u_0(s) - \sum w_j K(s, x_{ij})u(x_{ij})h_i = f(s),$$

where $w_j, x_{ij}, h_i$ are defined in Graham and Chandler [1]. In short, this is a composite $r$ point interpolatory rule on a mesh with $n+1$ points on $[0,1]$. Let $R$ be the order of the quadrature rule: $2r \ge R \ge r$.

Instead of the regularity condition (8) we assume that

$$\partial_s^R \partial_t^R K \in C \tag{9}$$

and $f \in C^R$. Again, by the behavior of the integral operator, we retrieve the loss of derivatives and obtain a full estimate in $C^R$-norm:

$$\|\partial^R(L_0 - L)u\| = \max_s | \int_0^1 \partial_s^R K(s,t)u(t)\,dt - \sum w_j \partial_s^R K(s, x_{ij})u(x_{ij})h_i|$$

$$\leq O(n^{-R}) \max_{t,s} |\partial_t^R(\partial_s^R K(s,t)u(t))| \leq c \cdot n^{-R}\|\partial^R u\|,$$

i.e., (3) holds in $C^R$-space. Hence, (5) and (6) give us

$$\|\partial^{2r}(u - \sum_{i=0}^{\gamma-1} u_i)\| \leq (c\|L_0^{-1}\|n^{-2r})^\gamma \|\partial^R u\|,$$

where $R = 2r$ is chosen in (9).

The similar framework and idea will be applied to the integral equation with singular kernel. We will have further works in this direction.

## References

[1] I. Graham and G. Chandler, *SIAM J. Numer. Anal.*, to appear.
[2] Q. Lin, *R.A.I.A.O., Anal. Numer.*, 1 (1982), 39–47.