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SPLINE FINITE DIFFERENCE METHODS AND THEIR EXTRAPOLATION FOR SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS*

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Abstract

In this paper we consider a class of singular two-point boundary value problem: $(x^\alpha u')' = f(x,u)$, $u(0) = A$, $u(1) = B$, $0 < \alpha < 1$. The asymptotic expression of the spline finite difference solution for the problem is obtained. By using the asymptotic expression, Richardson's extrapolation can be done and the accuracy of numerical solution can be improved. Some numerical examples are given in illustration of this theory.

§1. Introduction

Consider the singular two-point boundary value problem:

$$\begin{cases} (x^\alpha u')' = f(x,u), & 0 < x \le 1, \\ u(0) = A, & u(1) = B. \end{cases} \tag{1}$$

Here A and B are finite constants, and $\alpha \in (0,1)$ which may also take 1 or 2. We assume:

(A) for $(x,u) \in \{[0,1] \times R\}$, $f(x,u)$ is continuous, $\partial f/\partial u$ exists and is continuous, and $\partial f/\partial u \ge 0$.

It is well known that (1) has a unique solution. Problem (1) has been extensively discussed. In the linear case, Jamet [1] considered a standard three-point finite difference scheme with uniform mesh and has shown that the error is $O(h^{1-\alpha})$ in maximum norm. Ciarlet [2] considered the application of Rayleight-Ritz-Galerkin method and improved Jamet's result by showing that the error is $O(h^{2-\alpha})$ in uniform mesh for their Galerkin approximation. Gusttafsson [3] gave a numerical method for linear problems by representing the solutions as series expansions on a subinterval near the singularity and using the difference method for the remaining interval. He constructed a compact second order, fourth order scheme. Reddien [4] and Reddien and Schumaker [5] have considered the collocation method for singular two-point boundary value problems and studied the existence, uniqueness and convergence rates of these methods. Chawla

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and Katti [6] have constructed three kinds of difference methods and shown that these methods are $O(h^2)$-convergent. Recently, Iyengar and Jain [7] have discussed the construction of a spline function for a class of singular two-point boundary value problem: $x^{-\alpha}(x^\alpha u')' = f(x,u), u(0) = A,\ u(1) = B,\ 0 < \alpha < 1$ or $\alpha = 1,2$. The boundary conditions may also be of form $u'(0) = 0, u(1) = B$. They showed that the spline finite difference methods are $O(h^2)$-convergent when $0 < \alpha < 1, \alpha = 2$. Han[8] has shown, for $\alpha = 1$, that the spline finite difference method is also $O(h^2)$-convergent.

In this paper, first we give the spline finite difference scheme for the singular two-point boundary value problem which is similarly constructed in [7]. Secondly, we show that the asymptotic expression of the spline finite difference solution exists. So Richardson's extrapolation can be done. This will greatly increase the accuracy of numerical solution. Finally, some numerical examples are given in illustration of this theory.

§2. The Spline Finite Difference Method

Let Δ be an equidistant partition of $[0,1]$:

$$\Delta : 0 = x_0 < x_1 < \cdots < x_N = 1.$$

Set $\tilde{u}(x_j) = \tilde{u}_j$ and $f(x_j, \tilde{u}_j) = f_j$. We write

$$(x^\alpha \tilde{u}')' = \frac{f_{j-1}}{h}(x_j - x) + \frac{f_j}{h}(x - x_{j-1}) \quad x_{j-1} < x < x_j \tag{2}$$

where $h = x_{j+1} - x_j = 1/N$. Integrating (2), dividing by x^α, and then integrating we obtain

$$\tilde{u}(x) = \frac{f_{j-1}}{h}\Big(\frac{-x^{3-\alpha}}{2(3-\alpha)} + \frac{x_j x^{2-\alpha}}{2-\alpha}\Big) + \frac{f_j}{h}\Big(\frac{x^{3-\alpha}}{2(3-\alpha)} - \frac{x_{j-1}x^{2-\alpha}}{2-\alpha}\Big) + c_j x^{1-\alpha} + D_j$$

where c_j and D_j can be determined by interpolation conditions $\tilde{u}(x_{j-1}) = \tilde{u}_{j-1}, \tilde{u}(x_j) = \tilde{u}_j$. We get the spline approximation as

$$\begin{aligned}
\tilde{u}(x) = {} & \frac{1}{S_j}(x_j^{1-\alpha}\tilde{u}_{j-1} - x_{j-1}^{1-\alpha}\tilde{u}_j) + \frac{1}{S_j}(\tilde{u}_j - \tilde{u}_{j-1})x^{1-\alpha} + \frac{f_{j-1}}{h}\Big\{-\frac{x^{3-\alpha}}{2(3-\alpha)} + \frac{x_j x^{2-\alpha}}{2-\alpha} \\
& - \frac{1}{S_j}\Big[\frac{1}{2-\alpha}x_j(x_j^{2-\alpha} - x_{j-1}^{2-\alpha}) - \frac{1}{2(3-\alpha)}(x_j^{3-\alpha} - x_{j-1}^{3-\alpha})\Big]x^{1-\alpha} \\
& + \frac{1}{S_j}\Big[\frac{h}{2-\alpha}x_j^{2-\alpha}x_{j-1}^{1-\alpha} - \frac{h}{2(3-\alpha)}x_j^{1-\alpha}x_{j-1}^{1-\alpha}(x_j + x_{j-1})\Big]\Big\} \\
& + \frac{f_j}{h}\Big\{\frac{x^{3-\alpha}}{2(3-\alpha)} - \frac{x_{j-1}x^{2-\alpha}}{2-\alpha} + \frac{1}{S_j}\Big[\frac{1}{2-\alpha}x_{j-1}(x_j^{2-\alpha} - x_{j-1}^{2-\alpha}) \\
& - \frac{1}{2(3-\alpha)}(x_j^{3-\alpha} - x_{j-1}^{3-\alpha})\Big]x^{1-\alpha} - \frac{1}{S_j}\Big[\frac{h}{2-\alpha}x_j^{1-\alpha}x_{j-1}^{2-\alpha} \\
& - \frac{h}{2(3-\alpha)}x_j^{1-\alpha}x_{j-1}^{1-\alpha}(x_j + x_{j-1})\Big]\Big\}, \quad x_{j-1} < x < x_j
\end{aligned} \tag{3}$$

where $S_j = x_j^{1-\alpha} - x_{j-1}^{1-\alpha}$. Requiring that $\tilde{u}'(x)$ be continuous at node x_j we obtain the spline finite difference approximation of (1):

$$-\frac{1-\alpha}{S_j}\tilde{u}_{j-1} + \Big(\frac{1-\alpha}{S_j} + \frac{1-\alpha}{S_{j+1}}\Big)\tilde{u}_j - \frac{1-\alpha}{S_{j+1}}\tilde{u}_{j+1} = A_j^* f_{j-1} + B_j^* f_j + C_j^* f_{j+1},$$
$$j = 1, 2, \cdots, N-1 \quad (4)$$

where

$$A_j^* = -\frac{1}{h}\Big\{\frac{1}{2}x_j^2 - \frac{1-\alpha}{S_j}\Big[\frac{1}{2-\alpha}x_j(x_j^{2-\alpha} - x_{j-1}^{2-\alpha}) - \frac{1}{2(3-\alpha)}(x_j^{3-\alpha} - x_{j-1}^{3-\alpha})\Big]\Big\},$$

$$C_j^* = -\frac{1}{h}\Big\{\frac{1}{2}x_j^2 - \frac{1-\alpha}{S_{j+1}}\Big[\frac{1}{2-\alpha}x_j(x_{j+1}^{2-\alpha} - x_j^{2-\alpha}) - \frac{1}{2(3-\alpha)}(x_{j+1}^{3-\alpha} - x_j^{3-\alpha})\Big]\Big\},$$

$$B_j^* = -\frac{1}{h}\Big\{x_j x_{j-1} - \frac{1}{2}x_j^2 - \frac{1-\alpha}{S_j}\Big[\frac{1}{2-\alpha}x_{j-1}(x_j^{2-\alpha} - x_{j-1}^{2-\alpha})$$
$$- \frac{1}{2(3-\alpha)}(x_j^{3-\alpha} - x_{j-1}^{3-\alpha})\Big]\Big\} + \frac{1}{h}\Big\{x_j x_{j+1} - \frac{1}{2}x_j^2$$
$$- \frac{1-\alpha}{S_{j+1}}\Big[\frac{1}{2-\alpha}x_{j+1}(x_{j+1}^{2-\alpha} - x_j^{2-\alpha}) - \frac{1}{2(3-\alpha)}(x_{j+1}^{3-\alpha} - x_j^{3-\alpha})\Big]\Big\}.$$

When $\alpha = 0$, the scheme (4) reduces to

$$\tilde{u}_{j-1} - 2\tilde{u}_j + \tilde{u}_{j+1} = \frac{h^2}{6}(f_{j-1} + 4f_j + f_{j+1})$$

which is the same as the scheme obtained by cubic spline for $u''(x) = f(x, u(x))$.

Now, we consider the truncation error of the spline finite difference method. Note that

$$-\frac{1-\alpha}{S_j}u(x_{j-1}) + \Big(\frac{1-\alpha}{S_j} + \frac{1-\alpha}{S_{j+1}}\Big)u(x_j) - \frac{1-\alpha}{S_{j+1}}u(x_{j+1})$$
$$+ \frac{1-\alpha}{2-\alpha}\Big(\frac{x_{j+1}^{2-\alpha} - x_j^{2-\alpha}}{x_{j+1}^{1-\alpha} - x_j^{1-\alpha}} - \frac{x_j^{2-\alpha} - x_{j-1}^{2-\alpha}}{x_j^{1-\alpha} - x_{j-1}^{1-\alpha}}\Big)f_j = t_j$$

where

$$t_j = -B_{1j}f_j' - \frac{1}{2}B_{2j}f_j'' - \frac{1}{6}B_{3j}f'''(\xi_j)^{[9]},$$

$$B_{mj} = \frac{A_{mj}^+}{S_{j+1}} + \frac{A_{mj}^-}{S_{j+1}},$$

$$A_{mj}^{\pm} = \frac{1-\alpha}{m+1}\sum_{k=0}^{m+1}\frac{(-1)^k}{m+2-\alpha-k}\binom{m+1}{k}x_j^k\Big(x_{j\pm1}^{m+2-\alpha-k} - x_j^{m+2-\alpha-k}\Big),$$

$$A_j^* + B_j^* + C_j^* = -\frac{1-\alpha}{2-\alpha}\Big(\frac{x_{j+1}^{2-\alpha} - x_j^{2-\alpha}}{S_{j+1}} - \frac{x_j^{2-\alpha} - x_{j-1}^{2-\alpha}}{S_j}\Big) = -B_{0j},$$

$$B_{1j} + h(C_j^* - A_j^*) = 0.$$

Hence

$$-\frac{1-\alpha}{S_j}u(x_{j-1})+\Big(\frac{1-\alpha}{S_j}+\frac{1-\alpha}{S_{j+1}}\Big)u(x_j)-\frac{1-\alpha}{S_{j+1}}u(x_{j+1})-A_j^*f(x_{j-1},(x_{j-1}))$$
$$-B_j^*f(x_j,u(x_j))-C_j^*f(x_{j+1},u(x_{j+1}))$$
$$=-\frac{1-\alpha}{S_j}u(x_{j-1})+\Big(\frac{1-\alpha}{S_j}+\frac{1-\alpha}{S_{j+1}}\Big)u(x_j)-\frac{1-\alpha}{S_{j+1}}u(x_{j+1})+B_{0j}f(x_j,u(x_j))$$
$$-A_j^*[f(x_{j-1},u(x_{j-1}))-f(x_j,u(x_j))]-C_j^*[f(x_{j+1},u(x_{j+1}))-f(x_j,u(x_j))]$$
$$=-[B_{1j}+h(C_j^*-A_j^*)]f_j'-\frac{1}{2}[B_{2j}+h^2(C_j^*+A_j^*)]f_j''-\frac{1}{6}[B_{3j}+h^3(C_j^*-A_j^*)]f'''(\xi_j)$$
$$=\frac{h^3}{12}f_j''-\frac{1}{2}[B_{2j}+h^2(C_j^*+A_j^*)+\frac{1}{6}h^3]f_j''-\frac{1}{6}[B_{3j}-h^2B_{1j}]f'''(\xi_j). \tag{5}$$

With arguments precisely as given in [7] we can show

Theorem 1. *Assume that f satisfies (A) and $|f''(x)|\le M$. Then for the spline finite difference scheme (4), we have $\|U-\tilde U\|_\infty=O(h^2)$ for sufficiently small h, where U is the exact solution, and $\tilde U$ is the spline finite difference solution.*

§3. Asymptotic Expression of Spline Finite Difference Solution

In the following, we show that, under suitable conditions, the asymptotic expression of the spline finite difference solution exists for all $\alpha\in(0,1)$.

Let

$$\tilde u_j=u(x_j)+h^2v(x_j)+\eta_j\quad j=0,1,\cdots,N \tag{6}$$

where $v(x)$ satisfies the equation

$$\begin{cases}(x^\alpha v')'=\dfrac{\partial f}{\partial u}(x,u(x))v(x)+\dfrac{1}{12}\dfrac{d^2}{dx^2}f(x,u(x)),\\ v(0)=v(1)=0.\end{cases} \tag{7}$$

With the help of Theorem 1, we know that

$$|\eta_j|\le ch^2.$$

In the following, we shall show that, if $v(x)$ satisfies (7), then

$$|\eta_j|\le ch^4.$$

Substituting (6) into (4), and using Taylor's formula and (5), we have

$$-\frac{1-\alpha}{S_j}\eta_{j-1}+\Big(\frac{1-\alpha}{S_j}+\frac{1-\alpha}{S_{j+1}}\Big)\eta_j-\frac{1-\alpha}{S_{j+1}}\eta_{j+1}-A_j^*\frac{\partial f}{\partial u}(x_{j-1},u(x_{j-1}))\eta_{j-1}$$
$$-B_j^*\frac{\partial f}{\partial u}(x_j,u(x_j))\eta_j-C_j^*\frac{\partial f}{\partial u}(x_{j+1},u(x_{j+1}))\eta_{j+1}=t_j^* \tag{8}$$

where

$$t_j^* = -\frac{1}{2}[B_{2j} + \frac{1}{6}B_{0j}h^2 + h^2(C_j^* + A_j^*)]f_j'' - \frac{1}{6}[B_{3j} + 5B_{1j}h^2]f'''(\xi_j)$$
$$- \frac{1}{2}h^2[B_{2j} + h^2(C_j^* + A_j^*)]\frac{d^2}{dx^2}\Big(\frac{\partial f}{\partial u}(x, u(x))v(x) + \frac{1}{12}f''(x, u(x))\Big)\Big|_{x=\xi_{j1}}$$
$$+ O(h^5), \tag{9}$$

since for fixed x_j

$$\begin{cases} \lim_{h\to 0}\frac{B_{0j}}{h} = 1, \quad \lim_{h\to 0}\frac{B_{1j}}{h^3} = -\frac{\alpha}{12}x_j^{-1}, \\ \lim_{h\to 0}\frac{B_{2j}}{h^3} = \frac{1}{6}, \quad \lim_{h\to 0}\frac{B_{3j}}{h^5} = -\frac{\alpha}{30}x_j^{-1}, \\ \lim_{h\to 0}\frac{A_j^*}{h} = -\frac{1}{6}, \quad \lim_{h\to 0}\frac{C_j^*}{h} = -\frac{1}{6}, \\ \lim_{h\to 0}\dfrac{B_{2j} + \frac{1}{6}B_{0j}h^2 + h^2(C_j^* + A_j^*)}{h^5} = M_1 x_j^{-2} \end{cases} \tag{10}$$

where M_1 is a constant.

Let β be chosen such that $\alpha + \beta < 1$. We assume that

$$\max_{0<x\le 1} x^{-1+\beta}|f''(x)| \le M_2, \quad \max_{0<x\le 1} x^{\beta}|f'''(x)| \le M_2, \quad \max_{0<x\le 1} x^{1+\beta}|f^{(4)}(x)| \le M_2$$

for a suitable constant M_2.

With the help of (9) and (10) we know that for $k = 1(1)N-1$

$$|t_k^*(h)| \le ch^5 x_k^{-1-\beta}, \quad k = 1, 2, \cdots, N-1$$

where C is a constant (depending only on α).

Now, following the arguments precisely as given in [9] we can show that

$$\|\eta\| \le c^* h^4$$

where

$$C^* = C\max\Big(\frac{1}{\beta(1-\alpha-\beta)}, \frac{1}{(1-\alpha)^2 e}\Big).$$

We have thus established the following result:

Theorem 2. *Assume that f satisfies (A); further, let $f^{(4)}(x) \in C\{(0,1)\times R\}$. For fixed $\alpha \in (0,1)$, let β be chosen such that $\alpha + \beta < 1$, and assume $x^{-1+\beta}f''$, $x^{\beta}f'''$, $x^{1+\beta}f^{(4)} \in C\{[0,1]\times R\}$. Then, the spline finite difference solution can be expanded as*

$$\tilde{u}_j = u(x_j) + h^2 v(x_j) + \eta_j, \quad j = 0, 1, \cdots, N \tag{11}$$

where $v(x)$ satisfies equation (7), $|\eta_j| \le C^* h^4$.

From (11), we see

$$\tilde{u}_j^h = u(x_j) + h^2 v(x_j) + \eta_j^h, \quad \tilde{u}_{2j}^{h/2} = u(x_j) + \frac{h^2}{4} v(x_j) + \eta_{2j}^{h/2},$$

$$\frac{4\tilde{u}_{2j}^{h/2} - \tilde{u}_j^h}{3} = u(x_j) + O(h^4).$$

Richardson's extrapolation is $O(h^4)$-convergent.

§4. Numerical Illustration

To illustrate the present method and its fourth-order rate of convergence, we consider the following boundary value problems:

Example 1[12].

$$\begin{cases} (x^\alpha y')' = x^{5+2\alpha} \ln x, \quad 0 < x < 1, \\ y(0) = 1, \quad y(1) = -\dfrac{\alpha}{1-\alpha} - \dfrac{13+3\alpha}{(6+2\alpha)^2(7+\alpha)^2} \end{cases}$$

with the exact solution

$$y(x) = 1 - \frac{x^{1-\alpha}}{1-\alpha} + \frac{x^{7+\alpha}\ln x}{(6+2\alpha)(7+\alpha)} - \frac{(13+3\alpha)x^{7+\alpha}}{(6+2\alpha)^2(7+\alpha)^2}.$$

The maximum absolute errors are given in Table 1.

Table 1

N	Spline Scheme (4) (e_N)	e_N/e_{2N}	Extrapolation (E_N)	E_N/E_{2N}
($\alpha = 0.25$)				
8	9.045D–5	3.990	1.724D–6	15.88
16	2.267D–5	4.039	1.086D–7	15.97
32	5.612D–6	3.999	6.801D–9	15.99
64	1.403D–6	4.003	4.253D–10	16.00
128	3.505D–7	4.000	2.658D–11	
256	8.764D–8			
($\alpha = 0.5$)				
8	8.748D–5	4.184	2.245D–6	15.82
16	2.091D–5	3.986	1.419D–7	15.96
32	5.246D–6	4.016	8.895D–9	15.99
64	1.306D–6	4.004	5.563D–10	15.99
128	3.262D–7	4.001	3.479D–11	
256	8.153D–8			
($\alpha = 0.75$)				
8	8.414D–5	4.298	2.877D–6	15.76
16	1.958D–5	4.040	1.825D–7	15.94
32	4.845D–6	4.008	1.145D–8	15.98
64	1.209D–6	4.005	7.162D–10	15.95
128	3.019D–7	4.000	4.491D–11	
256	7.547D–8			

Continue

N	Spline Scheme (4) (e_N)	e_N/e_{2N}	Extrapolation (E_N)	E_N/E_{2N}
($\alpha = 0.9$)				
8	8.200D–5	4.348	3.320D–6	15.73
16	1.886D–5	4.090	2.111D–7	15.93
32	4.612D–6	4.001	1.325D–8	15.98
64	1.153D–6	4.006	8.293D–10	16.02
128	2.877D–7	4.000	5.175D–11	
256	7.192D–8			

Example 2[6]**.**

$$\begin{cases} (x^\alpha u')' = \beta x^{\alpha+\beta-2}[(\alpha+\beta-1)+\beta x^\beta]u, & 0 < x < 1, \\ u(0) = 1, \quad u(1) = e \end{cases}$$

with the exact solution

$$u(x) = \exp(x^\beta).$$

The maximum absolute errors are given in Table 2.

Table 2

N	Spline Scheme (4) (e_N)	e_N/e_{2N}	Extrapolation (E_N)	E_N/E_{2N}
($\alpha = 0.5, \beta = 4$)				
4	2.480D–1	4.930	9.638D–3	15.59
8	5.031D–2	4.233	6.183D–4	15.76
16	1.188D–2	4.059	3.923D–5	15.94
32	2.928D–3	4.015	2.462D–6	15.99
64	7.293D–4	4.004	1.539D–7	16.01
128	1.821D–4	4.001	9.613D–9	
256	4.553D–5			
($\alpha = 0.75, \beta = 3.75$)				
4	2.347D–1	4.846	9.098D–3	15.81
8	4.844D–2	4.164	5.756D–4	15.83
16	1.163D–2	4.048	3.635D–5	15.95
32	2.874D–3	4.012	2.279D–6	15.99
64	7.164D–4	4.003	1.425D–7	16.01
128	1.790D–4	4.001	8.901D–9	
256	4.473D–5			
($\alpha = 0.5, \beta = 4.75$)				
4	3.920D–1	5.227	1.824D–2	15.08
8	7.500D–2	4.307	1.209D–3	15.51
16	1.741D–2	4.070	7.796D–5	15.85
32	4.278D–3	4.022	4.919D–6	15.96
64	1.064D–3	4.005	3.082D–7	15.99
128	2.656D–4	4.001	1.928D–8	
256	6.637D–5			

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