Journal of Computational Mathematics, Vol.11, No.3, 1993, 215–224.

ON THE CONVERGENCE OF A C^0 FINITE ELEMENT METHOD FOR THIN PLATE BENDING*1)

Zhan Chong-xi
(*Northwest University , Xian, China*)

§1. Introduction

Using the finite element method to solve Kirchhoff's equation of thin plate bending usually requires that the finite elements have C^1 continuity. This requirement causes a rapid increase in the degrees of freedom of the elements and in the cost of the solution. Mixed, hybrid and more general nonconforming methods are thus introduced to attack the difficulty. Those methods, though free from the requirement and some what convenient in calculation have some disadvantages, and so new methods are still being searched for by engineers and scientists.

In a recent paper [1], Ortiz and Morris proposed a new method of using the C^0 finite elements. At each step of the method, a variational problem, which contains the derivatives of the unknown function of only the first order, needs to be solved. But unlike the usual mixed methods[2, 3], the method does not solve the deflection w and moments M_{ij} or w and Δw simultaneously. The potential energy of the bending plate is expressed as a functional of the solutions $u_i = \partial w/\partial x_i$, $u = 1, 2$, in the method and then the energy functional is minimized under the constraint $\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}$. This constrained extreme problem is solved by using the penalty method. It was reported in [1] that excellent accuracy was attained in the numerical test.

The purpose of this paper is to analyse the convergence properties of Ortiz and Morris's method (OM method) and to estimate the error bounds of the finite element approximation obtained by using the OM method. Also, we will study how to choose the penalty parameter such that it matches the partition parameter to get the optimal accuracy.

For the convenience of our description, we outline the OM method in the following. Assume, for the sake of simplicity, the plate is clamped at the boundary. For other boundary conditions, we refer the readers to the original paper [1]. Suppose the plate occupies a convex domain $\Omega \subset R^2$ with piecewise smooth boundary $\partial\Omega$. We are going

* Received September 6, 1991.
1) The Project Supported by National Natural Science Foundation of China.

to minimize the potential energy

$$\tilde{P}(w)=\frac{1}{2}\int_\Omega D\Big[(1-v)\sum_{i,j=1}^{2}\Big(\frac{\partial^2 w}{\partial x_i\partial x_j}\Big)^2+\nu\Big(\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}\Big)^2\Big]dx-\int_\Omega qw\,dx. \tag{1.1}$$

On space $H_0^2(\Omega)$, where $w(x)=x(x_1,x_2)$ is the deflection, $q=q(x)$ is the load and $dx=dx_1dx_2$. Set

$$u_i=\frac{\partial w}{\partial x_i},\quad i=1,2;\quad u=(u_1,u_2).$$

Suppose $\varphi(x)$ is the solution of the following boundary value problem:

$$\begin{cases}-\Delta\varphi=q, & x\in\Omega,\\ \varphi=0, & x\in\partial\Omega.\end{cases} \tag{1.2}$$

Let $n=(n_1,n_2)$ be the outward normal direction of $\partial\Omega$. Using Green's formula, we see, since $w=0$ on $\partial\Omega$,

$$\begin{aligned}\int_\Omega qw\,dx&=\int_\Omega-\Delta\varphi w\,dx=\int_\Omega\Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial w}{\partial x_1}+\frac{\partial\varphi}{\partial x_2}\frac{\partial w}{\partial x_2}\Big)dx-\int_{\partial\Omega}w\frac{\partial\varphi}{\partial n}ds\\&=\int_\Omega\Big(u_1\frac{\partial\varphi}{\partial x_1}+u_2\frac{\partial\varphi}{\partial x_2}\Big)dx=\int_\Omega u\cdot\nabla\varphi\,dx.\end{aligned} \tag{1.3}$$

Substitute it into (1.1), and the potential energy becomes

$$P(u)=\frac{1}{2}\int_\Omega D\Big[(1-\nu)\sum_{i,j=1}^{2}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2+\nu\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)^2\Big]dx-\int_\Omega u\cdot\nabla\varphi\,dx. \tag{1.4}$$

The necessary and sufficient condition for the existence of a function w such that $\frac{\partial w}{\partial x_i}=u_i$, for a given function $u\in V=H_0^1(\Omega)\times H_0^1(\Omega)$, is that u has null rotation:

$$Bu=\operatorname{rot}u=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}=0. \tag{1.5}$$

To minimize functional $P(u)$ on space V under constraint (1.5), we use the penalty method. Set $\lambda>0$ large enough, and then minimize

$$\begin{aligned}P_\lambda(u)&=P(u)+\frac{\lambda}{2}\int D(Bu)^2dx\\&=\frac{1}{2}\int_\Omega D\Big[(1-\nu)\sum_{i,j=1}^{2}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2+\nu\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)^2+\lambda\Big(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\Big)^2\Big]dx\\&\quad-\int_\Omega u\cdot\nabla\varphi\,dx\end{aligned} \tag{1.6}$$

on space V. Suppose $u_\lambda=(u_{\lambda,1},u_{\lambda,2})\in V$ is the minimum point of $P_\lambda(u)$. Substitute u_λ for u on the right-hand side of the following equation, and solve the problem obtained. We get an approximation of deflection w:

$$\begin{cases} \Delta w = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2}, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

Therefore, solving Kirchhoff's equation becomes solving problems (1.2), (1.6) and (1.7) successively. Of course, we can solve these problems approximately by using any C^0 finite element.

It is clear, for any given $\lambda > 0$, that the solution u_λ of (1.6) does not necessarily satisfy (1.5). Even though the real solution of (1.7) can be obtained, the solved deflection w_λ (with u_λ on the right-hand side of equation (1.7)) is still an approximation of the real deflection w. Actually, we can only solve these three problems approximately by using the finite element method. Therefore, we must study whether the approximate solutions converge to the real solution as $\lambda \to \infty$ and the partition parameter $h \to 0$, and estimate the error bound.

§2. The Convergence of the Penalty Method

In this section, we assume that for a given λ problem (1.6) can be solved precisely and problems (1.2), (1.7) can also be solved precisely. We will use the usual notations for Sobolev spaces and the seminorms and norms. If the seminorms or norms are calculated on domain Ω, the subscript Ω will be omitted. The seminorms and norms on the product space $H^m(\Omega) \times H^m(\Omega)$ are defined as

$$|u|_l = (|u_1|_l^2 + |u_2|_l^2)^{1/2}, \quad 0 \le l \le m;$$

$$\|u\|_l = \Big(\sum_{i=0}^{l} |u|_i^2\Big)^{1/2}, \quad 0 \le l \le m.$$

Notation $(\cdot\,,\,\cdot)$ is used to indicate the inner product on space $L^2(\Omega)$ or $L^2(\Omega) \times L^2(\Omega)$, i.e.

$$(u, v) = \int_\Omega uv dx, \quad \forall\, u, v \in L^2(\Omega);$$

$$(u, v) = \int_\Omega u \cdot v dx = \int_\Omega (u_1 v_1 + u_2 v_2) dx, \quad \forall\, u, v \in L^2(\Omega) \times L^2(\Omega).$$

Clearly, $B = \text{rot}$ is a linear operator defined on $V = H_0^1(\Omega) \times H_0^1(\Omega)$. $B : V \to L^2(\Omega)$. Since, $\forall v \in V$,

$$\|Bv\|_0 = \Big\{ \int_\Omega \Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big)^2 dx \Big\}^{1/2}$$

$$\le \Big\{ 2 \int_\Omega \Big(\Big(\frac{\partial v_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial v_1}{\partial x_2}\Big)^2 + \Big(\frac{\partial v_2}{\partial x_1}\Big)^2 + \Big(\frac{\partial v_2}{\partial x_2}\Big)^2 \Big) dx \Big\}^{1/2} \le C \|v\|_1, \tag{2.1}$$

where C is a constant independent of v, B is a bounded and hence continuous operator.

By the way, letter C will denote a generic constant and may take different values at different places.

The bending stiffness D and Poisson ratio ν are assumed to be constant, $D > 0, 0 < \nu < 1/2$. Define a bilinear form on V:

$$a(u,v) = D\int_\Omega \Big((1-\nu)\sum_{i,j=1}^{2}\Big(\frac{\partial u_i}{\partial x_j}\Big)\Big(\frac{\partial v_i}{\partial x_j}\Big) + \nu\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)\Big(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\Big)\Big)dx.$$

Since

$$a(v,v) = D\int_\Omega \Big(\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial v_2}{\partial x_2}\Big)^2 + 2\nu\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + (1-\nu)\Big(\Big(\frac{\partial u_1}{\partial x_2}\Big)^2 + \Big(\frac{\partial v_2}{\partial x_1}\Big)^2\Big)\Big)dx$$
$$\geq \frac{D}{2}\int_\Omega \Big(\Big(\frac{\partial v_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial v_1}{\partial x_2}\Big)^2 + \Big(\frac{\partial v_2}{\partial x_1}\Big)^2 + \Big(\frac{\partial v_2}{\partial x_2}\Big)^2\Big)dx = \frac{D}{2}|v|_1^2,$$

and by the Poincaré-Friedrichs inequality

$$\|v\|_0 \leq c|v|_1, \quad \forall\, v \in V,$$

we get the V-ellipticity of $a(\cdot,\cdot)$, i.e. there exists a constant $\alpha > 0$, such that

$$\alpha\|v\|_1^2 \leq a(u,v), \quad \forall\, v \in V. \tag{2.2}$$

Obviously, $a(\cdot,\cdot)$ is bounded on V, i.e. there exists a constant M independent of u and v, such that

$$|a(u,v)| \leq M\|u\|_1\|v\|_1, \quad \forall\, u,v \in V. \tag{2.3}$$

Let the null space of B be V_0, i.e. $V_0 = N(B)$. The problem of minimizing $P(u)$ on V subject to constraint (1.5) is the same problem of minimizing $P(u)$ on V_0 and hence is equivalent to the problem:

$$\text{find } u \in V_0, \text{ such that } a(u,v) = (v,\nabla\varphi), \quad \forall\, v \in V_0. \tag{2.4}$$

V_0 is obviously a closed subspace of V, so (2.2) and (2.3) indicate that problem (2.4) has a unique solution.

For any $\lambda > 0$, we define a "penalized" bilinear form as

$$C_\lambda(u,v) = a(u,v) + \lambda D\int_\Omega BuBv\,dx. \tag{2.5}$$

The V-ellipticity of $C_\lambda(\cdot,\cdot)$ is a consequence of (2.2). Since

$$\Big|\lambda D\int_\Omega BuBv\,dx\Big| \leq \lambda D\int_\Omega \Big|\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big)\Big|dx$$
$$\leq C\lambda D|u|_1|v|_1 \leq C\lambda D\|u\|_1\|v\|_1,$$

we see, $\forall u, v \in V$, that

$$|C_\lambda(u,v)| \leq M_\lambda\|u\|_1\|v\|_1, \tag{2.6}$$

where constant M_λ depends on λ. Minimizing $P_\lambda(u)$ on V is equivalent to the problem:

$$\text{find } u_\lambda \in V, \text{ such that } c_\lambda(u_\lambda,v) = (v,\nabla\varphi), \quad \forall\, v \in V. \tag{2.7}$$

Set

$$V_1 = \{v \in V;\ \text{such that}\, a(v,u) = 0, \forall u \in V_0\}.$$

Then every $v \in V$ can be uniquely decomposed as

$$v = v^{(0)} + v^{(1)}, \quad v^{(i)} \in V_i, \quad i = 0,1. \tag{2.8}$$

We need the following

Lemma 2.1. *There exists a constant $\beta > 0$, independent of v, such that $\forall v \in V_1$,*

$$\|Bv\|_0 \geq \beta\|v\|_1. \tag{2.9}$$

Proof. Since $B : V \to L^2(\Omega)$ is a bounded linear operator defined on V, the dual operator of B, say B', is well defined on $(L^2(\Omega))' = L^2(\Omega)$ and is also a bounded linear operator. If $v = (v_1, v_2) \in V, \varphi \in L^2(\Omega)$, then

$$\int_\Omega Bu\varphi dx = \int_\Omega \Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big)\varphi dx = \int_\Omega \Big(v_1\frac{\partial \varphi}{\partial x_2} - v_2\frac{\partial \varphi}{\partial x_1}\Big)dx,$$

where partial derivatives $\dfrac{\partial\varphi}{\partial x_1}, \dfrac{\partial\varphi}{\partial x_2}$ are understood as distributional derivatives. According to the theory of Sobolev space, $\dfrac{\partial\varphi}{\partial x_1}, \dfrac{\partial\varphi}{\partial x_2} \in H^{-1}(\Omega) = (H_0^1(\Omega))'$. The integral on the right-hand side of the above equation is understood as the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Therefore

$$B'\varphi = \Big(\frac{\partial\varphi}{\varphi x_2}, -\frac{\partial\varphi}{\partial x_1}\Big) \in H^{-1}(\Omega) \times H^{-1}(\Omega) = V', \quad \forall \varphi \in L^2(\Omega). \tag{2.10}$$

In [6], it is proved that the range of the gradient operator, grad $\in \mathcal{L}(L^2(\Omega), (H^{-1}(\Omega))^N)$, is a closed subspace of $(H^{-1}(\Omega))^N$, which means that (when $N = 2$) set $\Big\{\Big(\dfrac{\partial\varphi}{\partial x_1}, \dfrac{\partial\varphi}{\partial x_2}\Big) \in H^{-1}(\Omega) \times H^{-1}(\Omega); \varphi \in L^2(\Omega)\Big\}$ is closed in $H^{-1}(\Omega) \times H^{-1}(\Omega)$. By (2.10), we conclude that the range of B', $R(B')$, is a closed subspace of V'. Using the closed range theorem[4], the range of $B, R(B)$, is a closed subspace of $L^2(\Omega)$. If there are two funcitons v and $v' \in V_1$ such that $Bv = Bv'$, then $v - v' \in V_0$, and hence $v = v'$. So, B is a 1-1 mapping defined on V_1 onto $R(B)$. Then by using the open mapping theorem[4], $B^{-1} : R(B) \to V_1$ is also a bounded linear operator. The lemma is proved.

Now, we are in the position to prove

Theorem 2.1. *Assume that $\{\lambda_i\} \subset R^+$ and $\lambda_i < \lambda_j$ when $i < j, \lambda_i \to \infty (i \to \infty)$. The general term of sequence $\{\lambda_i\}$ is indicated by λ. If u and u_λ are the solutions of problems* (2.4) *and* (2.7) *respectively, then if λ is large enough, we have*

$$\|u - u_\lambda\|_1 \leq C\lambda^{-1}\|\varphi_0\|. \tag{2.11}$$

Proof. If $v \in V$, decompose u_λ and v according to (2.8):

$$u_\lambda = u_\lambda^{(0)} + u_\lambda^{(1)}; \quad v = v^{(0)} + v^{(1)}.$$

Then

$$c_\lambda(u_\lambda, v) = a(u_\lambda^{(0)} + u_\lambda^{(1)}, v^{(0)} + v^{(1)}) + \lambda D \int_\Omega B(u_\lambda^{(0)} + u_\lambda^{(0)})B(v^{(0)} + v^{(1)})dx$$

$$= a(u_\lambda^{(0)}, v^{(0)}) + a(u_\lambda^{(1)}, v^{(1)}) + \lambda D \int_\Omega Bu_\lambda^{(1)} Bv^{(1)} dx$$

$$= (v^{(0)} + v^{(1)}, \nabla\varphi). \tag{2.12}$$

If $v \in V_0$, noticing $Bv = 0$, we see

$$c_\lambda(u_\lambda - u, v) = a(u_\lambda - u, v) = 0, \quad \forall\, v \in V_0,$$

which shows $u_\lambda - u \in V_1$. Because of $u \in V_0$ and the uniqueness of decomposition (2.8), $u = u_\lambda^{(0)}$. Thus the component of u_λ in V_0 is u, for any λ. Hence

$$a(u_\lambda^{(0)}, v^{(0)}) = (v^{(0)}, \nabla\varphi)$$

by (2.4). Substituting the above equality into (2.12), we get

$$a(u_i^{(1)} v_\bullet^{(1)}) + \lambda D \int_\Omega Bu_\lambda^{(1)} Bv^{(1)} dx = (v^{(1)}, \nabla\varphi), \quad \forall v^{(1)} \in V_1,$$

or, $\forall v \in V_1$,

$$\int_\Omega Bu_\lambda^{(1)} Bv dx = \frac{1}{\lambda D}(-a(u_\lambda^{(1)}, v) + (v, \nabla\varphi)).$$

Using (2.9) twice, we have

$$\|u_\lambda^{(1)}\|_1 = C\|Bu_\lambda^{(1)}\|_0 = \sup_{v\in V_1} \frac{\left|\int_\Omega Bu_\lambda^{(1)} Bv dx\right|}{\|Bv\|_0} \le C\frac{1}{\lambda} \sup_{v\in V_1} \frac{|(v,\nabla\varphi)| + |a(u_\lambda^{(1)}, v)|}{\|v\|_1}. \tag{2.13}$$

Green's formula shows that, since $v|_{\partial\Omega} = 0$,

$$(v, \nabla\varphi) = \int_\Omega (v_1 \frac{\partial\varphi}{\partial x_1} + v_2 \frac{\partial\varphi}{\partial x_2}\Big) dx = \int_{\partial\Omega} (V_1 n_1 + v_2 n_2)\varphi ds$$

$$- \int_\Omega \Big(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2})\varphi dx = -\int_\Omega \Big(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\Big)\varphi dx.$$

Substituting the last equality into (2.13) and using (2.3), we get

$$\|u_\lambda^{(1)}\|_1 \le C\lambda^{-1}\|\varphi\|_0 + C'\lambda^{-1}\|u_\lambda^{(1)}\|_1.$$

If λ_i is large enough such that $C'\lambda^{-1} \le 1/2$, we see

$$\|u_\lambda^{(1)}\|_1 = \|u_\lambda - u\|_1 \le C\lambda^{-1}\|\varphi\|_0.$$

The theorem is proved.

We now examine what happens when function u on the right-hand side of (1.7) is substituted by u_λ. Suppose we then get a solution w_λ from (1.7). For $\xi(x), \eta(x) \in H^1(\Omega)$, set

$$b(\xi, \eta) = \int_\Omega \Big(\frac{\partial\xi}{\partial x_1}\frac{\partial\eta}{\partial x_1} + \frac{\partial\xi}{\partial x_2}\frac{\partial\eta}{\partial x_2}\Big)dx. \tag{2.14}$$

Problem (1.7) is equivalent to the problem:

$$\text{find}\, w \in H_0^1(\Omega),\ \text{ such that }\ b(w,\xi) = \int_\Omega \Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)\xi dx \quad \forall \xi \in H_0^1(\Omega). \tag{2.15}$$

Therefore,

$$b(w - w_\lambda, \xi) = \int_\Omega \Big(\frac{\partial(u_1 - u_{\lambda,1})}{\partial x_1} + \frac{\partial(u_2 - u_{\lambda,2})}{\partial x_2}\Big)\xi dx, \quad \forall \xi \in H_0^1(\Omega).$$

Then the $H_0^1(\Omega)$-ellipticity of bilinear form $b(\cdot,\cdot)$ gives

$$\begin{aligned}
\|w - w_\lambda\|_1^2 &\le Cb(w - w_\lambda, w - w_\lambda) \\
&= c\int_\Omega \Big(\frac{\partial(u_1 - u_{\lambda,1})}{\partial x_1} + \frac{\partial(u_2 - u_{\lambda,2})}{\partial x_2}\Big)(w - w_\lambda)dx \\
&= \Big\{c\int_{\partial\Omega}(w - w_\lambda)((u_1 - u_{\lambda,1})n_1 + (u_2 - u_{\lambda,2})n_2)ds \\
&\quad - \int_\Omega \Big((u_1 - u_{\lambda,1})\frac{\partial(w - w_\lambda)}{\partial x_1} + (u_2 - u_{\lambda,2})\frac{\partial(w - w_\lambda)}{\partial x_2}\Big)dx\Big\} \\
&= C\Big\{-\int_\Omega (u - u_\lambda)\cdot\nabla(w - w_\lambda)dx\Big\} \\
&\le C\|u - u_\lambda\|_0\|w - w_\lambda\|_1.
\end{aligned}$$

We have proved

Theorem 2.2. *Suppose w and w_λ are the solutions of (1.7) corresponding to the right hand funcitons u and u_λ respectively. Then*

$$\|w - w_\lambda\|_1 \le C\|u - u_\lambda\|_0 \le C\lambda^{-1}\|\varphi\|_0. \tag{2.16}$$

Remark. If an estimate of $\|u - u_\lambda\|_0$, which has higher convergence rate than that of $\|u - u_\lambda\|_1$ given by (2.11), can be obtained, it is obvious that an estimate of $\|w - w_\lambda\|_1$ better than that of (2.16) can be obtained. The problem remains open.

§3. The Convergence of the Finite Element Approximations

In this section, we are going to investigate the error produced by using the finite element method to solve problems (1.2), (1.6) and (1.7) approximately. For simplicity, let us assume that domain Ω is a polygon. As usual, make a triangulation on Ω, say T_n, which satisfies the condition of quasi-uniform partition and the partition parameter $h < 1$. Suppose that S_h^k is the space of finite element functions (piecewise polynomials). Let $P_k(K)$ be the set of all polynomials defined on $K \in T_h$ with degrees $\le k$ and $P_k = S_h^k|_k$. We always assume $P_k(K) \subset P_K$ for some integer $k \ge 1$, $S_h^k \subset C^0(\Omega)$ and $v|_{\partial\Omega} = 0$, $\forall v \in S_h^k$.

If φ_h is the finite element approximation of φ, the solution of problem (1.2), in S_h^k, from the well known results of finite element analysis[2], we have estimate

$$\|\varphi - \varphi_h\|_l \le ch^{k+1-l}\|\varphi\|_{k+1}, \quad l = 0, 1. \tag{3.1}$$

Let us then estimate the error of the approximation of u. Set $V_h = S_h^{k_1} \times S_h^{k_1}$, where k_1 is a positive integer not necessarily equal to k. Assume that $\tilde{u}_{\lambda h}$ is the solution of the problem

$$\tilde{u}_{\lambda h} \in V_h, \quad c_\lambda(\tilde{u}_{\lambda h}, v) = (v, \nabla\varphi), \quad \forall v \in V_h. \tag{3.2}$$

Since the well known result like (3.1):

$$\|u_\lambda - \tilde{u}_{\lambda h}\|_l \le Ch^{k_1+1-l}\|u_\lambda\|_{k_1+1}, \quad l = 0, 1 \tag{3.3}$$

can not be used here directly, because the constant C on the right-hand side of (3.3) may depend on the coefficients of $c_\lambda(\cdot,\cdot)$ and hence on λ, and so may the constant factor on the right-hand side of the ragularity inequality:

$$\|u_\lambda\|_{k+1} \le C\|\nabla\varphi\|_{k-1} \le C\|\varphi\|_k,$$

we have to make a more careful analysis.

Let us introduce the "penalized norm" in V:

$$\|v\|_{(\lambda)} = (C_\lambda(v, v))^{1/2}, \quad \forall v \in V.$$

Obviously, $\forall v, w \in V$,

$$\|v\|_{(\lambda)}^2 \le C_\lambda(v, v), \quad |c_\lambda(v, w)| \le \|v\|_{(\lambda)}\|w\|_{(\lambda)}. \tag{3.4}$$

The following lemma holds:

Lemma 3.1. *If λ is large enough,*

$$\|u_\lambda - \tilde{u}_{\lambda h}\|_1 \le C\lambda^{1/2} \inf_{v \in V_h} \|u_\lambda - v\|_1, \tag{3.5}$$

where constant C is independent of λ and h.

Proof. As usual, if $v \in V_h$,

$$\|u_\lambda - \tilde{u}_{\lambda h}\|_1 \le \|u_\lambda - v\|_1 + \|v - \tilde{u}_{\lambda h}\|_1. \tag{3.6}$$

By equality

$$c_\lambda(u_\lambda - \tilde{u}_{\lambda h}, v) = 0, \quad \forall v \in V_h$$

and (3.4), we see, $\forall v \in V_h$,

$$\begin{aligned}
\|v - \tilde{u}_{\lambda h}\|_{(\lambda)}^2 &\le C_\lambda(v - \tilde{u}_{\lambda h}, v - \tilde{u}_{\lambda h}) \\
&= c_\lambda(v - u_\lambda, v - u_{\lambda h}) + c_\lambda(u_\lambda - \tilde{u}_{\lambda h}, v - \tilde{u}_{\lambda h}) \\
&\le \|v - u_\lambda\|_{(\lambda)}\|v - \tilde{u}_{\lambda h}\|_{(\lambda)}.
\end{aligned} \tag{3.7}$$

Hence

$$\|v-\tilde{u}_{\lambda h}\|_1 \le \|v-\tilde{u}_{\lambda h}\|_{(\lambda)} \le \|u_\lambda - v\|_{(\lambda)} = (a(u_\lambda - v, u_\lambda - v) + \lambda D\|B(u_\lambda - v)\|_0^2)^{1/2}$$
$$\le (M\|u_\lambda - v\|_1^2 + c_1\lambda\|u_\lambda - v\|_1^2)^{1/2} \le c\lambda^{1/2}\|u_\lambda - v\|_1, \tag{3.8}$$

where we have used the assumption that λ is large enough such that $c_1\lambda \ge M$. The lemma is proved.

If the solution of (1.2) satisfies $\varphi \in H^{k_1}(\Omega)$, lemma 3.1 says

$$\|u_\lambda - \tilde{u}_{\lambda h}\|_1 \le C(\lambda^{-1/2} + \lambda^{1/2}h^{k_1})\|\varphi\|_{k_1}. \tag{3.9}$$

In fact, since

$$\|u_\lambda - v\|_1 \le \|u_\lambda - u\|_1 + \|u - v\|_1, \tag{3.10}$$

taking function v of the last inequality to be u_I, the interpolant of u in V_h, and using theorem 2.1, we obtain (3.9).

Now let $u_{\lambda h}$ satisfy

$$u_{\lambda h} \in V_h, \quad c_\lambda(u_{\lambda h}, v) = (v, \nabla\varphi_h), \quad \forall v \in V_h. \tag{3.11}$$

Calculating $u_{\lambda h}$ is our work. Since

$$\|\tilde{u}_{\lambda h} - u_{\lambda h}\|_1 \le C \sup_{v\in V_h} \frac{|c_\lambda(\tilde{u}_{\lambda h} - u_{\lambda h}, v)|}{\|v\|_{(\lambda)}} \le C \sup_{v\in V_h} \frac{|(v, \nabla(\varphi - \varphi_h))|}{\|v\|_1}$$
$$\le C\|\varphi - \varphi_h\|_0 \le ch^{k+1}\|\varphi\|_{k+1}, \tag{3.12}$$

we have

Theorem 3.1. *Assume that u and $u_{\lambda h}$ are the solutions of (2.4) and (3.11) respectively. Then λ and k_1 can be chosen such that*

$$\|u - u_{\lambda h}\|_1 \le Ch^{(k+1)/2}\|\varphi\|_{k+1}, \tag{3.13}$$

where constant C is independent of λ and h.

Proof. By using inequality

$$\|u - u_{\lambda h}\|_1 \le \|u - u_\lambda\|_1 + \|u_\lambda - \tilde{u}_{\lambda h}\|_1 + \|\tilde{u}_{\lambda h} - u_{\lambda h}\|_1$$

and combining (2.11), (3.9) and (3.12), we see

$$\|u - u_{\lambda h}\|_1 \le C(\lambda^{-1}\|\varphi\|_0 + (\lambda^{-1/2} + \lambda^{1/2}h^{k_1})\|\varphi\|_{k_1} + h^{k+1}\|\varphi\|_{k+1}).$$

Set $\lambda = h^{-(k+1)}, k_1 = k+1$. (3.13) follows.

Now, let us assume that $\tilde{w}_{\lambda h}$ satisfies

$$\begin{cases} \Delta\tilde{w}_{\lambda h} = \dfrac{\partial(u_{\lambda h})_1}{\partial x_1} + \dfrac{\partial(u_{\lambda h})_2}{\partial x_2}, & x \in \Omega, \\ \tilde{w}_{\lambda h} = 0, & x \in \partial\Omega; \end{cases}$$

and $w_{\lambda h}$ is the solution of the problem:

$$w_{\lambda h} \in S_h^{k_2}, \quad b(w_{\lambda h}, \xi) = \int_\Omega \Big(\frac{\partial(u_{\lambda h})_1}{\partial x_1} + \frac{\partial(u_{\lambda h})_2}{\partial x_2}\Big)\xi dx, \quad \forall\xi \in S_h^{k_2}, \tag{3.14}$$

where $b(\cdot,\cdot)$ is defined in (2.14) and $k_2 > 0$ is an integer. We can easily see

$$\|w - \tilde{w}_{\lambda h}\|_1 \le C\|u - u_{\lambda h}\|_0, \tag{3.15}$$

where w is the solution of (1.7). It is well known that

$$\|\tilde{w}_{\lambda h} - w_{\lambda h}\|_1 \le C \inf_{v \in S_h^{k_2}} \|\tilde{w}_{\lambda h} - v\|_1. \tag{3.16}$$

Since

$$\|\tilde{w}_{\lambda h} - v\|_1 \le \|\tilde{w}_{\lambda h} - w\|_1 + \|w - v\|_1,$$

if we take v of the last inequality to be w_I, the interpolant of w in $S_h^{k_2}$, we see

$$\|\tilde{w}_{\lambda h} - w_{\lambda h}\|_1 \le C(\|u - u_{\lambda h}\|_0 + ch^{k_2}\|w\|_{k_2+1}).$$

Let k_2 be chosen as

$$k_2 = \begin{cases} \frac{1}{2}(k+1), & \text{when } k+1 \text{ is even,} \\ \frac{1}{2}(k+1) + \frac{1}{2}, & \text{when } k+1 \text{ is odd.} \end{cases} \tag{3.17}$$

We get

Theorem 3.2. *If w and $w_{\lambda h}$ are the solutions of* (1.7) *and* (3.14) *respectively and k_2 is defined in* (3.17), *we have estimate*

$$\|w - w_{\lambda h}\|_1 \le Ch^{(k+1)/2}\|\varphi\|_{k+1},$$

where constant C is independent of λ and h.

Theorem 3.1 shows that the error order of the OM method is the same as that of the usual penalty method[5].

References

[1] M. Ortiz and G.R. Morris, C^0 finite element discretization of Kirchhoff's equation of thin plate bending. *Int. J. Numer. Math. Eng.*, **26** : 7 (1988), 1551–1566.

[2] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, 1978.

[3] C. Johnson, On the convergence of a mixed finite element method for plate bending problems, *Numer. Math.*, **21** : 1 (1973), 43–62.

[4] K. Yosida, Functional Analysis, Springer-Verlag, 1978.

[5] I. Babuška and M.Zlamal, Nonconforming element in finite element method with penalty, *SIAM J. Numer. Anal.*, **10** : 5 (1973), 863–879.

[6] V. Girault and P.A. Raviart, Finite Element Methods for Navier-Stokes Equations, Springer-Verlag, 1986.