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NOTES ON SUPERCONVERGENCE AND ITS RELATED TOPICS*

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Abstract

Some essences of superconvergence phenomena and interior relations among a posteriori error estimates, superconvergence and correction are discovered in this paper and a correction scheme with the fifth order accuracy is presented.

§1. Relations to Superconvergence

For years, much effort has been focused on superconvergence in the numerical solution of equations. In the context of finite element methods, it was found that in many cases the convergence rate of the finite element approximations at some specific points e.g. optimal stress points is higher than the usual global rate of convergence predicted by theory. Obviously, this kind of information is very useful and valuable when further manipulation of the computed solution is desired. Thus, superconvergence is of considerable interest both from a theoretical and a practical viewpoint.

An extension survey of work in this topic has been provided by Krizek and Neittaanmäki[3] and Lin and Zhou[8]. Of the large number (over 200) of papers cited that are concerned with various superconvergence phenomena at varied isolated "optimal stress points", and few papers deal with global superconvergence[4-6]. No papers, however, concerns with the interior reasons why the superconvergence phenomena exist and the interior relations which the superconvergence results relate to. In this section, we shall discover the reasons and the relations.

Let $u \in H_0^1$ be the exact solution of a model problem, an elliptic boundary value problem of second order:

$$a(u,v) = (f,v), \qquad \forall\, v \in H_0^1 \tag{1}$$

on a domain Ω, where $a(\cdot,\cdot)$ is the bilinear form associted with the differential operator. The Ritz projection $R_h u$ onto S_h, the linear finite element space with respect to a triangulation T_h of Ω with mesh size h, is defined by

$$a(R_h u, v) = a(u,v) \qquad \forall\, v \in S_h. \tag{2}$$

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For simplicity, we assume u is smooth enough and a norm is the L^∞-norm.

Proposition 1. Let $\Omega' \subset \Omega$ be a subdomain. The following assertions are equivalent

(i) $\nabla(R_h u - i_h u) = O(h^2|\ln h|) \quad \text{in}\, \Omega'$, (3)

(ii) $\nabla(i'_{2h} R_h u - u) = O(h^2|\ln h|) \quad \text{in}\, \Omega'$, (4)

(iii) $\nabla(R_h u - u) = \nabla(R_h u - i'_{2h} R_h u) + O(h^2|\ln h|) \quad \text{in}\, \Omega'$ (5)

where i_h and i'_{2h} denote the standard nodal linear and quadratic interpolations which satisfy[8]

$$i'_{2h} i_h = i'_{2h}, \qquad i_h i'_{2h} = i_h.$$

Proof. From the stability of interpolation operator and the formula

$$\nabla(R_h u - i_h u) = \nabla(i_h(i'_{2h} R_h u - u)), \tag{6}$$

one sees that (ii) implies (i). Conversely, a little modification of (i) yields (ii)[4-6]. While (iii) is an alternative version of (ii) only. This completes the proof.

Remark. (1°) If Ω is a convex polygonal domain, and Ω' covered by local almost uniform meshes in a piecewise sense is a subdomain away from corner points, then (i) holds true.

(2°) The equivalence of (i) and (ii) implies that $i'_{2h} R_h u$, the interpolated linear element, yields the same accuracy in gradient as the interpolated solution $i'_{2h} u$ or the direct quadratic finite element solution under certain geometric restrictions on the mesh. Conversely, from the view of interpolation theory, the employ of quadratic interpolation i'_{2h} is necessary.

(3°) (iii) is nothing but the a posteriori error estimates, which means also that superconvergence results (i) and (ii) are somewhat of direct corollaries from the defect correction. A post–processing in the finite element method made by Babuska and Miller[1] should be mantioned here.

§2. A Correction Scheme

Defect correction techniques for the finite element method were, as we know, initiated by Prof. Rannacher and developed by Prof. Blum during the last four years, see a survey in this topic[9]. From another point of view, a different type of correction scheme was presented to get a high accuracy approximation in 1991[7,8]. This kind of correction technique can be interpreted as the multiplications of certain contractive operators, from which a fourth order accuracy for the finite element gradient is natural obtained on a subdomain globally, see [7, 8]. Similarly, by means of the multiplications of three contractive operators we constract here a correction scheme with the fifth order accuracy stated as follows

Propostion 2. Let Ω be decomposed into the macro-triangles without interior macro-vertices and each macro-triangle be uniformly subdivided into subtriangles with mesh size h. Then there holds, in Ω

$$R_h u - (i'_{2h} + i'_{4h})R_h u + R_h(i'_{2h} + i'_{4h})R_h u + i'_{2h}R_h i'_{4h}R_h u - R_h i'_{2h}R_h i'_{4h}R_h u$$
$$= u + O(h^{\min(5,3+1/\alpha)-\varepsilon}) \quad (7)$$

where i'_{4h} is the standard nodal quadruplicate interpolation satisfies[7,8]

$$i'_{4h}i_h = i'_{4h} \qquad i_h i'_{4h} = i_h,$$

$\alpha\pi$ is the maximal interior angle of $\Omega, \varepsilon > 0$ small, Ω' away from the corner points.

Proof. Let $L_h u = (i'_{2h}R_h - I)(i'_{4h}R_h - I)u$ and Ω'' away from the corner points satisfy $\Omega' \subset \Omega'' \subset \Omega$. One sees that[7,8]

$$||L_h u||_{0,\infty,\Omega''} \le ch^4|\ln h|^2, \quad (8)$$

$$||L_h u||_{1,2,\Omega} \le ch^{\min(4,2+1/\alpha)-\varepsilon}. \quad (9)$$

Thus, form the local stability with respect to maximum norm one obtains

$$||(R_h - I)L_h u||_{0,\infty,\Omega'} \le ch|\ln h|||L_h u||_{1,\infty,\Omega''} + ch||L_h u||_{1,2,\Omega}$$
$$\le ch^{\min(5,3+1/\alpha)-\varepsilon} \quad (10)$$

while $(R_h - I)L_h u$ is nothing but the left hand of (7), which proves Proposition 2.

Remark. 1) It is easy to see that an alternative version of Proposition 2 is a global error expansion with terms of $O(h^2), O(h^3), O(h^4)$ and $O(h^{\min(5,3+1/\alpha)-\varepsilon})$ (cf. [8]).

2) We point out here that global high accuracy approximations can be also obtained from the defect correction scheme stated in [9] if interpolation postprocesses are added.

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