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A SHARP ESTIMATE OF A SIMPLIFIED VISCOSITY SPLITTING SCHEME*

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Abstract

A viscosity splitting method for solving the initial boundary value problems of the Navier-Stokes equation, introduced by Zheng and Huang, is considered. We give an improved and sharp estimate in the space $L^\infty(0,T;(L^2(\Omega))^2)$.

§1. Introduction

Let Ω be a bounded domain in R^2. For simplicity we assume that it is a simply connected bounded domain, and its boundary $\partial\Omega$ is sufficiently smooth. Denote by $x=(x_1,x_2)$ a point in R^2. The usual notations $H^s(\Omega), W^{m,p}(\Omega)$ for Sobolev spaces, and $\|\cdot\|_s, \|\cdot\|_{m,p}$ for their norms are applied through out this paper. It is known that $L^2(\Omega)=H^0(\Omega)$.

In [1] the viscosity splitting method for solving the two-dimensional initial boundary value problem of the Navier-Stokes equation

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\frac{1}{\rho}\nabla P=\nu\triangle u+f,\qquad x\in\Omega,t>0,\tag{1.1}$$

$$\nabla\cdot u=0,\qquad x\in\Omega,t>0,\tag{1.2}$$

$$u|_{x\in\partial\Omega}=0,\tag{1.3}$$

$$u|_{t=0}=u_0(x)\tag{1.4}$$

was considered, where $u=(u_1,u_2)$ is the velocity, P is the pressure, the positive constants ν,ρ are the density and viscosity respectively, and ∇ is the gradient, $\triangle=\nabla^2, \nabla\cdot u_0=0, u_o|_{x\in\partial\Omega}=0$. The following scheme was considered: divide the interval $[0,T]$ into equal subintervals with length k; then we solve $\tilde u_k(t),\tilde P_k(t),u_k(t),P_k(t)$ on each interval $[ik,(i+1)k), i=0,1,\cdots$, according to the following procedure:

First step. Solve a problem on interval $[ik,(i+1)k)$

$$\frac{\partial\tilde u_k}{\partial t}+(\tilde u_k\cdot\nabla)\tilde u_k+\frac{1}{\rho}\nabla\tilde P_k=f,\tag{1.5}$$

$$\nabla\cdot\tilde u_k=0,\tag{1.6}$$

$$\tilde u_k\cdot n|_{x\in\partial\Omega}=0,\tag{1.7}$$

$$\tilde u_k(ik)=u_k(ik-0)\tag{1.8}$$

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where n is the unit outward normal vector and $u_k(-0) = u_0$.

Second step. Solve a problem on interval $[ik, (i+1)k)$

$$\frac{\partial u_k}{\partial t} + \frac{1}{\rho} \nabla P_k = \nu \triangle u_k, \tag{1.9}$$

$$\nabla \cdot u_k = 0, \tag{1.10}$$

$$u_k|_{x\in\partial\Omega} = 0, \tag{1.11}$$

$$u_k(ik) = \tilde{u}_k(i+1)k - 0). \tag{1.12}$$

Zheng and Huang proved that this scheme converges, and for any $0 < \varepsilon < \frac{1}{4}$, the rate of convergence is $O(k^{\frac{3}{4}-\epsilon})$ in the space $L^\infty(0,T;(L^2(\Omega))^2)$, where k is the length of the time step.

We now consider the same scheme and give an improved and sharp estimate. Our main result is the following

Theorem. *If* $u_0 \in (H^3(\Omega))^2 \cap (H_0^1(\Omega))^2, \nabla \cdot u_0 = 0$, $f \in L^\infty(0,T;(H^3(\Omega))^2) \cap W^{2,\infty}(0,T;(H^{\frac{1}{2}}(\Omega))^2)$, *$u$ is the solution of problem* (1.1) – (1.4), *$\tilde{u}_k, u_k$ is the solution of problem* (1.5) – (1.12), $0 \le s < 3/2$, *then*

$$\sup_{0\le t\le T} \|\tilde{u}_k(t)\|_{s+1} \le M, \tag{1.13}$$

$$\sup_{0\le t\le T} (\|u(t) - u_k(t)\|_0, \|u(t) - \tilde{u}_k(t)\|_0) \le M'k, \tag{1.14}$$

where the constants M, M' depend only on the domain Ω, constants ν, s, T, and functions f, u_0 and u.

§2. Preliminaries

We will use the Helmholtz operator P and the Stokes operator A frequently. It is known that

$$(L^2(\Omega))^2 = X \oplus G$$

where

$$X = \text{ Closure in } (L^2(\Omega))^2 \text{ of } \{u \in (C_0^\infty(\Omega))^2; \nabla \cdot u = 0\},$$
$$G = \{\nabla P; P \in H^1(\Omega)\}$$

P is the orthogonal projection $P : (L^2(\Omega))^2 \to X$, which is a bounded operator from $H^s(\Omega))^2$ to $(H^s(\Omega))^2$ for any nonnegative s. A is defined as $A = -P\triangle$ with domain $D(A) = X \cap \{u \in (H^2(\Omega))^2; u|_{\partial\Omega} = 0\}$ which admits the following properties:

$$\|A^\alpha e^{-tA}\| \le Ct^{-\alpha}, \qquad \alpha \ge 0, t > 0, \tag{2.1}$$

$$\frac{1}{C}\|u\|_{2\alpha} \le \|A^\alpha u\|_0 \le C\|u\|_{2\alpha}, \qquad \forall u \in D(A^\alpha), \alpha \ge 0 \tag{2.2}$$

and if $0 \le s < \frac{1}{2}$ and $u \in X \cap (H^s(\Omega))^2$, then $u \in D(A^{\frac{s}{2}})$; if $1 \le s < 3/2$ and $u \in D(A) \cap (H^{s+1}(\Omega))^2$, then $u \in D(A^{\frac{s+1}{2}})$.

We first consider the linear problem. Assume $f_1 = P(f - u \cdot \nabla)u)$ is known, where u is the solution of (1.1)-(1.4), and u, f are sufficiently smooth. Then (1.5) becomes

$$\frac{\partial \widetilde{u}_k}{\partial t} = f_1. \tag{2.3}$$

Lemma 1. *If $u_0 \in (H^3(\Omega))^2 \cap (H_0^1(\Omega))^2, \nabla \cdot u_0 = 0$, f is sufficiently smooth, u is the solution of problem (1.1) – (1.4) and $\widetilde{u}^*, u^*$ is the solution of problem (2.3), (1.6) – (1.12), $0 \leq s < 3/2$, then*

$$\sup_{0 \leq t \leq T} (\|u(t) - u^*(t)\|_0, \|u(t) - \widetilde{u}^*(t)\|_0) \leq C_1 k \tag{2.4}$$

where the constant C_1 depends only on the domain Ω, constants ν, s, T, functions f, u_0 and u.

Lemma 2. *If $u_0 \in (H^3(\Omega))^2 \cap (H_0^1(\Omega))^2, \nabla \cdot u_0 = 0$, f is sufficiently smooth, u is the solution of problem (1.1) – (1.4), and $\widetilde{u}_k, u_k$ is the solution of problem (1.5) – (1.12), $0 \leq s < 3/2$, then, for $0 \leq t \leq T$,*

$$\sup_{0 \leq t \leq T} \|\widetilde{u}_k(t)\|_{s+1} \leq M \tag{2.5}$$

where the constant M depends only on the domain Ω, constants ν, s, T, functions f, u_0 and u.

The above results were proved in [1].

Following the argument in [2], we prove a similar lemma.

Lemma 3. *If $v, w \in (C^1(\overline{\Omega}))^2, v = (v_1, v_2)$, then*

$$\|e^{-\nu t A} P \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(v_i w)\|_0 \leq C t^{-1+\frac{1}{q}} \|v\|_{0,r} \|w\|_0 \tag{2.6}$$

and

$$\|e^{-\nu t A} P \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(v_i w)\|_0 \leq C t^{-1+\frac{1}{q}} \|w\|_{0,r} \|v\|_0 \tag{2.7}$$

where

$$q > 0, r > 0, \frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad t > 0.$$

Proof. Let the left hand side of (4.17)-(4.18) be a. Then

$$a = \sup_{\|\varphi\|_0 = 1} \Big\{ \int_\Omega \varphi e^{-\nu t A} P \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(v_i w) dx \Big\}.$$

Since P is an orthogonal projection on $(L^2(\Omega))^2$, hence

$$a = \sup_{\|\varphi\|_0 = 1} \Big\{ \int_\Omega P\varphi e^{-\nu t A} P \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(v_i w) dx \Big\}$$

and $e^{-\nu t A}$ is self-adjoint in the sense of

$$\int_\Omega \varphi \cdot e^{-\nu t A} \psi dx = \int_\Omega \psi \cdot e^{-\nu t A} \varphi dx \quad \forall \varphi, \psi \in X.$$

Therefore

$$a = \sup_{\|\varphi\|_0=1} \left\{ \int_\Omega e^{-\nu t A} P\varphi P \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(v_i w) dx \right\}.$$

Again by the orthogonality of P,

$$a = \sup_{\|\varphi\|_0=1} \left\{ \int_\Omega e^{-\nu t A} P\varphi \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(v_i w) dx \right\}.$$

By Green's formula and by observing $e^{-\nu t A} P\varphi|_{\partial\Omega} = 0$, we get

$$\begin{aligned} a &= \sup_{\|\varphi\|_0=1} \left\{ \int_\Omega \sum_{i=1}^{2} \frac{\partial}{\partial x_i} e^{-\nu t A} P\varphi \cdot (v_i w) dx \right\} \\ &\le \sup_{\|\varphi\|_0=1} C \sum_{i=1}^{2} \left\| \frac{\partial}{\partial x_i} e^{-\nu t A} P\varphi \right\|_{0,q} \|v_i\|_{0,r} \|w\|_0. \end{aligned} \tag{2.8}$$

From the imbedding theorem,

$$\left\| \frac{\partial}{\partial x_i} e^{-\nu t A} P\varphi \right\|_{0,q} \le \|e^{-\nu t A} P\varphi\|_{2-\frac{2}{q}} \le C\|A^{1-\frac{1}{q}} e^{-\nu t A} P\varphi\|_0 \le C t^{-1+\frac{1}{q}} \|\varphi\|_0$$

which together with (2.8) yields (2.6). The proof of (2.7) is similar.

§3. Proof of the Theorem

Let u^* and $\widetilde{u}^*$ be the solutions of (2.3), (2.6)–(2.12). Then,

$$\frac{\partial(\widetilde{u}^* - \widetilde{u}_k)}{\partial t} = P(((\widetilde{u}_k - u) \cdot \nabla)u + (\widetilde{u}_k \cdot \nabla)(\widetilde{u}_k - u)), \ ik \le t < (i+1)k,$$

$$\widetilde{u}^*(ik) - \widetilde{u}_k(ik) = u^*(ik - 0) - u_k(ik - 0).$$

Since

$$((\widetilde{u}_k \cdot \nabla)(\widetilde{u}^* - u_k), \widetilde{u}^* - u_k) = 0,$$

we have

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|\widetilde{u} - \widetilde{u}_k\|_0^2 &= (P(((\widetilde{u}_k - u) \cdot \nabla)u + (\widetilde{u}_k \cdot \nabla)(\widetilde{u}_k - u)), \widetilde{u}^* - \widetilde{u}_k) \\ &= (((\widetilde{u}_k - u) \cdot \nabla)u + (\widetilde{u}_k \cdot \nabla)(\widetilde{u}^* - u), \widetilde{u}^* - \widetilde{u}_k). \end{aligned}$$

Using (1.13) gives

$$\begin{aligned} \frac{\partial}{\partial t}\|\widetilde{u}^* - \widetilde{u}_k\|_0 &\le C(\|u\|_{\frac{5}{2}}(\|\widetilde{u}_k - \widetilde{u}^*\|_0 + \|\widetilde{u}^* - u\|_0) + \|\widetilde{u}_k\|_2\|\widetilde{u}^* - u\|_1) \\ &\le C(\|\widetilde{u}_k - \widetilde{u}^*\|_0 + \|\widetilde{u}^* - u\|_1) \end{aligned}$$

and then

$$\begin{aligned} \|(\widetilde{u}_k - \widetilde{u}^*)(t)\|_0 &\le e^{C(t-ik)}(\|(\widetilde{u}_k - \widetilde{u}^*)(ik)\|_0 + k \max_{0\le\tau<t} \|\widetilde{u}^*(\tau) - u(\tau)\|_1 \\ &\le C(\|\widetilde{u}^*(ik) - \widetilde{u}_k(ik)\|_0 + k). \end{aligned} \tag{3.1}$$

Let $f_2 = ((\widetilde{u}_k - u)\cdot\nabla)u + (\widetilde{u}_k\cdot\nabla)(\widetilde{u}_k - u)$. Then

$$u^*(t) - u_k(t) = \sum_{i=0}^{[\frac{t}{k}]}\int_{ik}^{(i+1)k} e^{-\nu(t-ik)A}Pf_2(s)ds.$$

Then

$$\begin{aligned}\|u^*(t) - u_k(t)\|_0 \le & \sum_{i=0}^{[\frac{t}{k}]-1}\int_{ik}^{(i+1)k}\|e^{-\nu(t-ik)A}Pf_2(s)\|_0 ds \\ & + \int_{[\frac{t}{k}]k}^{([\frac{t}{k}]+1)k}\|e^{-\nu(t-ik)A}Pf_2(s)\|_0 ds.\end{aligned}$$

By Lemma 3,

$$\begin{aligned}\|u^*(t) - u_k(t)\|_0 \le & C\sum_{i=0}^{[\frac{t}{k}]-1}\int_{ik}^{(i+1)k}(t-ik)^{-1+\frac{1}{q}}(\|u\|_{0,r}\|u-\widetilde{u}_k\|_0 \\ & + \|\widetilde{u}_k\|_{0,r}\|u-\widetilde{u}_k\|_0) + \int_{[\frac{t}{k}]k}^{([\frac{t}{k}]+1)k}\|f_2(s)\|_0 ds \\ & \le C\sum_{i=0}^{[\frac{t}{k}]-1}\int_{ik}^{(i+1)k}(t-ik)^{-1+\frac{1}{q}}(\|\widetilde{u}^* - \widetilde{u}_k\|_0 + k)ds + Ck \\ & \le C\sum_{i=0}^{[\frac{t}{k}]-1}\int_{ik}^{(i+1)k}(t-s)^{-1+\frac{1}{q}}(\|u^* - u_k\|_0 + k)ds + Ck.\end{aligned}$$

Set $\psi(t) = \sup_{0\le\tau<t}\|u^*(\tau) - u_k(\tau)\|_0$. Then

$$\|u^*(t) - u_k(t)\|_0 \le C\int_0^t (t-\tau)^{-1+\frac{1}{q}}\psi(\tau)d\tau + Ck.$$

Taking the supremum with respect to t, we obtain

$$\psi(t) \le C\int_0^t (t-\tau)^{-1+\frac{1}{q}}\psi(\tau)d\tau + Ck.$$

The corresponding Volterra integral equation is

$$y(t) = C\int_0^t (t-\tau)^{-1+\frac{1}{q}}y(\tau)d\tau + Ck.$$

It can be checked that

$$\psi(t) \le y(t), \quad y(t) \le Ck,$$

which together with (3.1) and (2.4) gives (1.14).

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