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IMPLICIT DIFFERENCE METHODS FOR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER*[1)]

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Abstract

This paper is a sequel to [2]. A two parameter family of explicit and implicit schemes is constructed for the numerical solution of the degenerate hyperbolic equations of second order. We prove the existence and the uniqueness of the solutions of these schemes. Furthermore, we prove that these schemes are stable for the initial values and that the numerical solution is convergent to the unique generalized solution of the partial differential equation.

§1. The Problem and the Difference Schemes

Consider the initial boundary value problem for the degenerate hyperbolic equation of second order

$$\frac{\partial^2 u}{\partial t^2} - x^p a(x,t)\frac{\partial^2 u}{\partial x^2} = b(x,t)\frac{\partial u}{\partial x} + f\Big(x,t,u,\frac{\partial u}{\partial t}\Big), \quad (x,t) \in Q, \tag{1}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le L, \tag{2}$$

$$\partial u/\partial t\,|_{t=0} = u_1(x), \quad 0 \le x \le L, \tag{3}$$

$$u(0,t) \text{ is finite on } 0 \le t \le T, \tag{4}$$

$$u(L,t) = 0, \quad 0 \le t \le T. \tag{5}$$

Here the domain $Q = \{0 < x < L, 0 < t \le T\}$ and $p \ge 1$.

Suppose that the following assumptions are valid for the coefficients in equation (1) and the initial functions:

(A1) $a(x,t)$ is differentiable with respect to x and t on $\bar{Q}$. And there exist constants A_0, A_1, A and C_a, for any $(x,t) \in \bar{Q}$, such that $0 < A_0 \le a(x,t) \le A_1, x^p a(x,t) \le A$, $|\,\partial a/\partial x\,| \le C_a$ and $|\,\partial a/\partial t\,| \le C_a$.

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(A2) $b(x,t)$ is a continuous function of $x \in [0,L]$. Furthermore, there are

$$| b(x,t) | \le B\sqrt{x^p}, \quad (x,t) \in \bar{Q},$$

$$| b(x,t_1) - b(x,t_2) | \le C_b\sqrt{x^p} \, | t_1 - t_2 |, \quad x \in [0,L], \quad t_1,t_2 \in [0,T],$$

where B and C_b are positive constants.

(A3) $f(x,t,u,w)$ is a continuous function of $(x,t,u,w) \in \bar{Q} \times R^2$. And for any $(x,t) \in \bar{Q}$ and $u, w \in R$ there are

$$\left|\frac{\partial}{\partial x} f(x,t,u,w)\right| \le C_f(1+ | u | + | w |), \quad \left|\frac{\partial}{\partial u} f(x,t,u,w)\right| \le C_f$$

where C_f is a positive constant.

(A4) $f(x,t,u,w)$ is semi-bounded for w, i.e., there exists a positive constant C_p such that

$$[f(x,t,u,w_1) - f(x,t,u,w_2)](w_1 - w_2) \le C_p \, | w_1 - w_2 |^2,$$

$$\forall u, w_1, w_2 \in R, (x,t) \in \bar{Q}.$$

(A5) For $x = L$, there is a positive constant C_L such that

$$| f(L,t_1,0,0) - f(L,t_2,0,0) | \le C_L \, | t_1 - t_2 |, \quad t_1, t_2 \in [0,T].$$

(A6) The initial functions satisfy the consistent condition, i.e., $u_0(L) = u_1(L) = 0$. In addition, $u_0(x)$ and $u_1(x)$ are Lipschitz continuously differentiable.

Under the assumptions (A1)–(A6), the existence and uniqueness of the generalized solution of the problem (1)–(5) have been proved by M.L. Krasnov[3].

Solve the problem (1)–(5) by means of the finite difference method. Divide the interval $[0,L]$ and $[0,T]$ into J and $N+1$ parts respectively. The space step is $h = L/J$ and the time step is $k = T/(N+1)$. Let $\omega_k = \{t^n = nk; n = 0,1,\cdots,N+1\}$ and $\omega_h = \{x_j = jh; j = 0,1,\cdots,J\}$. The set of all net points on the domain $\bar{Q}$ is denoted by $\omega_h \times \omega_k$. Let $V(x,t)$ be a discrete function defined on $\omega_h \times \omega_k$ and $V_j^n = V(x_j,t^n)$. Using the same symbols in [2], we denote $V_{\alpha,j}^n, V_{\bar{\alpha},j}^n$ and $V_{\mathring{\alpha},j}^n$ as V_j^n's forward, backward and centered difference quotients on variable $\alpha (= x$ or $t)$. The symbols $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively, are L_2 and L_∞ discretized norms with respect to x.

Now, let $V(x,t)$ be the difference approximate solution. We construct a two parameter family of explicit-implicit difference schemes

$$V_{t\bar{t},j}^n - x_j^p a_j^n \bar{V}_{x\bar{x},j}^n = b_j^n \bar{V}_{\mathring{x},j}^n + f_j^n, \quad 1 \le j \le J-1, 1 \le n \le N, \tag{6}$$

$$V_{t\bar{t},0}^n = f_0^n, \quad 1 \le n \le N, \tag{7}$$

$$V_J^n = 0, \quad 1 \le n \le N, \tag{8}$$

$$V_j^0 = u_0(x_j), \quad 0 \le j \le J, \tag{9}$$

$$V_{t,j}^1 = u_1(x_j), \quad 0 \le j \le J, \tag{10}$$

where

$$\bar{V}^n = rV^{n+1} + sV^n + (1-r-s)V^{n-1}, \tag{11}$$

$$f_j^n = f(x_j, t^n, V_j^n, V_{\hat{t},j}^n). \tag{12}$$

Here, r and s are parameters. Let $\lambda = s/2$ and $\sigma = r + s/2$, then

$$\bar{V}^n = -\lambda k^2 V_{\bar{t}t}^n + \sigma V^{n+1} + (1-\sigma)V^{n-1}. \tag{11'}$$

Hence, the scheme (6) can be rewritten as

$$\begin{aligned} V_{\bar{t}t,j}^n - x_j^p a_j^n\{-\lambda k^2 V_{\bar{x}x\bar{t}t,j}^n + \sigma V_{\bar{x}x,j}^{n+1} + (1-\sigma)V_{\bar{x}x,j}^{n-1}\} \\ = b_j^n\{-\lambda k^2 V_{\mathring{x}\bar{t}t,j}^n + \sigma V_{\mathring{x},j}^{n+1} + (1-\sigma)V_{\mathring{x},j}^{n-1}\} + f_j^n. \end{aligned} \tag{6'}$$

The scheme (6') is an implicit one if λ is not equal to σ. Otherwise, the second order accuracy of the scheme is preserved when $\sigma = 1/2$.

§2. Several Lemmas

For the discrete functions, we have the following interpolation formulae[1].

Lemma 1. *For any discrete function $\{y_j^n\}$ defined on the set $\omega_h \times \omega_k$, the following inequality holds:*

$$\|y^n\|_\infty^2 \le 2\|y^n\|\Big(\|y_{\bar{x}}^n\| + \frac{1}{L}\|y^n\|\Big),$$

or, in another form,

$$\|y^n\|_\infty^2 \le \Big(\frac{2}{L} + \frac{1}{\eta}\Big)\|y^n\|^2 + \eta\|y_{\bar{x}}^n\|^2$$

where η is an arbitrary positive constant. Besides, there is also

$$\|y^n\|^2 \le 2nk\sum_{m=1}^{n}\|y_{\bar{t}}^m\|^2 k + 2\|y^0\|^2.$$

Lemma 2. *Assume that the discrete function $\{y_j\}$ defined on ω_h satisfies the boundary condition $y_0 = 0$ or $y_J = 0$. Then we have*

$$\|y\|_\infty \le \sqrt{L}\|y_{\bar{x}}\|, \quad \|y\| \le L\|y_{\bar{x}}\|.$$

Lemma 3. *Let $\{y_j\}$ be a discrete function defined on ω_h. We have*

$$\|y_{\mathring{x}}\| \le \|y_{\bar{x}}\|.$$

Lemma 4. *By the scheme (6'), to solve the Cauchy problem of a linear constant coefficient wave equation $u_{tt} = a^2 u_{xx}$, a necessary condition for the stability of the scheme (6') is $\sigma \ge 1/2$.*

The proof of Lemma 4 is direct by the Fourier method. And according to this lemma, we can reasonably require that $\sigma \ge 1/2$ throughout this paper.

§3. Priori Estimates

In this section we are going to estimate the solution of the difference scheme (6′), (7)–(10). Multiplying both sides of (6′) by $-V^n_{\bar{x}xt,j}hk$, and summing up for j from 1 to $J-1$ and for n from 1 to $m(1 \le m \le N)$, we have

$$
\begin{aligned}
&-\sum_{n=1}^{m}\sum_{j=1}^{J-1} V^n_{\bar{t}t,j}V^n_{\bar{x}xt,j}hk - \lambda k^2 \sum_{n=1}^{m}\sum_{j=1}^{J-1} x_j^p a_j^n V^n_{\bar{x}x\bar{t}t,j}V^n_{\bar{x}xt,j}hk \\
&\quad + \sum_{n=1}^{m}\sum_{j=1}^{J-1} x_j^p a_j^n (\sigma V^{n+1}_{\bar{x}x,j} + (1-\sigma)V^{n-1}_{\bar{x}x,j})\, V^n_{\bar{x}xt,j}hk \\
&\quad = \lambda k^2 \sum_{n=1}^{m}\sum_{j=1}^{J-1} b_j^n V^n_{\dot{x}\bar{t}t,j}V^n_{\bar{x}xt,j}hk - \sum_{n=1}^{m}\sum_{j=1}^{J-1} b_j^n(\sigma V^{n+1}_{\dot{x},j} \\
&\quad + (1-\sigma)V^{n-1}_{\dot{x},j})V^n_{\bar{x}xt,j}hk - \sum_{n=1}^{m}\sum_{j=1}^{J-1} f_j^n V^n_{\bar{x}xt,j}hk. \qquad (13)
\end{aligned}
$$

We make the successive term estimation of (13) and get

$$
-\sum_{n=1}^{m}\sum_{j=1}^{J-1} V^n_{\bar{t}t,j}V^n_{\bar{x}xt,j}hk = \frac{1}{2}\|V^{m+1}_{\bar{x}\bar{t}}\|^2 - \frac{1}{2}\|V^1_{\bar{x}\bar{t}}\|^2 + \sum_{n=1}^{m} V^n_{\bar{t}t,0}V^n_{xt,0}k, \qquad (14)
$$

$$
\begin{aligned}
&-\lambda k^2 \sum_{n=1}^{m}\sum_{j=1}^{J-1} x_j^p a_j^n V^n_{\bar{x}x\bar{t}t,j}V^n_{\bar{x}xt,j}hk \ge -\frac{\lambda k^2}{2}\|\sqrt{a^m x^p}V^{m+1}_{\bar{x}xt}\|^2 \\
&\quad - |\lambda| A_1(\|\sqrt{x^p}V^1_{\bar{x}x}\|^2 + \|\sqrt{x^p}V^0_{\bar{x}x}\|^2) \\
&\quad - |\lambda| C_a \sum_{n=1}^{m}(\|\sqrt{x^p}V^n_{\bar{x}x}\|^2 + \|\sqrt{x^p}V^{n-1}_{\bar{x}x}\|^2)\, k, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=1}^{m}\sum_{j=1}^{J-1} x_j^p a_j^n (\sigma V^{n+1}_{\bar{x}x,j} + (1-\sigma)V^{n-1}_{\bar{x}x,j})\, V^n_{\bar{x}xt,j}hk = \frac{1}{4}\sum_{n=1}^{m}\sum_{j=1}^{J-1} x_j^p a_j^n [\,|V^{n+1}_{\bar{x}x,j}|^2 \\
&\quad - |V^{n-1}_{\bar{x}x,j}|^2 + (2\sigma-1)\,|V^{n+1}_{\bar{x}x,j} - V^{n-1}_{\bar{x}x,j}|^2]\, h \\
&\quad \ge \frac{1}{4}[\|\sqrt{a^m x^p}V^{m+1}_{\bar{x}x}\|^2 + \|\sqrt{a^m x^p}V^m_{\bar{x}x}\|^2\,] \\
&\quad - \frac{A_1}{4}[\|\sqrt{x^p}V^1_{\bar{x}x}\|^2 + \|\sqrt{x^p}V^0_{\bar{x}x}\|^2\,] - \frac{C_a}{2}\sum_{n=1}^{m}\|\sqrt{x^p}V^n_{\bar{x}x}\|^2\, k, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&\lambda k^2 \sum_{n=1}^{m}\sum_{j=1}^{J-1} b_j^n V^n_{\dot{x}\bar{t}t,j}V^n_{\bar{x}xt,j}hk \\
&\quad \le 2\,|\lambda| \sum_{n=1}^{m+1}\|V^n_{\bar{x}\bar{t}}\|^2 k + \frac{1}{4}\,|\lambda|\, B^2 \sum_{n=1}^{m+1}[\|\sqrt{x^p}V^n_{\bar{x}x}\|^2 + \|\sqrt{x^p}V^{n-1}_{\bar{x}x}\|^2\,]k, \qquad (17)
\end{aligned}
$$

$$
-\sum_{n=1}^{m}\sum_{j=1}^{J-1} b_j^n (\sigma V^{n+1}_{\dot{x},j} + (1-\sigma)V^{n-1}_{\dot{x},j})V^n_{\bar{x}xt,j}hk \le (\sigma + |1-\sigma|)\Big\{\Big(\frac{1}{8} + \frac{T^2}{2}
$$

$$+\frac{T}{4\eta_1}\Big)\sum_{n=1}^{m+1}\|V_{\bar{x}\bar{t}}^n\|^2k+\Big(B^2+\frac{C_b^2}{4}\Big)\sum_{n=1}^{m}[\|\sqrt{x^p}V_{\bar{x}x}^n\|^2+\|\sqrt{x^p}V_{\bar{x}x}^{n-1}\|^2]k$$
$$+B^2\eta_1[\|\sqrt{x^p}V_{\bar{x}x}^{m+1}\|^2+\|\sqrt{x^p}V_{\bar{x}x}^m\|^2]+R_1\Big\}, \tag{18}$$

and

$$-\sum_{n=1}^{m}\sum_{j=1}^{J-1}f_j^nV_{\bar{x}xt,j}^nhk\le\sum_{n=1}^{m}f_0^nV_{xt,0}^nk+\Big\{2+C_p+\frac{1}{2}C_f^2(3T^2L+T^2+3L)$$
$$+\Big(\frac{8}{L}+1\Big)T^2+\Big(\frac{8}{L}+\frac{1}{4\eta_2}\Big)2T\Big\}\sum_{n=1}^{m+1}\|V_{\bar{x}\bar{t}}^n\|^2k+\frac{1}{2}\Big(\frac{2}{L}\Big)^p\sum_{n=1}^{m}(\|\sqrt{x^p}V_{\bar{x}x}^n\|^2$$
$$+\|\sqrt{x^p}V_{\bar{x}x}^{n-1}\|^2)k+2\eta_2\Big(\frac{2}{L}\Big)^p[\|\sqrt{x^p}V_{\bar{x}x}^{m+1}\|^2+\|\sqrt{x^p}V_{\bar{x}x}^m\|^2]k, \tag{19}$$

where R_1 and R_2 are constants independent of h and k but η_1 and η_2 are arbitrary positive constants which will be given later.

Now, substituting (14)–(19) into (13) and noticing the boundary condition (7), we gain

$$\frac{1}{2}\|V_{\bar{x}\bar{t}}^{m+1}\|^2-\frac{\lambda k^2}{2}\|\sqrt{a^mx^p}V_{\bar{x}x\bar{t}}^{m+1}\|^2+\frac{1}{4}[\|\sqrt{a^mx^p}V_{\bar{x}x}^{m+1}\|^2+\|\sqrt{a^mx^p}V_{\bar{x}x}^m\|^2]$$
$$\le C_1\sum_{n=1}^{m+1}\|V_{\bar{x}\bar{t}}^n\|^2k+C_2\sum_{n=1}^{m+1}(\|\sqrt{x^p}V_{\bar{x}x}^n\|^2+\|\sqrt{x^p}V_{\bar{x}x}^{n-1}\|^2)k$$
$$+\{(\sigma+|1-\sigma|)B^2\eta_1+2(2/L)^p\eta_2\}(\|\sqrt{x^p}V_{\bar{x}x}^{m+1}\|^2+\|\sqrt{x^p}V_{\bar{x}x}^m\|^2)+C_3. \tag{20}$$

Here, C_1, C_2, C_3 and the following C_i $(i=4,5,\cdots)$ are constants independent of h and k.

Utilizing the following inequalities

$$k^2\|\sqrt{a^mx^p}V_{\bar{x}x\bar{t}}^{m+1}\|^2\le 2(\|\sqrt{a^mx^p}V_{\bar{x}x}^{m+1}\|^2+\|\sqrt{a^mx^p}V_{\bar{x}x}^m\|^2),$$
$$k^2\|\sqrt{a^mx^p}V_{\bar{x}x\bar{t}}^{m+1}\|^2\le\frac{4k^2}{h^2}A\|V_{\bar{x}\bar{t}}^{m+1}\|^2$$

and supposing $\varepsilon\le 1/4$ is a positive constant which will be given below, we know that the left-hand side of (20) is not smaller than

$$\frac{\varepsilon}{2}\|V_{\bar{x}\bar{t}}^{m+1}\|^2+A_0\varepsilon(\|\sqrt{x^p}V_{\bar{x}x}^{m+1}\|^2+\|\sqrt{x^p}V_{\bar{x}x}^m\|^2)$$
$$+\frac{h^2}{8A}\Big[1-\varepsilon-4\Big(\lambda+\varepsilon-\frac{1}{4}\Big)A\frac{k^2}{h^2}\Big]\|\sqrt{a^mx^p}V_{\bar{x}x\bar{t}}^{m+1}\|^2.$$

Taking $\eta_1=\dfrac{A_0\varepsilon}{4B^2(\sigma+|1-\sigma|)}$ and $\eta_2=\dfrac{A_0\varepsilon}{8(2/L)^p}$, from (20) we get

$$\varepsilon\|V_{\bar{x}\bar{t}}^{m+1}\|^2+\varepsilon A_0(\|\sqrt{x^p}V_{\bar{x}x}^{m+1}\|^2+\|\sqrt{x^p}V_{\bar{x}x}^m\|^2)$$
$$+\frac{h^2}{4A}\Big[1-\varepsilon+4\Big(\frac{1}{4}-\lambda-\varepsilon\Big)A\frac{k^2}{h^2}\Big]\|\sqrt{a^mx^p}V_{\bar{x}x\bar{t}}^{m+1}\|^2$$
$$\le C_4\sum_{n=1}^{m+1}\|V_{\bar{x}\bar{t}}^n\|^2k+C_5\sum_{n=1}^{m+1}(\|\sqrt{x^p}V_{\bar{x}x}^n\|^2+\|\sqrt{x^p}V_{\bar{x}x}^{n-1}\|^2)k+C_6. \tag{21}$$

If $\lambda < 1/4$, we take $\varepsilon = \min(1/4, 1/4 - \lambda)$. Then the third term on the left-hand side of (21) is not smaller than zero. So we obtain

$$\|V_{\bar{x}\bar{t}}^{m+1}\|^2 + A_0(\|\sqrt{x^p}V_{\bar{x}x}^{m+1}\|^2 + \|\sqrt{x^p}V_{\bar{x}x}^{m}\|^2) \le C_7 \sum_{n=1}^{m+1} \|V_{\bar{x}\bar{t}}^{n}\|^2 k$$
$$+ C_8 \sum_{n=1}^{m+1} (\|\sqrt{x^p}V_{\bar{x}x}^{n}\|^2 + \|\sqrt{x^p}V_{\bar{x}x}^{n-1}\|^2)k + C_9. \tag{22}$$

From (22), according to Gronwall's lemma, we obtain an estimate for the difference solution as follows:

$$\max_{1\le m\le N+1} \{ \|V_{\bar{x}\bar{t}}^{m}\|^2 + \|\sqrt{x^p}V_{\bar{x}x}^{m}\|^2 \} \le K_0, \tag{23}$$

where K_0 is a constant independent of h and k.

If $\lambda \ge 1/4$, we take $\varepsilon < 1/4$ and make the steplength k and the mesh length h satisfy

$$A\frac{k^2}{h^2} \le \frac{1-\varepsilon}{4(\lambda+\varepsilon-\frac{1}{4})}. \tag{24}$$

Thus, the third term on the left-hand side of (21) is not smaller than zero and we obtain (22) and conclude (24) again.

According to the above conclusions, we have the following lemma which is similar to the one in [2].

Lemma 5. *Let $\sigma \ge 1/2$. Under assumptions* (A1)–(A5), *suppose $\lambda < 1/4$, or $\lambda \ge 1/4$ with k and h satisfying condition* (24). *Then, the solution $\{V_j^n\}$ of the difference scheme* (6′), (7)–(10) *satisfies the following estimates when k is sufficiently small*

$$\|V_{\bar{x}\bar{t}}^{m+1}\| \le K_1, \qquad \|V_{\bar{x}}^{m+1}\| \le K_2, \qquad \|V^{m+1}\| \le K_3,$$
$$\|V^{m+1}\|_\infty \le K_4, \qquad \|V_t^{m+1}\| \le K_5, \qquad \|V_{\bar{t}}^{m+1}\|_\infty \le K_6,$$
$$\|\sqrt{x^p}V_{\bar{x}x}^{m+1}\| \le K_7, \qquad \|x^p a^{m+1}V_{\bar{x}x}^{m+1}\| \le K_8, \qquad \|V_{\bar{t}t}^{m}\| \le K_9,$$

where $m = 1, 2, \cdots, N$ and $K_i \quad (i = 1, 2, \cdots, 9)$ are the constants all independent of h and k.

§4. The Existence and the Uniqueness of the Difference Solution

In this section we are going to discuss the existence and the uniqueness of the solution of the difference scheme (6)–(10).

Having known the values of $\{V_j^n\}$ and $\{V_j^{n-1}\}$, we now solve $\{V_j^{n+1}\}$. In order to demonstrate the existence of $\{V_j^{n+1}\}$, we make a continuous mapping T_α such that

$$T_\alpha : \quad R^{J+1} \to R^{J+1} \quad , \qquad 0 \le \alpha \le 1.$$

$\forall z \in R^{J+1}$, let $v^{n+1} = T_\alpha z$ which is represented as

$$V_j^{n+1} - 2V_j^n + V_j^{n-1} = \alpha k^2 \Big\{ x_j^p a_j^n [-\lambda(z_{\bar{x}x,j} - 2V_{\bar{x}x,j}^n + V_{\bar{x}x,j}^{n-1}) + \sigma z_{\bar{x}x,j} + (1-\sigma)V_{\bar{x}x,j}^{n-1}]$$
$$+ b_j^n[-\lambda(z_{\dot{x},j} - 2V_{\dot{x},j}^n + V_{\dot{x},j}^{n-1}) + \sigma z_{\dot{x},j} + (1-\sigma)V_{\dot{x},j}^{n-1}]$$
$$+ f\Big(x_j, t^n, V_j^n, \frac{(z_j - V_j^{n-1})}{2k}\Big) \Big\}, \quad j = 1, 2, \cdots, J-1 \tag{25}$$

and

$$V_0^{n+1} - 2V_0^n + V_0^{n-1} = \alpha k^2 f\Big(x_0, t^n, V_0^n, \frac{z_0 - V_0^{n-1}}{2k}\Big), \tag{26}$$
$$V_J^{n+1} = 0. \tag{27}$$

Especially, when $\alpha = 0$, $\forall z \in R^{J+1}$ there are

$$V_j^{n+1} = 2V_j^n - V_j^{n-1}, \qquad j = 0, 1, \cdots, J-1$$

and $V_J^{n+1} = 0$.

Let V_j^{n+1} replace z_j in (25)–(27), and denote the new equations as (25′), (26′), (27′). According to Lemma 5, it is clear that, under the conditions of Lemma 5, all of the possible solutions $\{V_j^{n+1}\}$ of (25′)–(27′) are uniformly bounded for $\alpha \in [0,1]$. Using the fixed point principle, we immediately have the following lemma:

Lemma 6. *Under the conditions of Lemma 5, the difference scheme* (6′), (7)–(10) *has at least one solution.*

For the uniqueness, we have

Lemma 7. *Let* $\sigma \geq 1/2$. *Under assumptions* (A1)–(A5), *suppose* $\sigma \geq \lambda$, *or* $\sigma < \lambda$ *with* k *and* h *satisfying condition* (24). *Then, the solution of the difference scheme* (6′), (7)–(10) *is unique for sufficiently small* k.

Proof. Suppose $\{V_j^n\}$ and $\{V_j^{n-1}\}$ are known. Assume that there are two solutions $\{V_j^{n+1}\}$ and $\{\tilde{V}_j^{n+1}\}$. Let $W_j = V_j^{n+1} - \tilde{V}_j^{n+1}$ for $j = 0, 1, \cdots, J$. Then W_j must satisfy

$$W_j - (\sigma - \lambda)x_j^p a_j^n W_{\bar{x}x,j} k^2 = k^2(\sigma - \lambda) b_j^n W_{\dot{x},j} + k^2 \Big[f\Big(x_j, t^n, V_j^n, \frac{V_j^{n+1} - V_j^{n-1}}{2k}\Big)$$
$$- f\Big(x_j, t^n, V_j^n, \frac{\tilde{V}_j^{n+1} - V_j^{n-1}}{2k}\Big)\Big], \quad 0 < j < J, \tag{28}$$
$$W_0 = k^2 \Big[f\Big(0, t^n, V_0^n, \frac{V_0^{n+1} - V_0^{n-1}}{2k}\Big) - f\Big(0, t^n, V_0^n, \frac{\tilde{V}_0^{n+1} - V_0^{n-1}}{2k}\Big)\Big], \tag{29}$$
$$W_J = 0. \tag{30}$$

Under (A4), (29) multiplied by W_0 is rewritten as

$$W_0^2 \leq \frac{C_p}{2} k W_0^2.$$

Therefore $W_0 = 0$ when $k < 2/C_p$.

(i) If $\sigma \geq \lambda$, multiplying (28) by $W_j h/(x_j^p a_j^n)$, and then summing up for j from 1 to $J-1$, we get

$$\frac{2-kC_p}{2A_1}\sum_{j=1}^{J-1} x_j^{-p} \mid W_j \mid^2 h + (\sigma-\lambda)k^2 \sum_{j=1}^{J} \mid W_{\bar{x},j} \mid^2 h$$

$$\leq (\sigma-\lambda)k^2[\varepsilon_1 \sum_{j=1}^{J} \mid W_{\bar{x},j} \mid^2 h + \frac{B^2}{4\varepsilon_1 A_0^2}\sum_{j=1}^{J-1} x_j^{-p} \mid W_j \mid^2 h],$$

where ε_1 is an arbitrary positive constant. Taking $\varepsilon_1 = 1/8$, we have

$$\Big[1 - \frac{C_p k}{2} - 2(\sigma-\lambda)k^2\frac{A_1 B^2}{A_0^2}\Big]\sum_{j=1}^{J-1} x_j^{-p} \mid W_j \mid^2 h + \frac{7}{8}(\sigma-\lambda)A_1\|W_{\bar{x}}\|^2 \leq 0.$$

Hence, if k is sufficiently small, W_j vanishes for any $j = 1, 2, \cdots, J-1$.

(ii) If $\sigma < \lambda$ and k/h satisfies (24), multiplying (28) by $W_j h$ and summing up for j from 1 to $J-1$, we get

$$\Big(1 - \frac{k}{2}C_p\Big)\|W\|^2 - (\sigma-\lambda)k^2\sum_{j=1}^{J-1} x_j^p a_j^n W_{xx,j} W_j h \leq k \mid \sigma - \lambda \mid B\sqrt{L^p}\frac{k}{h}\|W\|^2. \tag{31}$$

From condition (24) and $\lambda > \sigma \geq 1/2$, it can be concluded that

$$A\frac{k^2}{h^2} \leq \frac{1-\varepsilon}{4(\lambda+\varepsilon-\frac{1}{4})} \leq \frac{2\lambda-\sigma}{8\lambda(\lambda-\sigma)}.$$

Therefore, the second term on the left-hand side of (31) is more than or equal to

$$-4A(\lambda-\sigma)\frac{k^2}{h^2}\|W\|^2 \geq \Big(-1+\frac{\sigma}{2\lambda}\Big)\|W\|^2.$$

Finally, from (31) and the above, we have

$$\Big[\frac{\sigma}{2\lambda} - \frac{C_p}{2}k - \frac{1}{2}B\,k\sqrt{\lambda L^p/A}\Big]\|W\|^2 \leq 0.$$

So, $\|W\|$ becomes zero if k is sufficiently small.

§5. Convergence and Stability

Now we are able to establish the main theorems about the convergence and stability for the difference scheme (6'), (7)–(10).

In order to show that the difference solution is convergent to the unique generalized solution of problem (1)–(5), some corresponding prior estimates have to be obtained. And we have them in Lemma 5. Using these estimates, we can discuss the state of limit sequences of the difference solution and its difference quotients. The proof of the following theorem is similar to the proof in [2].

Theorem 1. *Suppose $\sigma \geq 1/2$. Under assumptions* (A1)–(A6), *assume $\lambda < 1/4$, or $\lambda \geq 1/4$ with k and h satisfying condition* (24). *Then, as k and h tend to zero, the* *solution of the difference scheme* (6′), (7)–(10) *converges to a limiting function* $u(x,t)$ *in the space* $\hat{W}_2^{(2)}(Q)$, *where* $u(x,t)$ *is the unique generalized solution of equation* (1) *and satisfies initial-boundary conditions* (2)–(5) *in the common sense.*

Here the space $\hat{W}_2^{(2)}(Q)$ can be expressed as

$$\hat{W}_2^{(2)}(Q) = W_\infty^2(0,T;\ L_2(0,L)) \cap W_\infty^1(0,T;\ H^1(0,L)) \cap L_\infty(0,T;\ \hat{H}^2(0,L)),$$

where W_∞^2, W_∞^1, L_2, H^1 and L_∞ are common notations of Sobolev function spaces and

$$\hat{H}^2(0,L) = \left\{ f(x) \mid f \in H^1(0,L) \text{ and } \int_0^L x^p \mid f'' \mid^2 dx < \infty \right\}.$$

We now examine the stability of the difference scheme with respect to the initial values. Assume $\tilde{u}_0(x)$ and $\tilde{u}_1(x)$ are new initial values which also satisfy assumption (A6). The corresponding difference solution is $\{\tilde{V}_j^n\}$. Let $Z_j^n = V_j^n - \tilde{V}_j^n$ for $j = 0,1,\cdots,J$ and $n = 0,1,\cdots,N+1$. Then $\{Z_j^n\}$ satisfies the difference equations

$$\begin{aligned} Z_{\bar{t}t,j}^n - x_j^p a_j^n\{-\lambda k^2 Z_{\bar{x}x\bar{t}t,j}^n + [\sigma Z_{\bar{x}x,j}^{n+1} + (1-\sigma) Z_{\bar{x}x,j}^{n-1}]\} = b_j^n\{-\lambda k^2 Z_{\bar{x}\bar{t}t,j}^n \\ + [\sigma Z_{\bar{x},j}^{n+1} + (1-\sigma) Z_{\bar{x},j}^{n-1}]\} + f(x_j,t^n,V_j^n,V_{t,j}^n) - f(x_j,t^n,\tilde{V}_j^n,\tilde{V}_{t,j}^n), \\ j = 1,2,\cdots,J-1; \quad n = 1,2,\cdots,N, \end{aligned} \tag{32}$$

$$Z_{\bar{t}t,0}^n = f(0,t^n,V_0^n,V_{t,0}^n) - f(0,t^n,\tilde{V}_0^n,\tilde{V}_{t,0}^n), \quad n = 1,2,\cdots,N, \tag{33}$$

$$Z_J^n = 0, \quad n = 1,2,\cdots,N+1, \tag{34}$$

$$Z_j^0 = u_0(x_j) - \tilde{u}_0(x_j), \quad j = 0,1,\cdots,J, \tag{35}$$

$$Z_{\bar{t},j}^1 = u_1(x_j) - \tilde{u}_1(x_j), \quad j = 0,1,\cdots,J. \tag{36}$$

Under (A3) and (A4), we have

$$\begin{aligned} &[f(x_j,t^n,V_j^n,V_{t,j}^n) - f(x_j,t^n,\tilde{V}_j^n,\tilde{V}_{t,j}^n)] Z_{t,j}^n \\ &= Z_j^n Z_{t,j}^n \int_0^1 \frac{\partial f}{\partial u}(x_j,t^n,qV_j^n + (1-q)\tilde{V}_j^n)\, V_{t,j}^n) dq + \mid Z_{t,j}^n \mid^2 [f(x_j,t^n,\tilde{V}_j^n,V_{t,j}^n) \\ &- f(x_j,t^n,\tilde{V}_j^n,\tilde{V}_{t,j}^n)]/(V_{t,j}^n - \tilde{V}_{t,j}^n) \le C_f \mid Z_j^n Z_{t,j}^n \mid + C_p \mid Z_{t,j}^n \mid^2 . \end{aligned} \tag{37}$$

Multiplying (33) by $Z_{t,0}^n k$ and summing them up for n from 1 to m $(1 \le m \le N)$, we can get

$$\mid Z_{\bar{t},0}^{m+1} \mid^2 - \mid Z_{\bar{t},0}^1 \mid^2 \le (C_p + \frac{1}{2}C_f + C_f T) \sum_{n=1}^{m+1} \mid Z_{\bar{t},0}^n \mid^2 k + \frac{1}{2} C_f T \mid Z_0^0 \mid^2 .$$

Then, by induction, when k is small enough there holds

$$\mid Z_{\bar{t},0}^{m+1} \mid^2 \le C_{10} \mid Z_0^0 \mid^2 + C_{11} \mid Z_{\bar{t},0}^1 \mid^2, \quad m = 0,1,\cdots,N. \tag{38}$$

Multiply (32) by $Z_{t,j}^n hk$ and sum them up for $j = 1,2,\cdots,J-1$ and for $n = 1,2,\cdots,m$. Note (34), (36) and

$$\mid (a_j^n x_j^p)_{\bar{x}} \mid \le \bar{C}_a x_j^{p-1}$$

where $\bar{C}_a$ is a constant. As in Section 3, we can get the following estimate under the conditions of Lemma 5:

$$\|Z_{\bar{t}}^{m+1}\|^2 + \|\sqrt{x^p}Z_{\bar{x}}^{m+1}\|^2 \le C_{12}[\sum_{n=1}^{m+1} \|Z_{\bar{t}}^n\|^2 k + \sum_{n=0}^{m+1}(\|\sqrt{x^p}Z_{\bar{x}}^n\|^2 + \|x^{p-1}Z_{\bar{x}}^n\|^2)k]$$
$$+ C_{13}[\|u_0 - \tilde{u}_0\|^2 + \|\sqrt{x^p}(u_0 - \tilde{u}_0)_{\bar{x}}\|^2 + \|u_1 - \tilde{u}_1\|^2 + k^2\|\sqrt{x^p}(u_1 - \tilde{u}_1)_{\bar{x}}\|^2]. \quad (39)$$

If $p \ge 2$, the term $\|x^{p-1}Z_x^{n+1}\|^2$ in (39) can be estimated as

$$\|x^{p-1}Z_x^{n+1}\|^2 \le L^{p-2}\|\sqrt{x^p}Z_{\bar{x}}^{n+1}\|^2.$$

Consequently

$$\|Z_{\bar{t}}^{m+1}\|^2 + \|\sqrt{x^p}Z_x^{m+1}\|^2 \le C_{14}[\|u_0 - \tilde{u}_0\|^2 + \|\sqrt{x^p}(u_0 - \tilde{u}_0)_{\bar{x}}\|^2$$
$$+ \|u_1 - \tilde{u}_1\|^2 + k^2\|\sqrt{x^p}(u_1 - \tilde{u}_1)_x\|^2], \quad (40)$$

when k is sufficiently small. Using Lemma 1, we immediately get

$$\max_{1\le m\le N+1} \|Z^m\|^2 \le C_{15}[\|u_0 - \tilde{u}_0\|^2 + \|\sqrt{x^p}(u_0 - \tilde{u}_0)_x\|^2 + \|u_1 - \tilde{u}_1\|^2$$
$$+ k^2\|\sqrt{x^p}(u_1 - \tilde{u}_1)_x\|^2]. \quad (41)$$

So we have

Theorem 2. *Under the conditions of Lemma 5, if $p \ge 2$, then the difference scheme (6′), (7)-(10) is stable for initial values in the sense of (40) and (41).*

This theorem has answered the stability problem for $p \ge 2$ but not for $p < 2$. We now discuss the stability of the difference scheme for $p \in (0, 2)$ under some proper conditions.

If the initial disturbance satisfies

$$u_0(0) = \tilde{u}_0(0), \qquad u_1(0) = \tilde{u}_1(0)$$

as well as assumption (A6), from (35)-(36), we get

$$| Z_0^0 | = | Z_{t,0}^1 | = 0$$

and from (38) we get

$$| Z_{\bar{t},0}^{m+1} | = 0, \quad m = 0, 1, \cdots, N. \quad (38')$$

Multiplying (32) by $x_j^{-p} Z_{t,j}^n hk / a_j^n$ and summing up for j from 1 to $J - 1$ and for n from 1 to m respectively, we obtain

$$\sum_{n=1}^{m}\sum_{j=1}^{J-1}\frac{x_j^{-p}}{a_j^n}(| Z_{\bar{t},j}^{n+1} |^2 - | Z_{\bar{t},j}^n |^2)h - \sum_{n=1}^{m}\sum_{j=1}^{J-1}[-\lambda k^2 Z_{xxtt,j}^n + \sigma Z_{\bar{x}x,j}^{n+1} + (1-\sigma)Z_{\bar{x}x,j}^{n-1}]Z_{t,j}^n hk$$
$$\le \sum_{n=1}^{m}\sum_{j=1}^{J-1} | -\lambda k^2 Z_{\bar{x}tt,j}^n + \sigma Z_{\bar{x},j}^{n+1} + (1-\sigma)Z_{\bar{x},j}^{n-1} | \cdot | \sqrt{x_j^p} Z_{t,j}^n | hk. \quad (42)$$

We analyze (42) term by term and finally get the inequality

$$\max_{0\le m\le N}\Big\{\sum_{j=1}^{J-1} x_j^{-p} \mid Z_{\bar{t},j}^{m+1} \mid^2 h + \|Z_{\bar{x}}^{m+1}\|^2\Big\} \le C_{16}\{\sum_{j=1}^{J-1} x_j^{-p}[\mid (u_0 - \tilde{u}_0)_j \mid^2 + \mid (u_1 - \tilde{u}_1)_j \mid^2]h + \|(u_0 - \tilde{u}_0)_{\bar{x}}\|^2 + k^2\|(u_1 - \tilde{u}_1)_{\bar{x}}\|^2\}. \tag{43}$$

And we have

Theorem 3. *Under the conditions of Lemma 5 and $p \in (0,2)$, if the initial disturbance satisfies an additional restriction $u_i(0) = \tilde{u}_i(0)$ for $i = 0,1$, then the difference scheme (6′), (7)–(10) is stable for initial values in the sense of (43). In this case, the right-hand side of (43) makes sense.*

Besides, if a restriction to function f is added as follows:

(A7) $$f(x,t,u,w) \in C^2(\bar{Q} \times R^2)$$

the following theorem can hold.

Theorem 4. *Assume condition (A7) holds. Under the conditions of Lemma 5, the difference scheme (6′), (7)–(10) is stable for initial values in the norm of the space $\hat{W}_2^{(2)}(Q)$.*

Proof. Because (A3),(A4) and (A7) hold, we have

$$\begin{aligned}
&\sum_{j=1}^{J-1}[f(x_j,t^n,V_j^n,V_{t,j}^n) - f(x_j,t^n,\tilde{V}_j^n,\tilde{V}_{t,j}^n)]Z_{\bar{x}xt,j}^n h\\
&= -\sum_{j=1}^{J-1}\Big\{\int_0^1 \frac{\partial f}{\partial u}(x_j,t^n,qV_j^n + (1-q)\tilde{V}_j^n, V_{t,j}^n)dq \cdot Z_j^n\\
&+ \int_0^1 \frac{\partial f}{\partial w}(x_j,t^n,\tilde{V}_j^n, qV_{t,j}^n + (1-q)\tilde{V}_{t,j}^n)dq \cdot Z_{t,j}^n\Big\}_{\bar{x}} Z_{\bar{x}t,j}^n h\\
&- [f(0,t^n,V_0^n,V_{t,0}^n) - f(0,t^n,\tilde{V}_0^n,\tilde{V}_{t,0}^n)]Z_{\bar{x}t,1}^n.
\end{aligned}$$

Observing that $\|V^n\|_\infty, \|\tilde{V}^n\|_\infty, \|V_t^n\|_\infty$ and $\|\tilde{V}_{\bar{t}}^n\|_\infty$ are all uniformly bounded, we get

$$\begin{aligned}
&-\sum_{j=1}^{J-1}[f(x_j,t^n,V_j^n,V_{t,j}^n) - f(x_j,t^n,\tilde{V}_j^n,\tilde{V}_{t,j}^n)]Z_{\bar{x}xt,j}^n h \le C_{17}\sum_{j=1}^{J}(\mid Z_j^n \mid\\
&+ \mid Z_{t,j}^n \mid + \mid Z_{\bar{x},j}^n \mid \cdot \mid Z_{\bar{x}t,j}^n \mid)\cdot \mid Z_{\bar{x}t,j}^n \mid h + [f(0,t^n,V_0^n,V_{t,0}^n)\\
&- f(0,t^n,\tilde{V}_0^n,\tilde{V}_{t,0}^n)]Z_{xt,1}^n.
\end{aligned}$$

The following estimate can be obtained in the same way as in Section 3:

$$\max_{1\le m\le N+1}[\|Z_{\bar{x}t}^m\|^2 + \|\sqrt{x^p}Z_{\bar{x}x}^m\|^2] \le C_{18}(\|u_0 - \tilde{u}_0\|_{\hat{H}^2(0,L)}^2 + \|u_1 - \tilde{u}_1\|_{H^1(0,L)}^2). \tag{44}$$

This complete the proof of Theorem 4.

Remark. If we take $p \in (0,1)$ in equation (1), the boundary condition at $x = 0$ may be given as

$$u(0,t) = 0. \tag{4'}$$

Likewise, difference equation (7) will be replaced by

$$V_0^n = 0, \quad n = 0, 1, \cdots, N+1. \tag{7'}$$

We assume

(A5') At $x = 0$ there is

$$| f(0, t_1, 0, 0) - f(0, t_2, 0, 0) | \le \bar{C}_0 | t_1 - t_2 |, \quad \forall t_1, t_2 \in [0, T]$$

where $\bar{C}_0$ is a positive constant.

In addition we add a consistent condition $u_0(0) = u_1(0) = 0$ at $x = 0$ to assumption (A6) and denote (A6) as (A6'). Then, for $p \in (0,1)$, the lemmas and theorems described above except Theorem 2 may be derived similarly.

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