Journal of Computational Mathematics, Vol.11, No.2, 1993, 188–192.

ON THE GENERAL INTERPOLATION FORMULAS FOR DISCRETE FUNCTIONAL SPACES (I) *1)

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Abstract

The general interpolation formulas for discrete functional spaces of discrete functions with one index are presented in the note. These are the relationship among the discrete norms in the forms of the summation of powers, the maximum modulo, the Hölder and Lipschitz coefficients for the discrete functions.

§1

The imbedding theorems and the interpolation formulas for the functions of Sobolev's spaces are very important and useful in the linear and nonlinear theory of partial differential equations and systems[1–2]. Similarly the extensions of such interpolation formulas for the spaces of discrete functions play the extremely important part in the study of the finite difference approximations to the problems of linear and nonlinear partial differential equations and systems. In [3–5] some special interpolation formulas for the spaces of discrete functions defined on the finite and infinite segments and domains are established. These interpolation formulas for discrete functional spaces are used in the study of convergence and stability behavior for the finite difference schemes and in the construction of weak, generalized and classical solutions for the various problems of linear and nonlinear partial differential equations and systems.

In the present note, some general interpolation formulas for the discrete functional spaces of discrete functions with one index are presented. These general interpolation formulas give the connected relations among the discrete norms as the types of the summation of powers, the maximum modulo, the Lipschitz and Hölder quotients for different discrete functional spaces.

§2

Let us divide the finite interval $[0,l]$ into the small grid points $\{x_j = jh \mid j = 0,1,\cdots,J\}$, where $Jh = l < \infty, J$ is an integer and h is the stepsize. The discrete

* Received August 27, 1991.
1) The Project Supported by National Natural Science Foundation of China.

function $u_h = \{u_j | \ j = 0, 1, \cdots, J\}$ is defined on the grid points $\{x_j | \ j = 0, 1, \cdots, J\}$. Let us denote $\Delta_+ u_j = u_{j+1} - u_j$ or simply $\Delta u_j = u_{j+1} - u_j$ $(j = 0, 1, \cdots, J-1)$ and $\Delta_- u_j = u_j - u_{j-1} (j = 1, 2, \cdots, J)$. For the discrete functions $u_h = \{u_j | \ j = 0, 1, \cdots, J\}$ and its difference quotients $\delta^k u_h = \left\{ \frac{\Delta^k u_j}{h^k} \middle| j = 0, 1, \cdots, J-k \right\}$ of order $k = 0, 1, \cdots$, the discrete norms can be defined and denoted by the notations as follows: For $1 \le p < \infty$ we have

$$\|\delta^k u_h\|_p = \Big(\sum_{j=0}^{J-k} \Big| \frac{\Delta^k u_j}{h^k} \Big|^p h \Big)^{1/p} \tag{1}$$

and for $p = \infty$, we have

$$\|\delta^k u_h\|_\infty = \max_{j=0,1,\cdots,J-k} \Big| \frac{\Delta^k u_j}{h^k} \Big|, \tag{2}$$

where $k = 0, 1, \cdots$. For the norms of discrete functions u_h with negative index $p < 0$, we can define the norms in the following way. Let $s = \left[\frac{1}{|p|} \right]$ and $\lambda = \left\{ \frac{1}{|p|} \right\}$ be the integer and decimal parts of the positive value $\frac{1}{|p|}$ respectively, where s is an integer and $0 \le \lambda \le 1$ is a decimal number. For the case $\lambda = 0$, we define the norm of discrete functions with negative index as

$$\|\delta^k u_h\|_p = \max_{j=0,1,\cdots,J-(k+s)} \Big| \frac{\Delta^{k+s} u_j}{h^{k+s}} \Big| = \|\delta^{k+s} u_h\|_\infty, \tag{3}$$

where $k = 0, 1, \cdots$ and $s = \left[\frac{1}{|p|} \right] = \frac{1}{|p|}$. For the case $0 < \lambda \le 1$, we define the norm as

$$\|\delta^k u_h\|_p = \max_{\substack{r > m \\ r, m = 0, 1, \cdots, J-(k+s)}} \frac{\Big| \frac{\Delta^{k+s} u_r}{h^{k+s}} - \frac{\Delta^{k+s} u_m}{h^{k+s}} \Big|}{[(r-m)h]^\lambda}, \tag{4}$$

where $s = \left[\frac{1}{|p|} \right]$, $0 < \lambda = \left\{ \frac{1}{|p|} \right\} \le 1$ and $k = 0, 1, \cdots$.

For the case of definition (1), the norm of discrete function u_h is in the form of summation of powers. For the case of definitions (2) and (3), these are the discrete norms of maximum modulo type. For the case of definition (4), it is the discrete norm of type of Hölder coefficient. When $\lambda = 1$, it is of the type of Lipschitz coefficient.

The general interpolation formulas for the discrete functional spaces of discrete function with one index for the various types of norms can be stated in the following theorem.

Theorem 1. *For any discrete function $u_k = \{u_j | \ j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j = jh | \ j = 0, 1, \cdots, J\}$ of interval $[0, l]$ of finite length $l < \infty$ with positive and negative indices $-(n - k - 1/r) \le 1/p \le 1$ and the constants $1 \le q, r \le \infty$ and $0 \le k < n$, there is the relations among the norms as*

$$\|\delta^k u_h\|_p \le C(\|u_h\|_q^{1-\alpha} \|\delta^n u_h\|_r^\alpha + l^{1/p - 1/q - k} \|u_h\|_q) \tag{5}$$

with

$$\frac{1}{p} - k = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - n\Big) \tag{6}$$

and

$$0 \le \alpha \le 1, \tag{7}$$

where C is a constant independent of the constants p, q, r, the finite length $l < \infty$ of the interval, the mesh step $h > 0$ and the discrete function u_h.

The general interpolation formulas (5) is the expected extension of the interpolation formulas for the functions of Sobolev's spaces of discrete functions $u_h = \{u_j| \ j = 0, 1, \cdots, J\}$ with one index.

The similar so-called Sobolev's inequality for the discrete functional spaces of discrete functions can be stated as follows.

Theorem 2. *For any discrete function $u_h = \{u_j| \ j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j = jh| \ j = 0, 1, \cdots, J\}$ in the interval $[0, l]$ of finite length $l < \infty$ and the indices $1 \le q, r \le \infty$, $-(n - k - 1/r) \le 1/p \le 1$ and $0 \le k < n$, then for any positive small constant $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that*

$$\|\delta^k u_h\|_p \le \varepsilon \|\delta^n u_h\|_r + C(\varepsilon)\|u_h\|_q, \tag{8}$$

where

$$0 < \frac{k - 1/p + 1/q}{n - 1/r + 1/q} < 1 \tag{9}$$

and $C(\varepsilon)$ is canstant dependent on $\varepsilon > 0$ and independent of the parameters p, q, r, the finite length $l < \infty$ of the interval, the mesh step $h > 0$ and the discrete function u_h.

§3

For the norms of discrete functions corresponding to the maximum modulo and Hölder or Lipschitz coefficients of the differentiable functions, we adopt the notations as follows:

$$U_h^k = \max_{j=0,1,\cdots,J-k} \left|\frac{\Delta^k u_j}{h^k}\right| \tag{10}$$

and

$$U_h^{k,\lambda} = \max_{\substack{r>m \\ r,m=0,1,\cdots,J-k}} \frac{\left|\dfrac{\Delta^k u_r}{h^k} - \dfrac{\Delta^k u_m}{h^k}\right|}{[(r-m)h]^\lambda}, \tag{11}$$

where $k = 0, 1, \cdots$ and $0 \le \lambda \le 1$. Hence for norms of the discrete functions u_h with negative index $p < 0$, we have

$$\|u_h\|_p = U_h^{s,\lambda}, \tag{12}$$

where $s = \left[\frac{1}{|p|}\right], \lambda = \left\{\frac{1}{|p|}\right\}$ and thus $1/p = -(s + \lambda)$.

As a consequence of Theorem 1, we have the following interpolation relations among the maximum modulo and the Hölder coefficient for the discrete functions.

Theorem 3. *For any discrete function* $u_h = \{u_j|\ j = 0,1,\cdots,J\}$ *defined on the grid points* $\{x_j = jh|\ j = 0,1,\cdots,J\}$ *of the interval* $[0,l]$ *of finite length* $l = Jh < \infty$, *there are*

$$U_h^k \le C[(U_h^0)^{1-k/n}(U_h^n)^{k/n} + l^{-k}U_h^0] \tag{13}$$

and

$$U_h^{k,\lambda} \le C[(U_h^0)^{1-(k+\lambda)/n}(U_h^n)^{(k+\lambda)/n} + l^{-(k+\lambda)}U_h^0] \tag{14}$$

where $0 \le k < n, 0 \le \lambda < 1$ *and* C *is a constant independent of the finite length* $l < \infty$, *the mesh step* $h > 0$ *and the discrete function* u_k.

§4

For the discrete function $u_h = \{u_j|\ j = 0,\pm1,\pm2,\cdots\}$ defined on the grid points $\{x_j = jh|\ j = 0,\pm1,\pm2,\cdots\}$ of the real line $I\!R = (-\infty,+\infty)$ and the discrete function $u_h = \{u_j|\ j = 0,1,2,\cdots\}$ defined on the grid points $\{x_j = jh|\ j = 0,1,2,\cdots\}$ of the half real line $I\!R_+ = [0,\infty)$ with the mesh step $h > 0$, the norms of these discrete functions are defined to be the limit of the convergent infinite sums of the positive powers of discrete values on the grid points. For example for the discrete function $u_h = \{u_j|\ j = 0,1,2,\cdots\}$ and the index $1 \le p < \infty$, we have

$$\|\delta^k u_h\|_p = \Big(\sum_{j=0}^{\infty}\Big|\frac{\Delta^k u_j}{h^k}\Big|^p h\Big)^{1/p}, \tag{15}$$

where $k = 0,1,\cdots$. Similarly we have the other types of norms for the discrete functions defined on the grid points of infinite segment.

As the consequences of the theorems for discrete functions on the finite interval, we have the following theorem of interpolation formula for the discrete functions on the infinite interval.

Theorem 4. *For any discrete function* $u_h = \{u_j|\ j = 0,\pm1,\pm2,\cdots\}$ *defined on the grid points* $\{x_j = jh|\ j = 0,\pm1,\pm2,\cdots\}$ *of the real line* $I\!R = (-\infty,+\infty)$ *and any discrete function* $u_h = \{u_j|\ j = 0,1,2\cdots\}$ *defined on the grid points* $\{x_j = jh|\ j = 0,1,2,\cdots\}$ *of the half real line* $I\!R_+ = [0,\infty)$ *and for the parameters* $1 \le q,r < \infty$, $-\Big(n-k-\frac{1}{r}\Big) \le \frac{1}{p} \le 1$ *and* $0 \le k < n$, *there is*

$$\|\delta^k u_h\|_p \le C\|u_h\|_q^{1-\alpha}\|\delta^n u_h\|_r^{\alpha} \tag{16}$$

with

$$\frac{1}{p} - k = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - n\Big), \tag{17}$$

where C *is a constant independent of the constants* p,q,r, *the mesh step* $h > 0$ *and the discrete function* u_h *defined on the grid points of the whole real line* $I\!R$ *or the half real line* $I\!R_+$.

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