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STREAMLINE DIFFUSION METHODS FOR OPERATOR EQUATIONS*

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Abstract

To solve a class of operator equations numerically, some general streamline diffusion methods with satisfactory convergence properties are presented in this paper. It is proved that the approximation accuracy is only half a power of h, the mesh size, from being optimal when these methods are applied to mixed problems and convection-diffusion problems.

§1. Introduction

It was observed long ago that usual finite element methods for certain equations such as mixed and hyperbolic equations do not work well or even do not give reasonable results. It has to be of particular concern since in many interesting practical problems, well-approximations are desired. Trying to solve the problems, we present in this paper some streamline diffusion methods. It is shown that these methods work well and possess satisfactory convergence, for a class of operator equations, including mixed and hyperbolic types.

These methods are based upon an idea first introduced by Raithby[9] for finite difference methods and by Hughes and Brooks[7] for finite element methods for special problems in fluid dynamics by adding an artificial diffusion term acting only in the direction of the streamlines. This idea was also taken up by Johnson[8] with observation that such a streamline diffusion term can be introduced very naturally in the standard Galerkin method without modifying the equation. We refer to a series of work by Johnson[8] and his cooperators (see the literature cited therein) and by the author[10] in this direction.

This paper consists of four sections. In Section 2, general streamline diffusion methods are introduced, and existence, uniqueness and error analysis for these methods are discussed. In Section 3, some error estimates with half a power of h from being optimal are then given in the case of a boundary value problem of mixed type. In the last section, analogous results for convection-diffusion problems are presented.

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We shall point out that these methods are somewhat different from those stated in [8] and [10] when applied to convection-diffusion equations.

§2. Operator Equations

Suppose that W and X are two Hilbert spaces. We consider the following operator equation:

$$Lu = f, \tag{2.1}$$

where L is a linear operator from W into X, and $f \in X$.

Let $\{W^h : h \in (0,1)\}$ be a family of finite-dimensional subspaces of W. An approximation solution $u_h \in W^h$ of the problem (2.1) will be defined by the following discrete scheme:

Given $f \in X$, find $u_h \in W^h$ such that

$$A(u_h, v) = (f, lv + hLv), \forall v \in W^h, \tag{2.2}$$

where $A(u,v) = (Lu, lv + hLv)$, $l : W \to X$ is a linear operator and $(\cdot,\cdot)$ is the inner product of X.

Now we take the following assumption:

Assumption I. There exists a Hilbert space $(Y, \|\cdot\|_Y)$, into which W is imbedded boundedly, such that

$$(Lu, lu) \geq c\|u\|_Y^2, \ \forall u \in W. \tag{2.3}$$

If we set

$$|||u|||^2 = \|u\|_Y^2 + h\|Lu\|_X^2,$$

then (2.3) implies

$$A(u,u) \geq c|||u|||^2, \ \forall u \in W. \tag{2.4}$$

Let $\{\phi_i\}_{i=1}^n$ be a basis for W^h and

$$u_h = \sum_{i=1}^n z_i \phi_i.$$

Denote $z \doteq (z_1, \cdots, z_n)^T$ and $b = (b_1, \cdots, b_n)^T$ with

$$b_i = (f, l\phi_i + hL\phi_i).$$

Then u_h is given by the linear system:

$$\tilde{A}z = b,$$

where $\tilde{A} = (a_{ij})_{1 \leq i,j \leq n}$ and $a_{ij} = A(\phi_i, \phi_j)$, $1 \leq i,j \leq n$.

Lemma 2.1. *$\tilde{A}$ is invertible.*

Proof. Suppose that $\tilde{A}z = 0$ for some vector z. Setting $z_h = \sum_{i=1}^n z_i \phi_i$, we find that

$$o = z^T \tilde{A} z = A(z_h, z_h) \geq c\|z_h\|_Y^2, \tag{2.5}$$

which means $z_h = 0$. Now $\{\phi_i\}_{i=1}^n$ is linearly independent so that $z_h = 0$ implies $z = 0$. Since $\tilde{A}$ is a square matrix with trivial null space, $\tilde{A}$ is invertible.

Thus, we immediately obtain

Theorem 2.2. *There exists a unique* $u_h \in W^h$ *satisfiying* (2.2).

Theorem 2.3. *If* $u \in W$ *is a solution of* (2.1), *then*

$$|||u - u_h||| \le ch^{1/2} \inf_{v\in W^h} (h^{-1}\|l(u-v)\|_X + \|L(u-v)\|_X). \tag{2.6}$$

Proof. It is easy to see that there holds

$$A(u,v) = (f, lv + hLv),\ \forall v \in W^h,$$

hence

$$A(u - u_h, v) = 0,\ \forall v \in W^h. \tag{2.7}$$

Let $e = u - u_h$ and $\eta = u - v$ for $v \in W^h$. Using (2.4) and (2.7), we find

$$\begin{aligned} |||e|||^2 &\le cA(e,e) = cA(e,\eta) = c(Le, l\eta) + ch(Le, L\eta) \\ &\le c\|Le\|_X\|l\eta\|_X + ch\|Le\|_X\|L\eta\|_X \\ &\le ch^{-1/2}A(e,e)^{1/2} \le ch^{1/2}(h^{-1}\|l\eta\|_X + \|L\eta\|_X). \end{aligned}$$

This completes the proof of the theorem.

Remark 2.4. Time-dependent problems such as

$$u_t + Lu = f$$

can also be solved numerically by using another discrete scheme:

$$A(u_h, v) \equiv ((u_h)_t, lv) + (Lu_h, lv + hLv) = (f, lv + Lv),\ \forall v \in W^h$$

and similar results can be expected (cf. [10]).

§3. A Boundary Value Problem of Mixed Type

In this section, we specifically consider equations of the form

$$Lu \equiv k(y)u_{xx} + u_{yy} - c(x,y)u = f \tag{3.1}$$

in a bounded simply connected domain $\Omega \subset R^2$, where $\operatorname{sign} k(y) = \operatorname{sign} y$ and the domain Ω is bounded by the following curves: a piecewise smooth curve Γ_0 lying in the half plane $y > 0$ which intersects the line $y = 0$ at the points $A(-1,0)$ and $B(0,0)$, for $y > 0$ a piecewise smooth curve Γ_1 through A which meets the characteristic of (3.1) issued from B at the point C, and the curve Γ_2 which consists of the portion CB of the characteristic through B.

The boundary value problems we shall discuss may be stated as

$$\begin{cases} Lu = f, & \text{in } \Gamma, \\ u = 0, & \text{on } \Gamma_0 \cup \Gamma_1. \end{cases} \tag{3.2}$$

A number of authors have dealt with the finite element method for the numerical solutions of mixed type (3.2) (see [1–6]). However, all the approximation accuracies presented are, in general, one power of h from being optimal. The accuracy can actually be improved by half a power of h using the methods stated in Section 2 and the derivative in the "streamling direction" is in fact optimal. Therefore, the streamline diffusion method (2.2) should be somewhat better than all the earlier prosedures.

The following hypotheses are assumed in the forth coming discussion:

(i) $k(y)$ is continuously differentiable and satisfies sign $k(y) = \operatorname{sign} y$ and $yk'(y) \geq k(y)$ for $y \geq 0$.

(ii) $c(x,y)$ is continuously differentiable and satisfies $c|_{\Gamma_2} \geq 0$ and

$$xc_x + c \geq 0, \text{ for } y \leq 0,$$

$$(1+\alpha)c + xc_x + \alpha y c_y \geq 0, \text{ for } y \geq 0,$$

where $\alpha \in (1/2, 1)$.

(iii) The curve Γ_0 satisfies the condition $xn_1 + \alpha y n_2 \geq 0$, where $\vec{n} = (n_1, n_2)$ is an outward normal vector.

(iv) The curve Γ_1 satisfies the conditions $k(y)n_1^2 + n_2^2 \geq 0$ and $n_1 \leq 0$.

The Hilbert space Y is defined to be the completion of the set of infinitely differentiable functions on the closure $\overline{\Omega}$ of Ω vanishing on $\Gamma_0 \cup \Gamma_1$ with respect to the following norms:

$$\|u\|_Y^2 = \int_\Omega [|k|u_x^2 + u_y^2]\, dx dy + \int_{\Gamma_0 \cup \Gamma_1} \rho(x,y)[k(y)u_x^2 + u_y^2]\, ds + \int_{\Gamma_2} [(-k)^{1/2} u_x + u_y]^2 (-\alpha_1)\, dy$$

where $\rho(x,y) = \alpha_1 n_1 + \alpha_2 n_2$, $\alpha_1 = x$ and

$$\alpha_2 = \begin{cases} \alpha y, & \text{for } y \geq 0, \\ 0, & \text{for } y < 0. \end{cases}$$

$W = \{u \in H^2(\Omega) : u = 0 \text{ on } \Gamma_0 \cup \Gamma_1\}$ with $H^2(\Omega)$–norm, $X = L_2(\Omega)$ with the usual inner product of $L_2(\Omega)$. Let $l : W \to X$ be defined by

$$lv = \alpha_1 v_x + \alpha_2 v_y, \text{ for } v \in W.$$

Under the hypotheses stated above, it is clear that Assumption I holds true (see [1]).

Now we need another assumption

Assumption II. (i) There exists an $s \geq 0$ such that

$$u \in W \cap H^s(\Omega).$$

(ii) Let $\{W^h = W_h^{r,k} : h \in (0,1)\}$ consist of the family of finite dimensional subspaces $W_h^{r,k} \subset W \cap H^s(\Omega)$ with r and k nonnegative integers $k < r$ which possess the property that for any $w \in W \cap H^s(\Omega)$, there exists a $w^h \in W_h^{r,k}$ such that

$$\|w - w^h\|_{l,2} \leq ch^{s-l}\|w\|_{s,2}, \tag{3.3}$$

where $0 \le l \le k$, $l \le s < r$.

If Assumption II holds for $k \ge 2$, then we obtain an improved results from Theorem 2.3:

$$|||u - u_h||| \le ch^{s-3/2}\|u\|_{s,2}, \text{ for } s \ge 2. \tag{3.4}$$

Remark 3.1. Instead of (2.2), a conventional scheme can be used:

$$A(u_h, v) \equiv (Lu_h, lv + hL_0 v) = (f, lv + hL_0 v), \forall\, v \in W_h^{r,k},$$

where $L_0 v = k(y)u_{xx} + u_{yy}$.

§4. Convection-Diffusion Problems

We consider here stationary problems

$$Lu \equiv \nabla(\vec{\beta}u) + \alpha u - \varepsilon \Delta u = f \qquad \text{in}\, \Omega \tag{4.1}$$

together with boundary conditions:

$$u = 0 \ \text{on}\, \Gamma_- \quad \text{if} \quad \varepsilon = 0, \tag{4.2}$$

$$u = 0 \ \text{on} \ \partial\Omega \quad \text{if}\, \varepsilon > 0, \tag{4.3}$$

where Ω is a bounded domain in R^2, $\Gamma_- = \{x \in \partial\Omega : \vec{n} \cdot \vec{\beta} < 0\}$ with the coefficients α, $\vec{\beta} = (\beta_1, \beta_2)$ depending smoothly on x, and $\varepsilon \ge 0$ is a constant. We assume that

$$\alpha + \frac{1}{2} \nabla \vec{\beta} \ge \gamma \ \text{in}\, \Omega, \tag{4.4}$$

where $\gamma > 0$ is a constant, which ensures the stability of the problems.

Let $X = L_2(\Omega)$ with the usual inner product of $L_2(\Omega)$, $W = \{u \in H^1(\Omega) : u = 0 \,\text{on}\, \Gamma_-\}$ if $\varepsilon = 0$ and $W = H_0^1(\Omega) \cap H^2(\Omega)$ if $\varepsilon > 0$. Let Y be defined as the completion of the set of infinitely differentiable functions on the closure $\overline{\Omega}$ of Ω vanishing on Γ_- with respect to the following norms:

$$\|u\|_Y^2 = \|u\|_{0,2}^2 + \int_{\partial\Omega} u^2 |\vec{n} \cdot \vec{\beta}|\, ds$$

if $\varepsilon = 0$ and $Y = H_0^1(\Omega)$ with respect to the norms

$$\|u\|_Y^2 = \|u\|_{0,2}^2 + \varepsilon \| \nabla u\|_{0,2}^2$$

if $\varepsilon > 0$. Obviously, Assumption I holds true if $lv = v$ (see [8] and [10]).

Further, let $\{T_h\}$ be a family of quasi-uniform partition $T_h = \{e\}$ of Ω with size h. For a given positive r, we introduce finite element spaces:

$$W_h^{r,k} = \{v \in H^1(\Omega) : v|_e \in P_{r-1}(e), e \in T_h, v = 0 \quad \text{on} \quad \Gamma_-\}$$

if $\varepsilon = 0$ and

$$W_h^{r,k} = \{v \in H_0^1(\Omega) \cap H^2(\Omega) : v|_e \in P_{r-1}(e),\, e \in T_h\}$$

if $\varepsilon > 0$. Then, Assumption II holds true for $k \geq 1$ if $\varepsilon = 0$ and for $k \geq 5$ if $\varepsilon > 0$. Thus, we obtain an analogue of the results in [8], namely

$$|||u - u_h||| \leq ch^{s-1/2}||u||_{s,2}$$

for $1 \leq s \leq r$.

Similar results for time-dependent problems due to the scheme (2.2) or Remark 2.1 can be obtained (cf. [8] and [10]).

Remark 4.1. The schemes stated in this paper are different from the conventional ones.

The approximation schemes stated in this paper can also be applied to other problems (cf. [10]).

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