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THE HERMITE SCHEME FOR SEMILINEAR SINGULAR PERTURBATION PROBLEMS*[1)]

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Abstract

A numerical method for singularly perturbed semilinear boundary value problems is given. The method uses the fourth order Hermite scheme on a special discretization mesh. Its stability and convergence are investigated in the discrete L^1 norm.

§1. Introduction

We shall consider the following singularly perturbed boundary value problem:

$$T_u := -\varepsilon^2 u'' + c(x,u) = 0, \; x \in I = [0,1],$$

$$u(0) = u(1) = 0, \tag{1}$$

where $\varepsilon \in (0, \varepsilon^*]$ (usually $\varepsilon^* << 1$). Throughout the paper we shall assume:

$$c \in C^6(I \times I\!R), \tag{2.a}$$

$$c_u(x,u) > \gamma^2, \; (x,u) \in I \times I\!R, \; \gamma > 0. \tag{2.b}$$

These conditions guarantee that the problem (1) has a unique solution u_ε, $u_\varepsilon \in C^B(I \times I\!R)$, which exhibits two boundary layers at the endpoints of I. In particular, the following estimates hold, see [22]:

$$|u_\varepsilon^{(k)}(x)| \le M[1 + \varepsilon^{-k}(\exp(-\gamma x/\varepsilon) + \exp(\gamma(x-1)/\varepsilon))], \; x \in I, \; k = 0(1)6, \tag{3}$$

where M does not depend on ε.

Because of such a behaviour of u_ε it is necessary to use special methods to solve the problem numerically. We shall use a finite-difference scheme on a special non-equidistant discretization mesh which is dense in the layers. The mesh will guarantee that the local truncation errors of the scheme will be uniform (by "uniform" we shall always mean "uniform in ε"); hence the discretization will be uniformly consistent

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with the continuous problem. Then the uniform convergence (the convergence of the numerical solution towards the restriction of u_ε on the mesh) will follow if we show that our discretization is uniformly stable. Usually, as in Doolan, Miller, Schilders [7], Herceg, Vulanović [13], Herceg [8, 9, 10], Herceg, Petrović [12], Herceg, Vualnović, Petrović [14], Vulanović, Herceg, Petrović [26], Vulanović [21, 22, 23], the stability is shown in the maximum norm; hence the pointwise uniform convergence follows. The order of the convergence depends on the scheme used. Higher order convergence was proved in Vulanović, Herceg, Petrović [26], Herceg, Vulanović, Petrović [14] and Herceg [9], while Herceg [10], and Herceg, Petrović [12] used higher order schemes in the layers only. These papers, as well as Vulanović [22], Herceg, Vulanović [13] and Vulanović [23], use the approach of special discretization meshes. The concept of exponential fitting was used in Doolan, Miller, Schilders [7], Herceg [8], Vulanović [21], and Vulanović [23]. The method from [3] is based on piecewise linear interpolation, and for the use of spline-difference schemes see Surla [16, 17, 18, 19]. For other papers which deal with the numerical solution of the problem (1), see Herceg [9].

In this paper we shall use a discretization of the same type as in Herceg [9], Herceg, Vulanović, Petrović [14]. Basically, the Hermite scheme is used, but at some mesh points it is replaced by the standard central scheme. Such a switch is used in order to prove the uniform stability. For the same reason Herceg [9] and Herceg, Vulanović, Petrović [14] have a restriction on the nonlinearity of $c(x,u)$. Essentially, the following is required:

$$c_u(x,u) \leq \Gamma, \ x \in I, \ u \in I\!R; \quad 5\gamma^2 - 2\Gamma > 0. \tag{4}$$

Obviously, such an assumption is unpleasant, and our aim here will be to avoid it. We shall prove the uniform stability in the discrete L^1 norm (cf. Vulanović [24], [25] where this norm was used for discretizations of quasilinear singular perturbation problems) and for this (4) is not needed. Such a result was announced in Herceg [9] and Herceg, Vulanović, Petrović [14].

Thus we shall obtain the uniform convergence in the discrete L^1 norm. The L^1-error will be estimated by

$$M[\varepsilon n^{-4} + n^{-1}\exp(-pn)]$$

where n is the number of mesh steps, p is a positive constant independent of n and ε, and throughout the paper M denotes a positive generic constant independent of n and ε. From this we shall get that

$$M[n^{-3} + \varepsilon^{-1}\exp(-pn)]$$

is the upper bound for the maximal pointwise error. This is worse than Mn^{-4} from Herceg [9]. However, we point out that the numerical method which will be given here is essentially the same as the method from Herceg [9] (the deferent pointwise error estimates result from the different norms used); hence we might expect the uniform fourth order pointwise convergence to be still present. Our numercical experiments confirm that.

Let us finally note that problems of type (1) arise in practice as models for chemical catalysis reactions and the Michaelis-Menten process in biology, Bohl [1]. For other applications see [2] and [3].

§2. The Numerical Method

Let us introduce the discretization mesh I_h with the mesh points

$$x_i = \lambda(t_i),\ t_i = ih,\ i = 0, 1, \cdots, n;\quad h = \frac{1}{n},\ n = 2m, m \in I\!N \setminus \{1\},$$

where λ is a mesh generating function:

$$\lambda(t) = \begin{cases} \mu(t) = \dfrac{a\varepsilon t}{q-t}, & t \in [0, \alpha], \\ \mu'(\alpha)(t-\alpha) + \mu(\alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1]. \end{cases}$$

Here q is an arbitrary number from (0, 0.5) and $a \in (0, q/\varepsilon^*]$. The point α is determined from

$$\mu'(\alpha)(0.5-\alpha) + \mu(\alpha) = 0.5,$$

which reduces to a quadratic equation and α is easy to find:

$$\alpha = \frac{q - \sqrt{aq\varepsilon(1 - 2q + 2a\varepsilon)}}{1 + 2a\varepsilon}.$$

We have $\lambda \in C^1(I)$ and

$$0 < \lambda'(t) \le M,\ t \in I. \tag{5}$$

Such a mesh generating function was introduced in Vulanović [22] and used in our other papers, and in Herceg [9] in particular.

Let

$$h_i = x_i - x_{i-1},\ i = 1(1)n,\ \bar{h}_i = (h_i + h_{i+1})/2,\ i = 1(1)n-1.$$

It is obvious that

$$h_i \le h_{i+1},\ i = 1(1)m-1,\quad h_i = h_{n-i+1},\ i = 1(1)m. \tag{6}$$

By w^h, v^h etc. we shall denote mesh functions defined on $I_h/\{0,1\}$. They will be identified with $I\!R^{n-1}$- column vectors:

$$w^h = [w_1, w_2, \cdots, w_{n-1}]^T,\quad w_i := w_i^h.$$

In particular, we shall take

$$u_\varepsilon^h = [u_\varepsilon(x_1),\ u_\varepsilon(x_2), \cdots, u_\varepsilon(x_{n-1})]^T.$$

Let $\|\cdot\|_\infty$ and $\|\cdot\|_1$ be the usual norms in $I\!R^{n-1}$. The discrete L^1 norm is given by

$$\|w^h\|_1^h = \|Hw^h\|_1 = \sum_{i=1}^{n-1} \bar{h}_i |w_i|,\quad H = \operatorname{diag}(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_{n-1});$$

see [24, 25]. The corresponding matrix norms will be denoted in the same way. In particular, we have

$$\|A\|_1^h = \|HAH^{-1}\|_1,\ A \in I\!R^{n-1,n-1}.$$

We shall use two approximations of the differentiations operator T : the Hermitian approximation (see [9], [14]):

$$T_H w_i = -\varepsilon^2 D w_i + b_1(i)c_{i-1} + b_0(i)c_i + b_2(i)c_{i+1},\ i = 1(1)n-1$$

and the central approximation:

$$T_c w_i = -\varepsilon^2 D w_i + c_i,\ i = 1(1)n-1.$$

Here

$$\begin{aligned} Dw_i &= [(w_{i+1} - w_i)/h_{i+1} + (w_{i_1} - w_i)/h_i]/\bar{h}_i, \\ b_1(i) &= (h_i^2 - h_{i+1}^2 + h_i h_{i+1})/12 h_i \bar{h}_i, \\ b_2(i) &= (h_{i+1}^2 - h_i^2 + h_i h_{i+1})/12 h_{i+1} \bar{h}_i, \\ b_0(i) &= 1 - b_1(i) - b_2(i), \\ c_j &= c(x_j, w_j). \end{aligned}$$

These two approximations will be combined to make a discrete problem on the mesh I_h, corresponding to (1).

Let

$$Q = \frac{3}{5}(7 + \sqrt{14}) = 6.4449944\cdots$$

and

$$I_h' = \{x_i \in I_h :\ q - Qh < t_{i-1}, \alpha\ \text{or}\ 1 - \alpha < t_{i+1} < 1 - Qh\}$$

(note that I_h' may be empty). Note that [9] uses a different (smaller) value of Q. This is the only difference between the two methods, caused by technical reasons.

The discrete problem is given by

$$Fw^h = 0, \tag{7}$$

where

$$F_i w^h = \begin{cases} T_H w_i, & x_i \in I_h \setminus I_h', \\ T_c w_i, & x_i \in I_h' \quad i = 1(1)n-1. \end{cases}$$

First we shall investigate the stability of the operator F.

Theorem 1. *Let $c \in C^1(I \times I\!R)$ and let (2b) hold. Then the discrete problem has a unique solition w_ε^h which is a point of attraction of SOR-Nowton and Newton-SOR methods with the relaxation parameter $\omega \in (0,1]$. Moreover, the following stability inequality holds for any w^h, v^h :*

$$\|w^h - v^h\|_1^h \le \frac{12}{\gamma^2}\|Fw^h - Fv^h\|_1^h. \tag{8}$$

Proof. The Trechet derivative $F'(w^h)$ of F at arbitrary w^h is a tridiagonal matrix:

$$F'(w^h) = \begin{bmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & O \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & O & \cdot & \cdot & \cdot \\ & & & A_{n-1} & B_{n-1} \end{bmatrix}$$

with

$$A_i = -\varepsilon^2/h_i\bar{h}_i + a_1(i)c_u(x_{i-1,w_{i-1}}),$$

$$B_i = 2\varepsilon^2/h_ih_{i+1} + a_0(i)c_u(x_i, w_i),$$

$$C_i = -\varepsilon^2/h_{i+1}\bar{h}_i + a_2(i)c_u(x_{i+1}, w_{i+1}),$$

where

$$a_j(i) = \begin{cases} b_j(i), & x_i \in I_h \setminus I'_h, \\ 0, & x_i \in I'_h, \quad j = 1,2, \end{cases} \qquad a_0(i) = 1 - a_1(i) - a_2(i).$$

Let us write formally

$$C_0 = A_n = 0.$$

We shall show that

$$\|F'(w^h)^{-1}\|_1^h \le 12/\gamma^2 \tag{9}$$

and (8) will be immediate (note that $\|\cdot\|_\infty$ was used in Herceg [9] and because of that a condition of type (4) was needed). At the same time, (9) will imply the rest of the assertion; see Ortega, Rheinboldt [15].

In order to prove (9) we shall show

$$\sigma_i := |B_i| - |C_{i-1}|\bar{h}_i - |A_{i+1}|\bar{h}_{i+1}/\bar{h}_i \ge \gamma^2/12, \ i = 1(1)n-1. \tag{10}$$

This means that $HF'(w^h)H^{-1}$ is strictly diagonally dominant by columns and it follows that

$$\|(HF'(w^h)H^{-1})^{-1}\|_1 \le 12/\gamma^2;$$

see Varah [20]. This is in fact inequality (9) since

$$\|(HF'(w^h)H^{-1})^{-1}\|_1 = \|HF'(w^h)^{-1}H^{-1}\|_1 = \|F'(w^h)^{-1}\|_1^h;$$

cf. Vulanović [24, 25].

Let us prove (10) for $i = 1(1)m$ (for other indices the proof is analogous by the symmetry (6)). This proof is similar to the proof from Herceg [9]. It always holds that

$$B_i > 0.$$

If $x_{i-1},\ x_i,\ x_{i+1} \in I'_h$ we have

$$C_{i-1} \le 0, \ A_{i+1} \le 0$$

and it easily follows that

$$\sigma_i \ge \gamma^2.$$

Now suppose $x_{i-1},\ x_i, x_{i+1} \in I_h \setminus I'_h$ and let us consider the hardest possible situation when

$$C_{i-1} \ge 0, \ A_{i+1} \ge 0;$$

cf. Herceg [9]. Using estimates from Herceg [9] and Herceg, Vulanović, Petrović [14]:

$$b_0(i) \ge \frac{5}{6}, \ \frac{1}{6} \ge b_2(i) \ge \frac{1}{12}, \ b_2(i) \ge b_1(i) \ge -\frac{1}{6},$$

we get

$$\sigma \geq \frac{\gamma^2}{\bar{h}_i}[b_0(i)\bar{h}_i - |b_1(i+1)|\bar{h}_{i+1} - b_2(i-1)\bar{h}_{i-1}]$$

$$\geq \frac{\gamma^2}{\bar{h}_i}[\frac{5}{6}\bar{h}_i\frac{1}{6}\bar{h}_{i+1} - \frac{1}{6}\bar{h}_{i-1}] \geq [\frac{2}{3} - \frac{1}{6}\frac{\bar{h}_{i+1}}{\bar{h}_i}]\gamma^2 \geq \frac{\gamma^2}{12}.$$

The last inequality follows because we have

$$2\bar{h}_{i+1} \leq 7\bar{h}_i. \tag{11}$$

Indeed, if $t_{i-1} \geq \alpha$, it holds that $\bar{h}_{i+1} = \bar{h}_i$, and if, on the other hand, $t_{i-1} < \alpha$, we use

$$t_{i-1} \leq q - Qh \tag{12}$$

(which holds because $x_i \in I_h \setminus I'_h$) to get

$$\frac{\bar{h}_{i+1}}{\bar{h}_i} \leq \frac{\mu'(t_{i+1})}{\mu'(t_{i-1})} \leq \left(\frac{q - t_{i-1}}{q - t_{i+2}}\right)^2$$

(note that $t_{i+2} < q$). Finally, (11) follows since (12) is equivalent to

$$\left(\frac{q - t_{i-1}}{q - t_{i+2}}\right)^2 \leq \frac{7}{2}.$$

Note that for $i = 1$ we have $C_0 = 0$ and that situation is better than the one considered. This is also true for the case when $x_{i-1} \in I_h \setminus I'_h(x_{i-1},\ x_i \in I_h \setminus I'_h)$ and $x_i,\ x_{i+1} \in I'_h(x_{i+1} \in I'_h)$. Thus (10) is proved and so is the theorem.

Let us mention that besides SOR-Newton methods, AOR-Newton methods (see Cvetković, Herceg [4,5,6], Herceg, Cvetković [11]) may be used to solve the discrete problem (7).

Now let us consider the consistency error

$$r^h = Fu^h_\varepsilon.$$

Let

$$d = \varepsilon h^4 + h \cdot \exp(-pn),$$

where

$$p = \mathrm{aq}\gamma/Q.$$

Theorem 2. *Let* (2) *hold. Then it follows that*

$$\|r^h\|^h_1 \leq Md. \tag{13}$$

Proof. We shall estimate $\bar{h}_i|r_i|$, $i = 1(1)m - 1$, where r_i are components of r^h. For $i = m(1)n$ the estimates can be obtained analogously. We shall use (3) which reduces to

$$|u^{(k)}_\varepsilon(x)| \leq M[1 + \varepsilon^{-k}\exp(-\gamma x/\varepsilon)],\ x \in [0, 0.5],\ k = 0(1)6. \tag{14}$$

As in Herceg [9], Vulanović [22], Herceg, Vulanović, Pereović [14] and our other papers, we distinguish the following three cases:

1° $t_{i-1} \geq \alpha$,

2° $t_{i-1} < \alpha$ and $t_{i-1} \leq q - Qh$,

3° $q - Qh < t_{i-1} < \alpha$.

In cases 1^0 and 2^0 it holds that $x_i \in I_h \setminus I'_h$. By expanding r_i in the same way as in Herceg [9], and using the same technique we can prove

$$\bar{h}_i|r_i| \leq M\varepsilon h^5. \tag{15}$$

When case 3^0 holds we have $x_i \in I'_h$ and the central scheme is used:

$$r_i = \varepsilon^2[u''_\varepsilon(x_i) - Du_\varepsilon(x_i)].$$

Now we have to use a technique different from Herceg [9]. We use the following integral representation of r_i:

$$\bar{h}_i r_i = -\frac{1}{2h_{i+1}}\int_{x_i}^{x_{i+1}}(s - x_{x+1})^2 u'''_\varepsilon(s)ds - \frac{1}{2h_i}\int_{x_i}^{x_{i-1}}(s - x_{i-1})^2 u'''_\varepsilon(s)ds.$$

Then from (14) and (5) we get

$$\bar{h}_i|r_i| \leq M\Big[\varepsilon^2 h^2 + \frac{1}{\varepsilon h_{i+1}}\int_{x_i}^{x_{i+1}}(s - x_{i+1})^2 \exp(-\gamma s/\varepsilon)ds$$
$$+ \frac{1}{2h_i}\int_{x_{i-1}}^{x_i}(s - x_{i-1})^2 \exp(-\gamma s/\varepsilon)ds\Big]$$

and after two partial integrations there follows

$$\bar{h}_i|r_i| \leq M[\varepsilon^2 h^2 + \bar{h}_i \cdot \exp(-\gamma x_{i-1}/\varepsilon)] \leq M[\varepsilon^2 h^2 + h \cdot \exp(-aq\gamma/(q - t_{i-1}))]$$
$$\leq M[\varepsilon^2 h^2 + h \cdot \exp(-aq\gamma/Qh)] = M[\varepsilon^2 h^2 + h \cdot \exp(-pn)].$$

Because of

$$q - Qh < t_{i-1} < \alpha \leq q - M\sqrt{\varepsilon} < q,$$

case 3° can be true only if

$$\sqrt{\varepsilon} \leq Mh,$$

and it follows that

$$\bar{h}_i|r_i| \leq M[\varepsilon^2 h^4 + h \cdot \exp(-pn)].$$

Also note that case 3^0 can happen at 7 points at most. Thus we have

$$\|r^h\|_1^h \leq \sum_{i s.t.} \bar{h}_i|r_i| + Md, \quad x_i \in I_h \setminus I'_h$$

and from (15) we get (13).

By combining Theorems 1 and 2 we have the convergence theorem:

Theorem 3. *Let* (2) *hold. Then it follows that*

$$\|w_\varepsilon^h - u_\varepsilon^h\|_1^h \leq Md.$$

Since

$$\|w^h\|_1^h \geq \bar{h}_1\|w^h\|_\infty \geq M\varepsilon h\|w^h\|_\infty,$$

we have the following

Corollary. Let (2) hold. Then it follows that

$$\|w_\varepsilon^h - u_\varepsilon^h\|_\infty \leq M[h^3 + \varepsilon^{-1} \cdot \exp(-pn)].$$

The last result is almost the third order uniform pointwise convergence, but as we have said, we expect numerical results to show

$$\|w_\varepsilon^h - u_\varepsilon^h\|_\infty \le Mh^4.$$

Note that from Vulanović [24] it follows that

$$\|w_\varepsilon^h - u_\varepsilon^h\|_1^h \le Mh$$

holds for the central scheme on an arbitrary locally almost equidistant mesh. The result of Theorem 3 is much better because of the use of the special scheme on the special mesh.

§3. Numerical Results

We shall concider the following test example, Bohl [1]:

$$-\varepsilon^2 u' + \frac{u-4}{5-u} = 0, \ u(0) = u(1) = 0.$$

It is known, Bohl [1], that $0 \le u_\varepsilon \le 4$. So we have

$$\Gamma = 1 \ge c_u(x,u) \ge \frac{1}{25} = \gamma^2.$$

Note that (4) does not hold. This problem was considered in [2], [3], [4], [5], [6], [7] as well.

In the following Table we present the numerical order of convergence. Here we determine numerically the order of uniform convergence of our scheme as usual when the exact solution is not known:

$$\text{Ord} = \frac{\log E_n - \log E_{2n}}{\log 2},$$

where

$$E_n := \|u_\varepsilon^* - u\|_\infty$$

and for each fixed $\varepsilon, u_\varepsilon^*$ is the numerical solution to our problem with $n = 1024$ mesh points. This solution we compare with the other numerical solutions obtained by our method for $n = 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512$. The corresponding nonlinear systems are solved by one-step Newton-Gauss-Seidel methods. For numerical solutions u we consider the iterate u^k for which

$$\|Fu^k\|_\infty < 10^{-7} \text{ and } \|u^k - u^{k-1}\|_\infty < 10^{-11}.$$

In each case the start approximation is $u^0 = [0,\ 4,\ 4, \cdots, 4,\ 0]^T$.

All computations have been carried out on the ATARI 1040 ST with 48 bits accuracy in floating point.

Table

$n \setminus \varepsilon$	2^{-4}	2^{-8}	2^{-16}	2^{-24}
4	-	-	-	-
8	3.8985	6.6896	14.528	20.0523
16	4.1643	2.5064	1.5343	1.5318
32	3.7656	1.9034	2.6312	2.6157
64	3.8134	4.0804	4.5252	4.5490
128	3.9943	4.0018	4.0025	4.0026
256	4.0767	4.0170	4.0211	4.0210
512	4.0847	4.1704	4.0981	4.0980

References

[1] E. Bohl, Finite modele gewöhnlicher Randwertaufgaben, Teubner, Stuttgart, 1981.

[2] K. W. Chang and F. A. Howes, Nonlinear Singular Phenomena, Springer, New York, Berlin, Heidelberg, Tokyo, 1984.

[3] R.C.Y. Chin and R. Krasny, A hybrid asymptotic-finite element method for stiff two-point boundary value problems, *SIAM J. Sci. Stat. Comput.*, **4** (1983), 229–243.

[4] Lj. Cvetković and D. Herceg, Über die Konvergenz des VAOR-Verfahrens, *Z. angew. Math. Mech.*, **66**(1986), 405–406.

[5] Lj. Cvetković and D. Herceg, An improvement for the area of convergence of the AOR method, *Anal. Numer. Theor. Approx.*, **16** (1987), 109–115.

[6] Lj. Cvetković and D. Herceg, Nonlinear AOR method, *Z. angew. Math. Mech.*, **68** (1988), 486–487.

[7] E.P. Doolan, J.J.H. Miller and W.H.A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.

[8] D. Herceg, A uniformly convergent scheme with quasi-constant fitting factors, *Univ. u Novom Sadu, Zb. Rad. Prirod. -Mat. Fak. Ser. Mat.*, **11** (1981), 105–115.

[9] D. Herceg, Uniform fourth order difference scheme for a singular perturbation problem, *Numer. Math.*, **56** (1990), 675–693.

[10] D. Herceg, On numerical solution of singularly perturbed boundary value problem, in: Z. Bogte, ed., V Conference on Applied Mathematics, University of Ljubljana, Institute of Mathematics, Physics and Mechanics, Ljubljana, 1986, 59–66.

[11] D. Herceg and Lj. Cvetković, On the extrapolation method and USA algorithm, *Journal of Economic Dynamics and Control*, **13** (1989), 301–311.

[12] D. Herceg and N. Petrović, On numerical solution of a singularly perturbed boundary value problem II, *Univ. u Novom Sadu, Zb. Rad. Prirod. -Mat. Fak. Ser. Mat.*, **17** : 1 (1987), 163–186.

[13] D. Herceg and R. Vulanović, Some finite-difference schemes for a singular perturbation problem on a non-uniform mesh, *Univ. u Novom Sadu, Zb. Rad. Prirod. - Mat. Fak. Ser. Mat.*, **11** (1981), 117–134.

[14] D. Herceg, R. Vulanović and N. Petrović, Higher order schemes and Richardson extrapolation for singular perturbation problems, *Bull. Aust. Math. Soc.*, **39** (1989), 129–139.

[15] J. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York and London, 1970.

[16] K. Surla, A uniformly convergent spline difference scheme for singular perturbation problems, *Z. angew. Math. Mech.*, **66** (1986), 528–529.

[17] K. Surla, Numerical solution of singularly perturbed boundary value problems using adaptive spline function approximation, *Anal. Numer. Theor. Approx.*, **16** (1987), 175–189.

[18] K. Surla and D. Herceg, An exponential spline difference scheme for singular perturbation problems, in: J.W. Schmidt, H. Späth, eds., Splines in Numerical Analysis, Akademie-Verlag, Berlin, 1989, 171–180.

[19] K. Surla and M. Stojanović, A solving singularly perturbed boundary value problem by spline in tension, *J. Comput. Appl. Math.*, **24** (1988), 355–363.

[20] J. M. Varah, A lower bound for the smallest singular values of a matrix, *Lin. Alg. Appl.*, **11** (1975), 3–5.

[21] R. Vulanović, An exponentially fitted scheme on a nonuniform mesh, *Univ. u Novom Sadu, Zb. Rad. Prirod. -Mat. Fak. Ser. Mat.*, **12** (1982), 205–215.

[22] R. Vulanović, On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretization mesh, *Univ. u Novom Sadu, Zb. Rad. Prirod. -Mat. Fak. Ser. Mat.*, **13** (1983), 187–201.

[23] R. Vulanović, Exponential fitting and special meshes for solving singularly perturbed problems, in: B. Vrdoljak, ed., IV Conference on Applied Mathematics, Faculty of Civil Engineering, Split, 1985, 59–64.

[24] R. Vulanović, Quasilinear singular perturbation problems and the uniform L_1-convergence, *Z. angew. Math. Mech.*, **69** (1989), 130–132.

[25] R. Vulanović, Finite-difference schemes for quasilinear singular perturbation problems, *J. Comp. Apl. Math.*, **26** (1989), 345–365.

[26] R. Vulanović, D. Herceg and N. Petrović, On the extrapolation for a singularly perturbed boundary value problem, *Computing*, **36** (1986), 69–79.