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A BLOCK CHAOTIC AND ASYNCHRONOUS ALGORITHM FOR CONSISTENT SYSTEMS WITH INCOMPLETE DATA*

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Abstract

In this paper, we generalize the paracontracting matrices to pseudocontracting matrices. The convergence of (parallel) iteration

$$x_i = P_{j_i,x_{i-1}} x_{i-1}$$

and

$$x_{i+r_i} = \alpha_{j_i} x_{i+r_i-1} + (1-\alpha_{j_i}) P_{j_i,x_i} x_i$$

where $P_{j,x}$, $j = 1, \cdots, n$, are paracontracting and/or pseudocontracting matrices is analyzed. These iterations can also be applied to solve consistent systems with incomplete data.

§1. Introduction

Consider the following system:

$$Rx = f \tag{1.1}$$

where $R \in \mathbb{C}^{n\times m}$, and $m > n$. This sort of systems may arise in application of computed tomography, parallel beam reconstruction, and other fields. In these areas, m may be very large, $m \gg n$. If (1.1) is consistent, its solution set is $\tilde{x} + N(R)$, where $\tilde{x}$ is the unique minimum 2-norm solution of (1.1), and $N(R)$ is the nullspace of R.

Write R into the form:

$$R = \begin{bmatrix} R_1^T \\ \vdots \\ R_n^T \end{bmatrix} \tag{1.2}$$

where $R_i \in \mathbb{C}^m$, and construct paracontracting matrices:

$$P_i = I - \omega \frac{R_i R_i^T}{R_i^T R_i}, \qquad i = 1,\ 2,\ \cdots, n \tag{1.3}$$

where $\omega \in (0,\ 1)$. Based on the Kaczmarz algorithm (c.f. [5]), Elsner et al. proposed an asynchronous paracontracting method to solve (1.1) in [1].

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In this paper, we generalize the algorithm in [1] to the block case. The methods of proving the convergence in [1] are not valid. In Section 2, we present our main results on the block iterations. Then using these results, we propose block and asynchronous algorithms for solving (1.1) in Section 3.

§2. Block Paracontracting and/or Pseudocontracting Iterations

Through this paper, we denote $\|\cdot\|$ as the 2-norm. A^T, x^T denote the Hermitian transpose of the matrix A, vector x, respectively. $R(A)$ denotes the range of A. A matrix $P \in \mathbb{C}^{m\times m}$ is paracontracting (cf. [1], [6]) if

$$Px \neq x \iff \|Px\| < \|x\| \qquad \text{for all } x \in \mathbb{C}^m . \tag{2.1}$$

Lemma 2.1. *If P is paracontracting, for $\forall x \in R(I-P)$, there exists a constant $\gamma < 1$ such that $\|Px\| \leq \gamma\|x\|$.*

Let $R \in \mathbb{C}^{n\times m}$, $N = \{1,\cdots,n\}$. N_i, $i = 1, \cdots, p$ are p subsets of N such that $\cup_{i=1}^{p} N_i = N$. Note that we allow $N_i \cap N_j \neq \phi$ for $i \neq j$. n_i is the number of elements in N_i. r_j^T is the jth row of R. R_i is an $m \times n_i$ matrix with columns r_i for all $j \in N_i$. Assume $R_i^T R_i$ is nonsingular, and consider matrices

$$P_i = I - \omega R_i(R_i^T R_i)^{-1} R_i^T \qquad i = 1,\cdots,p. \tag{2.2}$$

In the following, we can prove that P_i is paracontracting. If we need to compute $P_i x$, then we need to solve an equation: $R_i^T R_i c = P_i^T x$. Our first idea is to use some C_i to approximate $R_i^T R_i$:

$$P_i = I - \omega P_i C_i^{-1} R_i^T \qquad i = 1,\cdots,p. \tag{2.3}$$

Theorem 2.2. *Let $P_i = I - \omega R_i C_i^{-1} R_i^T$. $0 < \omega < 1$. If $R_i^T R_i = C_i - (C_i - R_i^T R_i)_i$ is a P-regular splitting, then P_i is paracontracting.*

There is no difficulty to prove Theorem 2.2. We refer the readers to [3] for P-regular splitting. Let $R^T R = D + L + L^T$ where D is diagonal and L is strictly lower triangular. If $C = D$ (Jacobi type), or $C = D + L$ (Gauss-Seidel type), or $C = (D + \lambda L)/\lambda$ with $0 < \lambda < 2$ (SOR type), or $C = D + L + L^T$, then $R^T R = C - (C - R^T R)$ is a P-regular splitting.

The next idea is to choose some number to replace $R_i^T R_i$ in (2.2). Let $P_i = I - \omega\beta_i R_i R_i^T$. In this paper, we use only the contracting property. Then our second idea is: for arbitrary fixed $x \in \mathbb{C}^m$, choose an optimal $\beta_{i,x}$, which depends on x, to minimize $\|P_{i,x}\|$, where $P_{i,x} = I - \beta_{i,x} R_i R_i^T$.

Theorem 2.3. *Let $x \in \mathbb{C}^m$ be arbitrary and fixed. Then*

$$\beta_{i,x} = \begin{cases} (\|R_i^T x\|/\|R_i R_i^T x\|^2, & \textit{if } R_i^T x \neq 0 \\ 0, & \textit{if } R_i^T x \neq 0 \end{cases} \tag{2.4}$$

minimize $\|P_{i,x} x\|$, and $\|P_{i,x} x\| < \|x\| \iff P_{i,x} x \neq x$.

Theorem 2.4. *For any* $x \in R(R_i)$, $0 < \omega < 1$, *there exists a* $\gamma < 1$, *which is independent of* x, *such that*

$$\|P_{i,x}x\| = \|(I - \omega\beta_{i,x}R_iR_i^T)x\| \le \gamma\|x\|.$$

Proof. Let $R_i \in \mathbb{C}^{m\times q}$, with rank r. Now suppose $R_i = U\hat{\Sigma}V^T$ is its singular value decomposition, where $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{q\times q}$ is unitary, and

$$\hat{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Sigma = \text{diag}\,(\sigma_1, \cdots, \sigma_r)$, $\sigma_1 \ge \cdots \ge \sigma_r > 0$. Let $x \equiv U\tilde{y}$. We have

$$x \in R(R_i) \Leftrightarrow \tilde{y} = (y^T,\ 0^T)^T, \qquad y \in \mathbb{C}^r \tag{2.5}$$

and

$$\|R_i^Tx\| = \|\Sigma y\|, \quad \|x\| = \|y\|, \quad \text{and}\, \|R_iR_i^Tx\| = \|\Sigma^2 y\|. \tag{2.6}$$

From the definition of $P_{i,x}$, we can easily draw that

$$\|P_{i,x}x\|^2 = \|x\|^2 - \omega(2-\omega)\frac{\|R_i^Tx\|^4}{\|R_iR_i^Tx\|^2}.$$

Then

$$\sup_{x\in R(R_i)} \frac{\|P_{i,x}x\|^2}{\|x\|^2} = 1 - \omega(2-\omega) \inf_{x\in R(R_i)} \frac{\|R_i^Tx\|^4}{\|x\|^2\|R_iR_i^Tx\|^2}$$

but

$$\sup_{x\in R(R_i)} \frac{\|x\|\|R_iR_i^Tx\|}{\|R_i^Tx\|^2} = \sup_{z\in\mathbb{C}^r} \frac{\|y\|\|\Sigma^2y\|}{\|\Sigma y\|^2} = \sup_{z\in\mathbb{C}^r} \frac{\|\Sigma^{-1}z\|\|\Sigma z\|}{\|z\|^2}$$
$$\le \sup_{z\in\mathbb{C}^r} \frac{\|\Sigma^{-1}z\|}{\|z\|} \times \sup_{z\in\mathbb{C}^r} \frac{\Sigma z}{\|z\|} = \sigma_1/\sigma_r.$$

Set γ such that

$$\gamma^2 = 1 - \omega(2-\omega)\sigma_r/\sigma_1.$$

Then $\gamma < 1$, and for $\forall x \in R(R_i)$

$$\|P_{i,x}x\| \le \gamma\|x\|.$$

In general, for a fixed x, $P_{i,x}$ is not a paracontracting matrix. For example if $R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\beta_{i,x} = 1$. But for $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $P_{i,x}y = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$. Thereby, $\|P_{i,x}y\| > \|y\|$. So, $P_{i,x}$ is not paracontracting. But, from Theorems 2.3 and 2.4, $\|P_{i,x}x\| \le \|x\|$. So we call $P_{i,x}$ a pseudocontracing matrix.

Remark. If $R_i \in \mathbb{C}^m$, i. e. there is only one column in R_i, then for $\forall x :\ R_i^Tx \ne 0$, we have $\beta_{i,x} = 1/(R_i^TR_i)$. It is equivalent to the usual case in (1. 3), and $P_{i,x}$ is independent of the choice of x.

Remark. In Theorem 2.2, we assumed that $R_i^TR_i$ is nonsingular. However, in Theorem 2.3, the nonsingularity is not required.

Now we consider the following iteration:

$$x_i = P_{j_i,\, x_{i-1}} x_{i-1} \quad 1 \le j_i \le p, \qquad i = 1,\, 2,\, \cdots \tag{2.7}$$

where $P_{j,x}$ is defined by

$$P_{j,x} = \begin{cases} P_j = I - \omega R_j C_j^{-1} R_j^T & 0 < \omega < 1, \quad \text{or} \qquad (2.8a) \\ I - \omega \beta_{j,x} R_j R_j^T & 0 < \omega < 1 \qquad (2.8b) \end{cases}$$

In (2.8 a), we assumed that $R_j^T R_j$ is nonsingular and the splitting $R_j^T R_j = C_j - (C_j - R_j^T R_j)$ is P-regular; in (2.8 b) $\beta_{j,x}$ is defined as in (2.4). According to Lemma 2.1 and Theorem 2.4, if $x_{i-1} \in R(R_{j_i}$, there exists a $\gamma_{j_i} < 1$ such that

$$\|x_i\| = \|P_{j_i,\, x_{i-1}} x_{i-1}\| \le \gamma_{j_i} \|x\|_{i-1}.$$

Setting

$$\gamma \equiv \max_{1 \le i \le p} \gamma_i$$

we have $\gamma < 1$.

Theorem 2.5. *If $x_0 \in Span\{R_1,\, \cdots, R_p\} \equiv \mathbb{R}$, and $\{j_i\}_{i=1}^{\infty}$ is admissible, i. e. for every $j \in \{1,\, \cdots, p\}$, j appears in it infinitely often, then the vector sequence $\{x_i\}_{i=1}^{\infty}$ defined by (2.7) and (2.8) converges to the zero vector.*

Proof. For the vector sequence $\{x_i\}_{i=1}^{\infty}$, $\|x_i\| \le \|x_{i-1}\|$. Hence $\{x_i\}_{i=1}^{\infty}$ is a bounded sequence, and there exists a convergent subsequence $\{x_{p_i}\}_{i=1}^{\infty}$, such that $\lim_{i\to\infty} x_{p_i} = y$. If $y \ne 0$, we know $y \in \mathbb{R}$. For $i = 1, \cdots, p$, let $y = z_i + w_i$, where $z_i \in R(R_i)$, $w_i \perp R(R_i)$. Let

$$\sigma \equiv \min\{\|z_i\| \mid z_i \ne 0,\ 1 \le i \le p\}.$$

We have $\sigma > 0$ (otherwise $y \perp \mathbb{R}$, and hence $y = 0$). For any fixed $\varepsilon > 0$, $\exists I$, for $\forall i > I$, $\|x_{p_i} - y\| \le \varepsilon$. Now for a fixed $i > I$, denote $q = p_i$. r is a nonnegative integer such that

$$P_{j_{q+k},\, x_{q+k-1}} y = y \quad \text{for} \quad k < r, \quad \text{and} \quad P_{j_{q+r},\, x_{q+r-1}} y \ne y.$$

So $z_{j_{q+r}} \ne 0$. Because $\{j_i\}_{i=1}^{\infty}$ is admissible, r always exists. Then

$$\begin{aligned} \|x_{q+r-1} - y\| &= \|P_{j_{q+r-1},\, x_{q+r-2}} \cdots P_{j_{q+1},\, x_q}(x_q - y)\| \\ &= \|P_{j_{q+r-1},\, x_{q+r-2}-y} \cdots P_{j_{q+1},\, x_q - y}(x_q - y)\| \\ &\le \|x_q - y\| < \varepsilon. \end{aligned}$$

Let $x_{q+r-1} = z + w$, where $z \in R(R_{j_{q+r}})$, $w \perp R(R_{j_{q+r}})$. From the above inequality we have $\|z - z_{j_{q+r}}\| < \varepsilon$, $\|w - w_{j_{q+r}}\| < \varepsilon$. Hence

$$\|y\| \le \|x_{q+r}\| = \|P_{j_{q+r},\, x_{q+r-1}} x_{q+r-1}\| = \|w + P_{j_{q+r},\, x_{q+r-1}} z\|$$

$$= (\|w\|^2 + \|P_{j_{q+r},x} z\|^2)^{1/2} \le (\|w\|^2 + \gamma^2 \|z\|^2)^{1/2}$$

$$\le ((\|w_{j_{q+r}}\| + \varepsilon)^2 + \gamma^2 (\|z_{j_{q+r}}\| + \varepsilon)^2)^{1/2}.$$

From this we have

$$\|z_{j_{q+r}}\| \le C\varepsilon^{1/2}$$

where C is some constant which is dependent on y, γ and independent of ε. This contradicts $\|z_{j_{q+r}}\| \ge \sigma > 0$ if ε is small enough. So we have $y = 0$. From $\lim_{i\to\infty} \|x_i\| = \|y\| = 0$, we know $\lim_{i\to\infty} x_i = 0$.

Iteration (2. 7) is a recursive procedure. Because of the chaos of P_{j_i}'s, it fits only sequential computers (including vector computers), not MIMD computers. Now we consider the following asynchronous iteration:

$$x_{i+r_i} = \alpha_{j_i} x_{i+r_i-1} + (1-\alpha_{j_i}) P_{j_i,\, x_i} x_i \tag{2.10}$$

where $P_{i,x}$ is defined as in (2. 8), $\alpha_1, \cdots, \alpha_p \in (0,\ 1)$, and $\{j_i\}_{i=1}^{\infty}$ is a regulated sequence, i. e., there exists a $T > 0$ such that for any i, $\{1, \cdots, p\} \subseteq \{j_1, \cdot, \ j_{i+T-1}\}$.$r_i$, $i = 1,\ 2,\ \cdots$, are integers such that $1 \le r_i \le T$. Asynchronous implementetion of (2. 10) is: Assume that we have p processors Proc(1), $\cdots$, Proc(p) with shared memory. At time i, Proc(j_i) retrives the global approximation vector x_i, which resides in shared memory, and forms a local approximation vector $P_{j_i,\, x_i} x_i$. If the global approximation vecotr in the shared memory has been updated $r_i - 1$ times while Proc(j_i) has finished its local approximation, then the global approximation x_{i+r_i} is formed at time $i + r_i$ as in (2. 10).

Theorem 2.6. *Let*

$$x_s = \begin{cases} x \in \mathbb{R}, & s \le T, \\ \alpha_{j_i} x_{s-1} + (1-\alpha_{j_i}) P_{j_i,\, x_i} x_i, & s = i + r_i > T. \end{cases}$$

If $\{j_i\}_{i=T+1}^{\infty}$ *is regulated,* r_i *is the smallest positive integer such that* $j_{i+r_i} = j_i$, $\alpha_j \in (0,\ 1)$, $j = 1, \cdots, p$, *and* $P_{j,\, x}$ *is defined as in* (2.10), *then* $\lim_{i\to\infty} x_s = 0$.

Proof. Consider the vector $\xi_i \in \mathbb{C}^{pm}$ partitioned into p subvectors as follows:

$$\xi_i = ((\xi_i)_1^T, \ \cdots, \ (\xi_i)_p^T)^T$$

where $(\xi_i)_l$, $l = 1, \ \cdots, \ p$, are defined in the following way: before or at time i, Proc(l) finishes its last local approximation and obtains a local approximation vector. Substituting this vector into (2. 10), we can obtain a global approximation vector, that is $(\xi_i)_l$.

Let $i > T$. Because $j_{i+r_i} = j_i$, then $\xi_i = B_i \xi_{i-1}$, where B_i is a $pm \times pm$ matrix given in block form $((B_i)_{s,\,t})_{s,t=1}^{p}$, where

$$(B_i)_{s,t} = \begin{cases} \delta_{s,t} I, & \text{if } s \ne j_i, \text{or } s = j_i,\ t \ne j_i,\ \ j_{i-1}, \\ (I - \alpha_{j_i}) P_{j_i,\, x_i}, & \text{if } s = t = j_i \text{ and } j_i \ne j_{i-1}, \\ \alpha_{j_i} & \text{if } s = j_i,\ t = j_{i-1} \text{ and } j_i \ne j_{i-1}, \\ \alpha_{j_i} I + (1-\alpha_{j_i}) P_{j_i, x_i} & \text{if } s = t = j_i = j_{i-1} \end{cases}$$

where $\delta_{s,t} = 1$ if $s = t$ or 0 if $s \ne t$. And $x_i = (\xi_{i-1})_{j_i}$. Next, define a norm $|\cdot|$ on $\mathbb{C}^{pm}$ by

$$|\xi| \equiv \max_{1\le l\le p} \|(\xi)_l\|.$$

It is easy to show $|\xi_i| \le |\xi_{i-1}|$. So $\{\xi_i\}_{i=1}^{\infty}$ is a bounded sequence, and $\{\xi_{2Ti}\}_{i=1}^{\infty}$ has a convergent subsequence $\{\xi_{2T\nu_i}\}_{i=1}^{\infty}$, $\lim_{i\to\infty}\xi_{2T\nu_i} = \eta$. As well known, $(\eta)_l \in \mathbb{R}$ for $1 \le l \le p$. Suppose $\eta \ne 0$. Then for $\forall\varepsilon > 0$, $\exists I > 0$, integer, for all $i > I, |\xi_{2T\nu_i} - \eta| < \varepsilon$. Now for a fixed $i > I$, if we denote $2T\nu_i \equiv \theta$, $j_\theta \equiv j$, $j_{\theta+1} \equiv jj$, then we have $|\xi_\theta - \eta| < \varepsilon$, $\|(\xi_\theta)_l - (\eta)_l\| < \varepsilon$. Let

$$(\eta)_l = z_l + w_l,\ (\xi_\theta) = \tilde{z}_l + \tilde{w}_l$$

$$z_l,\ \tilde{z}_l \in R(R_l),\ w_l,\ \tilde{w}_l \perp R(R_l), \qquad 1 \le l \le p.$$

Therefore

$$\|z_l - \tilde{z}_l\| \le \varepsilon,\ \ \|w_l - \tilde{w}_l\| \le \varepsilon, \qquad 1 \le l \le p. \tag{2.11}$$

Now consider $\xi_{\theta+1} = B_{\theta+1}\xi_\theta$.

1) $\|(\eta)_{jj}\| < |\eta|$.

Suppose $\|(\eta)_{jj}\| = d_{jj}|\eta|$, where $d_{jj} < 1$ is a constant independent of θ. For simplicity of notation, we denote hereafter by d all positive constants smaller than one and independent of θ and ε, and by c all positive constants independent of θ and ε. In the following, the readers can easily distinguish these different d's and c's. Hence we have

$$\|(\xi_{\theta+1})_{jj}\| \le \alpha_{jj}\|(\xi_\theta)_j\| + (1-\alpha_{jj})\|(\xi_\theta)_{jj}\| \le \alpha_{jj}\|(\eta_\theta)_j\| + (1-\alpha_{jj})\|(\eta_\theta)_{jj}\| + \varepsilon$$

$$\le \alpha_{jj}|\eta| + (1-\alpha_{jj})d|\eta| + \varepsilon \le d|\eta| + \varepsilon.$$

The last d depends on η, $\{\alpha_i\}_{i=1}^{p}$.

2) $\|(\eta)_j\| < |\eta|$.

It is similar to the case 1), and

$$\|(\xi_{\theta+1})_{jj}\| \le d|\eta| + \varepsilon$$

where d depends on η, $\{\alpha_i\}_{i=1}^{p}$.

3) $\|(\eta)_{jj}\| = \|(\eta)_j\| = |\eta|$, but $z_{jj} \ne 0$.

$$\begin{aligned}\|(\xi_{\theta+1})_{jj}\| &= \|\alpha_{jj}(\xi_\theta)_j + (1-\alpha_{jj})P_{jj,\,(\xi_\theta)_{jj}}(\xi_\theta)_{jj}\| \\ &\le \alpha_{jj}(|(\eta) + \varepsilon)| + (1-\alpha_{jj})\|P_{jj,\,(\xi_\theta)_{jj}}(\tilde{z}_{jj} + \tilde{w}_{jj}\| \\ &\le \alpha_{jj}|\eta| + (1-\alpha_{jj})\|w_{jj}\| + \gamma\|z_{jj}\|) + c\varepsilon \\ &\le d|\eta| + c\varepsilon\end{aligned}$$

where d and c depend on η, $\{\alpha_i\}_{i=1}^{p}$ and R.

4) $\|\eta_{jj}\| = \|(\eta)_j\| = |\eta|$, $z_{jj} = 0$, but $\eta_{jj} \ne \eta_j$.

$$P_{jj,\,(\xi_\theta)_{jj}}(\xi_\theta)_{jj} = P_{jj,\,(\xi_\theta)_{jj}}(\tilde{z}_{jj} + \tilde{w}_{jj}) = \tilde{w}_{jj} + P_{jj,\,\tilde{z}_{jj}}\tilde{z}_{jj}.$$

From (2.11), we have $\|\tilde{z}_{jj}\| < \varepsilon$, $\|\tilde{w}_{jj} - w_{jj}\| < \varepsilon$. Hence,

$$\|(\xi_{\theta+1})_{jj}\| = \|\alpha_{jj}(\xi_\theta)_j + (1-\alpha_{jj})P_{jj,\,(\xi_\theta)_{jj}}(\xi_\theta)_{jj}\| \le \|\alpha_{jj}(\eta)_j + (1-\alpha_{jj})w_{jj}\| + c\varepsilon$$

$$\le \|\alpha_{jj}(\eta)_j + (1-\alpha_{jj})(\eta)_{jj}\| + c\varepsilon \le d\|\eta\| + c\varepsilon$$

where d depends on η and $\{\alpha_i\}_{i=1}^p$, depends on η, $\{\alpha_i\}_{i=1}^p$ and R.

5) $\eta_{jj} = \eta_j$, $\|\eta_j\| = |\eta|$, $z_{jj} = 0$.

$$\|(\xi_{\theta+1})_{jj} - (\eta)_{jj}\| = \|\alpha_{jj}(\xi_\theta)_j + (1-\alpha_{jj})P_{jj,\,(\xi_\theta)_{jj}}(\xi_\theta)_{jj} - (\eta)_{jj}\|$$
$$\leq \alpha_{jj}\|(\xi_\theta)_j - (\eta)_j\| + (1 - \alpha_{jj})\|P_{jj,\,z_{jj}}\tilde{z}_{jj} + w_{jj} - \tilde{w}_{jj}\| \leq \varepsilon.$$

Repeat the above steps until $\theta + 2T$. Because $\{j_i\}_{j=1}^\infty$ is regulated, if the following conditions are not valid:

i) $(\eta)_i = (\eta)_j$ for $1 \leq i,\ j \leq p$; ii) $(\eta)_i \perp R(R_i)$ for $1 \leq i \leq p$,

then

$$|\xi_{\theta+2T}| \leq d|\eta| + c\varepsilon \tag{2.12}$$

where constants $d < 1$, and c are independent of $\theta = 2T\nu_i$, but depend on T. On the other hand, if i) and ii) hold concurrently, then $(\eta)_i \perp \mathbb{R}$ for $i = 1, \cdots, p$. But this means $(\eta)_i = 0$, and therefore $\eta = 0$. So (2.12) always holds, but for $\eta \neq 0$ and ε small enough. So $\eta = 0$.

In light of the above results, we have $\lim_{i\to\infty} |\xi_i| = 0$, so $\lim_{i\to\infty} \xi_i = 0$. This is equivalent to $\lim_{i\to\infty} x_i = 0$.

§3. Algorithms

In this section, we propose two algorithms for solving (1.1). Partition f into the same block form as R, i.e. $R_j^T x = f_j$. Define

$$Q_j(x) = \begin{cases} (I - \omega R_j C_j^{-1} R_j^T)x + \omega R_j C_j^{-1} f_j, \\ (I - \omega \beta_{j,x}^{-} R_j^T R_j)x + \omega \beta_{j,x}^{-} R_j f_j \end{cases} \tag{3.1}$$

where $\beta_{j,x}^{-}$ is defined as $\beta_{j,x}^{-} = \left(\dfrac{\|f_j - R_j x\|}{\|R_j(f_j - R_j x)\|}\right)^2$, for $j = 1, \cdots, p$. Then the two algorithms are:

Block Chaotic Algorithm:

$$x_i = Q_{j_i}(x_{i-1}) \tag{3.2}$$

where $\{j_i\}_{i=1}^\infty$ is an admissible sequence, $0 < \omega < 1$, $C_j + C_j^T - R_j^T R_j$ is positive definite.

Block Asynchronous Algorithm:

$$x_{i+r_i} = \alpha_{j_i} x_{i+r_i-1} + (1 - \alpha_{j_i}) P_{j_i, x_i} x_i \tag{3.3}$$

where $\{j_i\}_{i=1}^\infty$ is a regulated sequence, $0 < \alpha_j < 1$. $Q_{j,\,x}(x)$ is defined in (3.1). Its parallel implementation is as in [1].

Theorem 3.1. *If (1.1) is consistent, then for any initial vector $x_0 \in \mathbb{C}^m$, iteration (3.1) and (3.4) converge to $\tilde{x}$, the unique minimum norm solution, plus the component of x_o in $N(R)$.*

Proof. Let $\hat{x}_i = x_i - \tilde{x} -$ the component of x_0 in $N(R)$. It is easy to show that (3.2) and (3.3) with initial vector x_0 are equivalent to (2.7) and (2.10) with initial vector $\hat{x}_0$. In light of Theorems 2.5 and 2.6, $\lim_{i\to\infty} \hat{x}_i = 0$. So we obtain the conclusion.

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