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SOLVING INVERSE PROBLEMS FOR HYPERBOLIC EQUATIONS VIA THE REGULARIZATION METHOD*

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Abstract

In the paper, we first deduce an optimization problem from an inverse problem for a general operator equation and prove that the optimization problem possesses a unique, stable solution that converges to the solution of the original inverse problem, if it exists, as a regularization factor goes to zero. Secondly, we apply the above results to an inverse problem determining the spatially varying coefficients of a second order hyperbolic equation and obtain a necessary condition, which can be used to get an approximate solution to the inverse problem.

§1. Introduction

Recently, more attention has been paid to various inverse problems for partial differential equations, which arise in a variety of applications such as heat conduction, blood flow in tumors, seismic data inversion, and flow of fluids in porous media. But most inverse problems are ill-posed in the sense of Hadamard. Many of them have no solution, or their solutions, if existing, are not unique. Besides, the solutions of many inverse problems are unstable. Namely, small variations of the data may produce large variations in the solution.

A general inverse problem we consider in the paper is to determine a parameter $q \in Q$, which is a vector-valued function, satisfying the operator equation

$$\Phi(u, q) = f, \tag{1}$$

on the basis of measurement data

$$z = \Lambda u \in \mathcal{K}, \tag{2}$$

where $\Phi \in C^k(Q \times V, F)$, $C^k(X, Y)$ denotes the Banach space of k-times continuously differentiable mappings on X to Y, X and Y are topological spaces, $f \in F$ is given, $u \in V$ is a state of the system (1), and Q, V, F and $\mathcal{K}$ are topological spaces.

The above-mentioned inverse problem usually is ill-posed in the sense of Hadamard.

The regularization method, introduced by Tikhonov [10] for solving Fredholm integral equation, is one of most popular means to solve ill-posed (in a sense of Hadamard)

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problems. Later, Tikhonov applied the method to various ill-posed problems and summarized some of his results in [11]. J. L. Lions applied that method to optimal control problems of distributed parameter systems governed by partial differential equations [9]. Other papers emphasizing the regularization method include [2,3,6,7,9,13].

In §2 we deduce an optimization problem from the inverse problem via a stabilizing functional. We prove that the optimization problem has a unique, stable solution, and that the solution converges to the true solution of the original inverse problem, if it exists, as a regularization factor goes to zero. Therefore, we can take that solution for an approximate solution to the original inverse problem, if it exists, or for a quasisolution to the original inverse problem, if it does not exist owing to some inexactness of the right term f.

In §3 we make an in-depth study of an inverse problem determining the spatially varying coefficients of a second order linear hyperbolic equation to show implementation of the regularization method solving an inverse problem for partial differential equations. First, we prove that the state of the system, which is the solution of the hyperbolic equation, is a smooth function of the parameter q, which is a vector consisting of the coefficients of the hyperbolic equation. Secondly, we make up a smooth functional with a cost functional and a stabilizing functional and then give a necessary condition, which is a variational inequality and can be applied to computation of the approximate solution of the inverse problem.

§2. The General Inverse Problem

We deduce an optimization problem from the above-mentioned general inverse problem. Suppose that $\forall q \in Q_{ad}$, which is a set in a function space to be defined later, we can get a solution to (1), $u = u(q)$, which denotes the dependence of u on q. Consider the cost functional

$$J_{ls}(q) \equiv \|\Lambda u(q) - z\|_{\mathcal{K}}^2, \tag{3}$$

where $\mathcal{K}$ is the observation space. Obviously, $J_{ls}(q) = 0$ if $(u(q), q)$ is the solution to the problem (1)–(2).

Next, we say a nonnegative, continuous functional, $\psi(q)$, is a stabilizing functional, if for any number $r > 0$ the set $\{q \in Q; \psi(q) \leq r\}$ is compact.

Now, define the smooth functional

$$J_{sm}(q) \equiv J_{ls}(q) + \beta\psi(q), \quad q \in Q_{ad}, \tag{4}$$

where the regularization factor β is a constant, positive number and $\psi(q)$ is a stabilizing functional. We have

Theorem 1. *Let Q, V, and $\mathcal{K}$ be Banach spaces and suppose that the following assumptions hold:*

H1. *$\forall q \in Q_{ad}$ there is a solution $u = u(q) \in V$ to equation* (1) *and $u \in V$ is continuous in $q \in Q$,* H2. *Q_{ad} is a closed set in Q,*

H3. *ψ is a stabilizing functional,*

H4. $\Lambda \in \mathcal{L}(V;\mathcal{K})$.

Then there is an optimal parameter $\hat{q} \in Q_{ad}$ minimizing the smooth functional (4).

Proof. Assume $\{q_n\} \subset Q_{ad}$ such that

$$\lim_{n\to+\infty} J_{sm}(q_n) = \inf_{q\in Q_{ad}} J_{sm}(q) \equiv j. \tag{5}$$

Obviously, $0 \le j < +\infty$. Hence, there is a constant $r > 0$ such that

$$\psi(q_n) \le r.$$

Therefore, by the assumption H3 there exists a subsequence of $\{q_n\}$, $\{q_{nk}\}$, such that

$$q_{nk} \xrightarrow{s} \hat{q}, \text{ in } Q.^{1)} \tag{6}$$

Moreover, $\hat{q} \in Q_{ad}$ by the assumption H2.

By the assumption H1, $\forall\, q_{nk}$ there is the solution $u_{nk} = u(q_{nk})$ of equation (1), so that

$$u(q_{nk}) \xrightarrow{s} u(\hat{q}), \text{ in } V. \tag{7}$$

Therefore, it follows that

$$J_{sm}(q_{nk}) \longrightarrow J_{sm}(\hat{q}). \tag{8}$$

Comparing (5) with (8) we have

$$J_{sm}(\hat{q}) = \inf_{q\in Q_{ad}} J_{sm}(q).$$

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied and that the set of solutions to the inverse problem (1)-(2), $\tilde{S}$, is not empty. Then from the collection of sets*

$$S_\beta \equiv \{\hat{q}_\beta;\ J_{sm,\beta}(\hat{q}_\beta) = j_\beta\}$$

we can select a subsequence $\{\hat{q}_\beta\}$ such that

$$\hat{q}_\beta \xrightarrow{s} \tilde{q}, \text{ in } Q$$

as the regularization factor $\beta \to 0$, where $\tilde{q} \in \tilde{S}$ and

$$J_{sm,\beta}(q) \equiv \|\Lambda u(q) - z\|^2 + \beta\psi(q), \quad j_\beta \equiv \inf_{q\in Q_{ad}} J_{sm,\beta}(q).$$

Proof. Take $\bar{q} \in \tilde{S}$ and we have

$$J_{ls}(\bar{q}) = 0 \le J_{ls}(\hat{q}_\beta) \le J_{ls}(\hat{q}_\beta) + \beta\psi(\hat{q}_\beta) = j_\beta \le J_{sm,\beta}(\bar{q}) = \beta\psi(\bar{q}). \tag{9}$$

Therefore, the set $S \equiv \cup_{0<\beta\le 1} S_\beta$ is bounded and then we can get a subsequence $\{\hat{q}_\beta\}$ from the collection of sets $\{S_\beta\}$ such that

$$\hat{q}_\beta \xrightarrow{s} \tilde{q}, \text{ in } Q, \tag{10}$$

1) "$x_n \xrightarrow{s} x$, in X" means that x_n converges strongly to x in X.

as $\beta \to 0$, and $\bar{q} \in Q_{ad}$ by the assumption **H2**.

Thus,

$$J_{ls}(\hat{q}_\beta) \longrightarrow J_{ls}(\bar{q}), J_{sm,\beta}(\hat{q}_\beta) \longrightarrow J_{ls}(\bar{q}) \tag{11}$$

as $\beta \to 0$. We hope to prove $\bar{q} \in \tilde{S}$, i.e. $\bar{q}$ is a solution to the inverse problem (1)–(2).

In addition, it follows from (9) and (10) that

$$\lim_{\beta \to 0} J_{ls}(\hat{q}_\beta) = J_{ls}(\bar{q}) = 0. \tag{12}$$

Combining (10) with (12) we get $\bar{q} \in \tilde{S}$.

Theorem 3. *Suppose that the assumptions of Theorem 1 are valid and that the following assumption holds:*

H5. *The solution to the inverse problem (1)–(2) is unique.*

Then, for any $\varepsilon > 0$, there are $\delta_1 = \delta_1(\varepsilon, z_o)$ and $\delta_2 = \delta_2(\varepsilon, z_o)$ such that the inequality

$$\|\hat{q} - \hat{q}_o\| < \varepsilon$$

holds, where $\hat{q}$ and $\hat{q}_o$ are optimal parameters minimizing the smooth functional

$$J_{sm,z}(q) \equiv \|\Lambda u(q) - z\|^2 + \beta\psi(q) \tag{13}$$

with z and z_o, respectively, whenever

$$\|z - z_o\| < \delta_1, \quad \textit{and} \quad 0 < \beta < \delta_2.$$

Proof. Set

$$F_\beta \equiv \{z \in \mathcal{K};\ \|z - z_o\|^2 \le \beta\ \},\ \ \forall\beta \in (0,1]. \tag{14}$$

To $z \in F_\beta$ there corresponds the unique solution $\tilde{q} = Az$ to the inverse problem (1)–(2), where A is the inverse operator of P. Set

$$E_\beta \equiv \{\tilde{q} \in Q_{ad};\ \ \tilde{q} = Az,\ z \in F_\beta\}. \tag{15}$$

By Theorem 1, for any $z \in F_\beta$ there is an optimal parameter $\hat{q} \equiv Bz$ minimizing (13), where B is an operator on F_β to G_β, and

$$G_\beta \equiv \{\hat{q} \in Q_{ad};\ \ \hat{q} = Bz,\ z \in F_\beta\}.$$

In order to show the dependence of $J_{sm,z}(q)$ on β we denote

$$J_{sm,z}(q;\beta) \equiv J_{sm,z}(q).$$

Set

$$G \equiv \cup_{\beta\in(0,1]} G_\beta.$$

Then $\forall\hat{q} \in G$ there is β such that $\hat{q} \in G_\beta$; hence, there is $z \in F_\beta$ such that

$$\beta\psi(\hat{q}) \le J_{sm,z}(\hat{q};\beta) \le J_{sm,z}(\tilde{q}_o;\beta) = \|\Lambda u(\tilde{q}_o) - z\|^2 + \beta\psi(\tilde{q}_o) = \|z - z_o\|^2 + \beta\psi(\tilde{q}_o),$$

where $\Lambda u(\tilde{q}_o) = z_o$. Therefore,

$$\psi(\hat{q}) \le \psi(\tilde{q}_o) + 1/\beta\|z - z_o\|^2 \le 1 + \psi(\tilde{q}_o).$$

By the definition of the stabilizing functional ψ we know the set G is compact.

Because the operator P is continuous and one-to-one, the restriction of P to G, P_G, is homomorphic by Tikhonov's lemma.[2)] In other words, the restriction of A to $F' \equiv P(G)$, $A_{F'}$, is continuous, i.e. $\forall\varepsilon > 0$, there is a $\eta = \eta(\varepsilon) > 0$ such that for $q_1, q_2 \in G$ satisfying $\|P(q_1) - P(q_2)\| < \eta$ we have

$$\|q_1 - q_2\| < \varepsilon.$$

Now, to estimate $\|P(\hat{q}) - z\|$ we have

$$\|P(\hat{q}) - z\|^2 = \|\Lambda u(\hat{q}) - z\|^2 \leq J_{sm,z}(\hat{q}) \leq J_{sm,z}(\tilde{q}_o)$$
$$= \|\Lambda u(\tilde{q}_o) - z\| + \beta\psi(\tilde{q}_o) = \|z - z_o\|^2 + \beta\psi(\tilde{q}_o).$$

Therefore, it follows that

$$\begin{aligned}\|P(\hat{q}) - P(\hat{q}_o)\|^2 &= \|\Lambda u(\hat{q}) - \Lambda u(\hat{q}_o)\|^2 = \|\Lambda u(\hat{q}) - z + z - z_o + z_o - \Lambda u(\hat{q}_o)\|^2\\ &\leq 3\{\|\Lambda u(\hat{q}) - z\|^2 + \|z - z_o\|^2 + \|z_o - \Lambda u(\hat{q}_o)\|^2\}\\ &\leq 3\{[\|z - z_o\|^2 + \beta\psi(\tilde{q}_o)] + \|z - z_o\|^2 + [\beta\psi(\tilde{q}_o)]\}\\ &= 6\{\|z - z_o\|^2 + \beta\psi(\tilde{q}_o)\}.\end{aligned}$$

Hence,

$$\|P(\hat{q}) - P(\hat{q}_o)\| \leq \sqrt{6}\sqrt{\|z - z_o\|^2 + \beta\psi(\tilde{q}_o)} \leq \sqrt{(3\|z - z_o\|)^2 + 6\beta\psi(q_o)}$$
$$\leq 3\|z - z_o\| + \sqrt{6\beta\psi(\tilde{q}_o)}.$$

Taking $\delta_2 = \min\{1, \eta^2/(24\psi(\tilde{q}_o))$ and $\delta_1 = \min\{\eta/6, \sqrt{\delta_2}\}$ we have

$$\|P(\hat{q}) - P(\hat{q}_o)\| < \eta.$$

Thus, it follows that

$$\|\hat{q} - \hat{q}_o\| < \varepsilon.$$

§3. An Inverse Problem to a Hyperbolic Equation

Consider the mixed initial-boundary value problem of a hyperbolic equation:

$$\begin{cases} L_q u = f(x,t), \ (x,t) \in D, \quad D \equiv \Omega \times (0,T),\\ u\,|_{\partial\Omega} = 0, \ t \in (0,T),\\ u\,|_{t=0} = u_o(x), \ \partial_t u\,|_{t=0} = u_1(x), \ x \in \Omega, \end{cases} \tag{16}$$

where

$$L_q u \equiv \partial_t^2 u - \partial_i(a_{ij}(x)\partial_j u) + b_i(x)\partial_i u + c(x)u, \tag{17}$$

$$q \equiv \{a_{11}, \cdots, a_{mm}, b_1, \cdots, b_m, c, f\}. \tag{18}$$

[2)] I.e. a one-to-one mapping is a homomorphism if it is continuous and its domain is compact, cf. [11].

An inverse problem associated with (16) is to estimate q on the basis of observation of the state $u(x,t)$:

$$z = \Lambda u. \tag{19}$$

We define the parameter space as

$$Q \equiv \prod_{i,j=1}^{m} W_\infty^1(\Omega) \times \prod_{i=1}^{m} L^\infty(\Omega) \times L^\infty(\Omega) \times W_2^{0,1}(D) \tag{20}$$

with a norm

$$\|q\|_Q \equiv \sum_{i,j=1}^{m} \|a_{ij}\|_{W_\infty^1} + \sum_{i=1}^{m} \|b_i\|_\infty + \|c\|_\infty + \|f\|_2 + \|\partial_t f\|_2, \tag{21}$$

where $W_p^r(\Omega)$ and $W_p^{0,1}(D)$ are Sobolev spaces, of which definitions can be found in [5].

Obviously, Q is a Banach space.

We define the admissible parameter set Q_{ad} as

$$Q_{ad} \equiv \{q \in Q;\ \|q\| \le M,\ \nu|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \forall \xi \in \mathbf{R}^m, \text{a.e. } x \in \Omega\}, \tag{22}$$

where M, μ, and ν are positive constants.

First, we have

Theorem 4. *Suppose $q \in Q_{ad}$ and that the following assumptions hold:*

1. $a_{ij} = a_{ji},\ \forall i,j = 1,\cdots,m,$
2. $\Lambda \in \mathcal{L}(\tilde{V},\mathcal{K})$, *where* $\tilde{V} \equiv W_2^2(D) \cap C(0,T;H_0^1(\Omega))$ *and* $\mathcal{K} \equiv L^2(\Omega)$,
3. $u_o \in H_0^1(\Omega) \cap W_2^2(\Omega)$ *and* $u_1 \in W_2^1(\Omega)$,
4. $\Omega \subset \mathbf{R}^m$ *is bounded,* $\partial\Omega \in C^2$, *where* $\partial\Omega$ *is the boundary of* Ω,

where $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_2^1(\Omega)$.

Then there is a unique solution $u = u(q)$ to (16), $u \in \tilde{V}$, and

$$u \in C^k(Q_{ad};V),\ \ k \in \mathbf{N} \cup \{0\}, \tag{23}$$

where $V \equiv W_2^1(D)$.

Furthermore, if $\delta q = \{\delta a_{11},\cdots,\delta a_{mm},\delta b_1,\cdots,\delta b_m,\delta c,\delta f\} \in Q$, then $\dot{u} \equiv u'(q)\delta q$, where u' is the Fréchet derivative of u at q, is determined by

$$\begin{cases} L_q\dot{u} = \delta f + \partial_i(\delta a_{ij}\,\partial_j u) - \delta b_i\,\partial_i u - \delta c\,u, & (x,t) \in D, \\ \dot{u}\,|_{\partial\Omega} = 0, & t \in (0,T), \\ \dot{u}\,|_{t=0} = 0,\ \ \partial_t\dot{u}\,|_{t=0} = 0, & x \in \Omega, \end{cases} \tag{24}$$

where L_q is defined by (17) and $u = u(q)$.

Proof. By [4] we know that to $q \in Q_{ad}$ there corresponds a unique solution $u = u(q)$ of (16) and $u \in \tilde{V}$.

We prove (23) in the case $k = 0$. The other cases follow by similar reasoning.

Take $q, \tilde{q} \in Q_{ad}$, where $\tilde{q} \equiv \{\tilde{a}_{11},\cdots,\tilde{a}_{mm},\tilde{b}_1,\cdots,\tilde{b}_m,\tilde{c},\tilde{f}\}$, and obtain the solutions corresponding to them:

$$u = u(q),\quad \tilde{u} = u(\tilde{q}). \tag{25}$$

That is, u and $\tilde{u}$ satisfy

$$L_q u \equiv \partial_t^2 u - \partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f, \tag{26}$$

$$L_{\tilde{q}}\tilde{u} \equiv \partial_t^2 \tilde{u} - \partial_i(\tilde{a}_{ij}\partial_j \tilde{u}) + \tilde{b}_i\partial_i \tilde{u} + \tilde{c}\tilde{u} = \tilde{f} \tag{27}$$

respectively, and the same initial-boundary value conditions. Set

$$v \equiv \tilde{u} - u, \quad \delta q \equiv \tilde{q} - q.$$

Then $v \in \tilde{V}$ satisfies

$$\begin{cases} L_{\tilde{q}} v = \delta f + \partial_i(\delta a_{ij}\partial_j u) - \delta b_i \partial_i u - \delta c u, \\ v \mid_{\partial\Omega} = 0, \\ v \mid_{t=0} = 0, \quad \partial_t v \mid_{t=0} = 0. \end{cases} \tag{28}$$

Now, set $w \equiv h^{-1} v$, where $h \equiv d - e^{\lambda x_1}$. Here d and λ are constants to be determined later. This leads us to

$$\begin{cases} \tilde{L}_{\tilde{q}} w = h^{-1}[\delta f + \partial_i(\delta a_{ij}\partial_j u) - \delta b_i \partial_i u - \delta c u], \\ w \mid_{\partial\Omega} = 0, \\ w \mid_{t=0} = 0, \quad \partial_t w \mid_{t=0} = 0, \end{cases} \tag{29}$$

where

$$\tilde{L}_{\tilde{q}} w \equiv \partial_t^2 w - \partial_i(\tilde{a}_{ij}\partial_j w) + \tilde{b}_i \partial_i w + \tilde{c} w + [\tilde{a}_{1i}\partial_i w + \partial_i(\tilde{a}_{i1} w) + (\lambda \tilde{a}_{11} - \tilde{b}_1) w]\lambda h^{-1} e^{\lambda x_1}.$$

Multiply the two sides of (29) by $w_t \equiv \partial_t w$, and integrate them over $D_t \equiv \Omega \times (0, t)$. Using Green's formula and considering $w_t \mid_{\partial\Omega} = \partial_t(w \mid_{\partial\Omega}) = 0$ we get

$$\begin{aligned} 1/2|w_t(t)|_2^2 &+ \int_{D_t} \{\tilde{a}_{ij}\partial_i w \partial_j w_t + \tilde{b}_i \partial_i w w_t + \tilde{c} w w_t\} \\ &+ \int_{D_t} \{(\lambda \tilde{a}_{11} - \tilde{b}_1) w w_t + \tilde{a}_{1i}\partial_i w w_t + \partial_i(\tilde{a}_{i1} w) w_t\}\lambda e^{\lambda x_1} h^{-1} \\ &= \int_{D_t} h^{-1}\{\delta f + \partial_i(\delta a_{ij}\partial_j u) - \delta b_i \partial_i u - \delta c u\} w_t, \end{aligned} \tag{30}$$

where $|w_t(t)|_2^2 \equiv \int_\Omega |w_t(x,t)|^2 dx$. Then

$$\begin{aligned} |w_t(t)|_2^2 &+ \int_\Omega \{\tilde{a}_{ij}\partial_i w \partial_j w + \lambda \tilde{a}_{11} w^2 \lambda e^{\lambda x_1} h^{-1}\}\Big|_t + \int_\Omega \{\tilde{c} + (\partial_i \tilde{a}_{i1} - \tilde{b}_1)\lambda e^{\lambda x_1} h^{-1}\} w^2 \Big|_t \\ &+ 2\int_{D_t} \{\tilde{b}_i + 2\tilde{a}_{i1}\lambda e^{\lambda x_1} h^{-1}\}\partial_i w \partial_t w \\ &= 2\int_{D_t} h^{-1}\{\delta f + \partial_i(\delta a_{ij}\partial_j u) - \delta b_i \partial_i u - \delta c u\}\partial_t w. \end{aligned} \tag{31}$$

It is easy to obtain the following estimates:

$$\left|-\int_{D_t}\tilde{b}_1 w^2\right| \le M\|w\|_{2,t}^2, \quad \left|\int_{D_t}\partial_i\tilde{a}_{i1}w^2\right| \le Mm\|w\|_{2,t}^2,$$

$$\left|\int_{D_t}\tilde{c}w^2\right| \le M\|w\|_{2,t}^2, \quad \left|\int_{D_t}2\tilde{b}_i\partial_i w\partial_t w\right| \le M(\|\nabla w\|_{2,t}^2 + m\|\partial_t w\|_{2,t}^2),$$

$$\left|\int_{D_t}2\tilde{a}_{i1}\partial_i w\partial_t w\right| \le M(\|\nabla w\|_{2,t}^2 + m\|\partial_t w\|_{2,t}^2), \tag{32}$$

where $\|\psi\|_{2,t}^2 \equiv \int_0^t\int_\Omega |\psi|^2 dxdt$. Suppose

$$\tilde{x}_1 \equiv \sup\{x_1;\ (x_1,\cdots,x_m)\in\Omega\}, \quad \hat{x}_1 \equiv \inf\{x_1;\ (x_1,\cdots,x_m)\in\Omega\}.$$

Note that we can assume $\tilde{x}_1 < 0$ without loss of generality.

Because $e^{\lambda x_1}h^{-1}(x_1)$ is increasing in x_1, we can take $k(> 2)$ such that $ke^{\lambda\hat{x}_1} > e^{\lambda x_1} + 1$. Then take d such that $ke^{\lambda\hat{x}_1} \ge d \ge e^{\lambda\tilde{x}_1} + 1$. Thus, $h \ge 1$, $e^{\lambda x_1}h^{-1}(x_1) \ge 1/(k-1)$, and $\lambda e^{\lambda x_1} \le 1/|x_1| \le c_2$, where c_2 is dependent only on Ω.

Therefore, by the assumption of Q_{ad} and (32) the above argument leads us to

$$\text{LHS of (31)} \ge |\partial_t w(t)|_2^2 + \nu|\nabla w(t)|_2^2 + \lambda\nu/(k-1)|w(t)|_2^2$$

$$- M|w(t)|_2^2 - (m+1)Mc_2|w(t)|_2^2 - \int_0^t 3Mc_2(|\nabla w(t)|_2^2 + m|\partial_t w(t)|_2^2)dt$$

$$= |\partial_t w(t)|_2^2 + \nu|\nabla w(t)|_2^2 + [\lambda\nu/(k-1) - M - (m+1)Mc_2]|w(t)|_2^2$$

$$- 3Mc_2\int_0^t(|\nabla w(t)|_2^2 + m|\partial_t w(t)|_2^2)dt.$$

Take λ so large that $\lambda\nu/(k-1) - M - (m+1)Mc_2 \ge c_1 \equiv \min\{1,\nu\}$. Then it follows from the above that

$$\text{LHS of (31)} \ge c_1\{|\partial_t w(t)|_2^2 + |\nabla w(t)|_2^2 + |w(t)|_2^2\}$$

$$- 3mMc_2\int_0^t\{|\partial_t w(t)|_2^2 + |\nabla w(t)|_2^2 + |w(t)|_2^2\}dt$$

$$= c_1|||w(t)|||^2 - 3mMc_2\int_0^t |||w(t)|||^2 dt, \tag{33}$$

where $|||w(t)|||^2 \equiv |\partial_t w(t)|_2^2 + |\nabla w(t)|_2^2 + |w(t)|_2^2$.

By the assumption of Q_{ad} and the energy inequality of hyperbolic equations we have

$$\|u\|_{\tilde{V}} \le C, \tag{34}$$

where C is independent of $q \in Q_{ad}$.

Therefore, by the Cauchy inequality we obtain

$$\text{RHS of (31)} \le CM \int_{D_t} \Big\{ |\delta f|^2 + \sum_{ij} (|\delta a_{ij}|^2 + |\partial_i \delta a_{ij}|^2) + \sum_i |\delta b_i|^2 + |\delta c|^2 \Big\} dt$$

$$+ \int_0^t |w_t(t)|_2^2 dt \le c_3 \int_{D_t} |\delta f|^2 + c_4 t \Big\{ \sum_{ij=1}^m \|\delta a_{ij}\|^2_{W^1_\infty(\Omega)} + \sum_{i=1}^m \|\delta b_i\|_2^2$$

$$+ \|\delta c\|_2^2 \Big\} + \int_0^t |\partial_t w(t)|_2^2 dt, \qquad \text{a.e. } t \in (0,T).$$

Combining the above with (33) we get

$$c_1 \|w(t)\|^2 - 4mMc_2 \int_0^t |||w(t)|||^2 dt \le c_3 \int_{D_t} |\delta f|^2 dx dt + c_4 t \Big\{ \sum_{i,j=1}^m \|\delta a_{ij}\|^2_{W^1_\infty(\Omega)}$$

$$+ \sum_{i=1}^m \|\delta b_i\|_2^2 + \|\delta c\|_2^2 \Big\} + \int_0^t |\partial_t w(t)|_2^2 dt, \quad \text{a.e. } t \in (0,T).$$

Let $y(t) \equiv \int_0^t |||w(t)|||^2 dt$, and in view of the definition of Q, (20), this is equivalent to

$$\frac{dy(t)}{dt} - c_5 y(t) \le c_6 \|\tilde{q} - q\|_Q^2, \ \text{ a.e. } \ t \in (0,T).$$

Multiply the above by $e^{-c_5 t}$ and integrate it over $(0,t)$:

$$\int_0^t |||w(t)|||^2 \, dt \le c_6 \|q' - q\|_Q^2 e^{c_5 T}, \ \text{ a.e. } \ t \in (0,T).$$

Hence,

$$\|w\|_V^2 \le c_6 e^{c_5 T} \|\tilde{q} - q\|_Q^2.$$

So, the solution $u(q)$ of the problem (16) depends continuously on q.

Finally, we prove (24).

Suppose $\tilde{q} = q + \delta q$. Therefore, $\tilde{u} = u(\tilde{q})$ satisfies

$$\mathcal{L}_{\tilde{q}} \tilde{u} = \tilde{f}, \ \ \tilde{u}\, |_{\partial\Omega} = 0, \ \ \tilde{u}\, |_{t=0} = u_0, \ \ \partial_t \tilde{u}\, |_{t=0} = u_1.$$

So, $v \equiv \tilde{u} - u$ satisfies (28). Thus $w \equiv \tilde{u} - u - \dot{u}$ satisfies

$$\begin{cases} \mathcal{L}_q w = \partial_i(\delta a_{ij} \partial_j v) - \delta b_i \partial_i v - \delta c v, \\ w\, |_{\partial\Omega} = 0, \\ w\, |_{t=0} = 0, \ \ \partial_t w\, |_{t=0} = 0. \end{cases} \tag{35}$$

By the energy inequality of a hyperbolic equation [4] we can obtain from (28)

$$v = O(\|\delta f + \partial_i(\delta a_{ij}\, \partial_j u) - \delta b_i\, \partial_i + \delta c\, u\|).$$

So, it follows from the above and (34) that

$$v = O(\|\delta q\|),$$

and, similarly, from (35) we have

$$w = O(\|\delta q\|^2).$$

Thus, $\dot{u} = u'(q)\delta q$.

Consider $\psi(q)$, given by

$$\psi(q) \equiv \sum_{|k|=0}^{l} \beta_k \|D_x^k q\|_2^2,$$

where $D_x^k = \partial_1^{k_1} \cdots \partial_m^{k_m}$, $|k| = k_1 + \cdots + k_m$, and l satisfies the inequality $l > m/2$.

Moreover, suppose that

$$\tilde{Q} \equiv [W_2^l(\Omega)]^{m^2+m+1} \times W_2^l(D)$$

and

$$\tilde{Q}_{ad} \equiv \{q \in Q_{ad};\ \|q\|_{\tilde{Q}} \le M\}, \tag{36}$$

where Q_{ad} is defined by (22). It follows from the Rellich-Kondrachov compactness theorem in Sobolev spaces [1] that $\tilde{Q}_{ad}$ is compact in Q, which is defined by (20). Therefore, $\psi(q)$ is a stabilizing functional. The observation space $\mathcal{K} \equiv L^2(\Omega)$ and the measurement is

$$z(x) = u(x,T), \quad x \in \Omega. \tag{37}$$

Altogether, the smooth functional $J_{sm,\beta}(q)$ is defined by

$$J_{sm,\beta}(q) \equiv \int_\Omega |z - u(T;q)|^2 dx + \beta\psi(q). \tag{38}$$

Theorem 5. *Suppose that the assumptions of Theorem 4 are satisfied and that $z \in C(\Omega) \subset \mathcal{K}$. Then a necessary condition for q_β to be an optimal parameter minimizing (38) over $\tilde{Q}_{ad}$ is that the variational inequality holds:*

$$\begin{aligned} &\int_D \delta a_{ij}\partial_j u \partial_i v + \int_D (\delta b_i \partial_i u + \delta c u - \delta f)v + \sum_{|k|=0}^{l} \beta_{|k|}(D_x^k a_{ij}^\beta, D_x^k \delta a_{ij}) \\ &+ \sum_{|k|=0}^{l} \beta_{|k|}(D_x^k b_i^\beta, D_x^k \delta b_i) + \sum_{|k|=0}^{l} \beta_{|k|}(D_x^k c^\beta, D_x^k \delta c) \\ &+ \sum_{|k|=0}^{l} \beta_{|k|}((D_x^k f^\beta, D_x^k \delta f)) \ge 0, \quad \forall q \in \tilde{Q}_{ad},\ \ \delta q \equiv q - q_\beta, \end{aligned} \tag{39}$$

where $u = u(q_\beta)$ is defined by (16), $q_\beta \equiv \{a_{11}^\beta, \cdots, a_{mm}^\beta, b_1^\beta, \cdots, b_m^\beta, c^\beta, f^\beta\}$, $\delta q \equiv \{\delta a_{11}, \cdots, \delta a_{mm}, \delta b_1, \cdots, \delta b_m, \delta c, \delta f\}$,

$$(h,g) \equiv \int_\Omega h(x)g(x)dx, \quad ((a,b)) \equiv \int_D a(x,t)b(x,t)dx\,dt,$$

and $v = v(q_\beta)$ is governed by

$$\begin{cases} L_q^* v \equiv \partial_t^2 v - \partial_j(a_{ij}^\beta \partial_i v) - \partial_i(b_i^\beta v) + c^\beta v = 0, \\ v \mid_{\partial\Omega} = 0, \\ v \mid_{t=T} = 0, \quad \partial_t v \mid_{t=T} = u(T;q_\beta) - z. \end{cases} \tag{40}$$

Proof. By Theorem 4 $J_{sm,\beta}(q)$ is Fréchet-differentiable. If q_β is an optimal parameter minimizing $J_{sm,\beta}(q)$ over $\tilde{Q}_{ad}$, then

$$J_{sm,\beta}(q) \geq J_{sm,\beta}(q_\beta), \quad \forall\, q \in \tilde{Q}_{ad}.$$

Because $\tilde{Q}_{ad}$ is convex, we have

$$J_{sm,\beta}(q_\beta + \theta(q - q_\beta)) \geq J_{sm,\beta}(q_\beta), \quad \forall\, \theta \in (0,1)$$

for fixed $q \in \tilde{Q}_{ad}$. Hence,

$$1/\theta\{J_{sm,\beta}(q_\beta + \theta(q - q_\beta)) - J_{sm,\beta}(q_\beta)\} \geq 0, \quad \forall\, \theta \in (0,1).$$

Let $\theta \to +0$. Then

$$J'_{sm}(q_\beta)\delta q \geq 0, \quad q \in \tilde{Q}_{ad}, \ \ \delta q \equiv q - q_\beta. \tag{41}$$

Thus,

$$\begin{aligned} &-\int_\Omega [z - u(T; q_\beta)]u'(q_\beta)\delta q dx + \sum_{|k|=0}^{l} \beta_{|k|}(D_x^k a_{ij}^\beta, D_x^k \delta a_{ij}) \\ &+ \sum_{|k|=0}^{l} \beta_{|k|}(D_x^k b_i^\beta, D_x^k \delta b_i) + \sum_{|k|=0}^{l} \beta_{|k|}(D_x^k c^\beta, D_x^k \delta c) \\ &+ \sum_{|k|=0}^{l} \beta_{|k|}((D_x^k f^\beta, D_x^k \delta f)) \geq 0, \quad \forall q \in \tilde{Q}_{ad}, \ \ \delta q \equiv q - q_\beta. \end{aligned} \tag{42}$$

Next, denoting $\dot{u} \equiv u'(q_\beta)\delta q$, from Theorem 4 we have

$$\begin{cases} L_{q_\beta}\dot{u} = \delta f + \partial_i(\delta a_{ij}\partial_j u(q_\beta)) - \delta b_i \partial_i u(q_\beta) - \delta c u(q_\beta), \\ \dot{u}\,|_{\partial\Omega} = 0, \\ \dot{u}\,|_{t=0} = 0, \quad \dot{u}\,|_{t=0} = 0. \end{cases} \tag{43}$$

By [4] the problem (40) has a unique solution $v = v(q_\beta) \in \hat{V} \equiv H^2(D)$. Also, there is a unique solution $\dot{u}$ to (43). Therefore,

$$\begin{aligned} -\int_\Omega [z - u(T; q_\beta)]u'(q_\beta)\delta q \, dx &= \int_\Omega \partial_t v(T; q_\beta)\dot{u}(T; q_\beta)\, dx \\ &= \int_D \partial_t\{\partial_t v(q_\beta)\dot{u}(q_\beta)\}\, dx = \int_D \{\partial_t^2 v(q_\beta)\dot{u}(q_\beta) + \partial_t v(q_\beta)\partial_t \dot{u}(q_\beta)\} \\ &= \int_D \{\partial_j(a_{ij}^\beta \partial_i v) + \partial_i(b_i^\beta v) - c^\beta v\}\dot{u} + \int_\Omega v\partial_t \dot{u}\,|_{t=0}^T - \int_D v\partial_t^2 \dot{u} \\ &= -\int_D \{a_{ij}^\beta \partial_i v \partial_j \dot{u} + b_i^\beta \partial_i \dot{u}\, v + c^\beta v\dot{u}\} - \int_D v\partial_t^2 \dot{u} \\ &= -\int_D v\{\partial_t^2 \dot{u} - \partial_i(a_{ij}^\beta \partial_j \dot{u}) + b_i^\beta \partial_i \dot{u} + c^\beta \dot{u}\} \\ &= -\int_D v\{\delta f + \partial_i(\delta a_{ij}\partial_j u(q_\beta)) - \delta b_i \partial_i u(q_\beta) - \delta c u(q_\beta)\} \end{aligned}$$

$$= \int_D \{\delta a_{ij} \partial_j u(q_\beta) \partial_i v + \delta b_i \partial_i u(q_\beta)\, v + \delta c\, u(q_\beta) v - \delta f\, v\}. \tag{44}$$

Substituting (44) into (42) we obtain (39).

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