Journal of Computational Mathematics, Vol.11, No.2, 1993, 103–112.

AN INVERSE PROBLEM FOR THE BURGERS EQUATION*

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Abstract

In this paper the Generalized Pulse-Spectrum Technique (GPST) is extended to solve an inverse problem for the Burgers equation. We prove that the GPST is equivalent in some sense to the Newton-Kantorovich iteration method. A feasible numerical implementation is presented in the paper and some examples are excuted. The numerical results show that this procedure works quite well.

§1. Introduction

In this paper, we shall consider an inverse problem of the Burgers equation[1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) = f. \tag{1.1}$$

We assume $f \neq 0$ in order to obtain exact solutions which will be used to compare with our numerical results. The coefficient function $\nu = \nu(x)$ has practical senses as conductivity or viscosity, etc[2,3]. Our purpose in the paper is to investigate the identification of the coefficient $\nu(x)$ through the Burgers equation and some initial and boundary conditions. This inverse problem has obviously both theoretical interest and practical importance.

In the next section we shall present an iterative algorithm for identifying numerically the coefficient $\nu(x)$ by using the Generalized Pulse-Spectrum Technique (GPST). The GPST has been applied to many inverse problems and proved to be a versatile and efficient numerical algorithm[4–6]. In Section 3, we consider the same problem as an inverse problem of an abstract operator equation and prove that the algorithm presented in the previous section is equivalent in some sense to the Newton-Kantorovich iterative method. Numerical implementation of the algorithm is discussed in Section 4. Finite difference methods are applied to both the direct and inverse problems, which result in a linear system containing $\Delta\nu$, the improvement of the approximation of ν, as its

* Received February 11, 1987. Revised manuscript received May 8, 1992.

unknown. Some regularization methods are used to treat the ill-posedness of the inverse problem. Several examples are given in Section 5, which show that the numerical algorithm presented in the paper works very well. Finally, a brief discussion of the algorithm and its perfoemance is given.

§2. The Numerical Algorithm

Consider the following initial-boundary value problem of the Burgers equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} - \dfrac{\partial}{\partial x}\Big(\nu(x)\dfrac{\partial u}{\partial x}\Big) = f, \\ u(x,0) = U_0(x), & 0 < x < 1,\ 0 < t \le T. \\ u(0,t) = r_0(t), \quad u(1,t) = r_1(t), \end{cases} \tag{2.1}$$

In order to identify $\nu(x)$ through equation (2.1) we need some auxiliary condition which we assume in the paper as follows

$$Bu(x,t) = \frac{\partial}{\partial x}u(0,t) = r(t). \tag{2.2}$$

The GPST algorithm is used to solve numerically the above inverse problem. First choose a function, say $\nu_0(x)$, as the initial approximation of $\nu(x)$ and then use the procedure described below to obtain the first approximation $\nu_1(x)$.

Suppose that the nth iterative approximation of $\nu(x)$, say $\nu_n(x)$, has been obtained and that $u_n(x,t)$ is the corresponding solution to (2.1) with $\nu(x)$ replaced by $\nu_n(x)$; this means u_n is the solution to the following equation,

$$\begin{cases} \dfrac{\partial u_n}{\partial t} + u_n\dfrac{\partial u_n}{\partial x} - \dfrac{\partial}{\partial x}\Big(\nu_n(x)\dfrac{\partial u_n}{\partial x}\Big) = f, \\ u_n(x,0) = U_0(x), \\ u_n(0,t) = r_0(t), \quad u_n(1,t) = r_1(t). \end{cases} \tag{2.3}$$

Assume that the $(n+1)$st approximation $\nu_{n+1}(x)$ and the corresponding $u_{n+1}(x,t)$ are as follows

$$\nu_{n+1}(x) = \nu_n(x) + \delta\nu_n(x), \tag{2.4}$$

$$u_{n+1}(x,t) = u_n(x,t) + \delta u_n(x,t). \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.1) and substracting (2.3) from it, we have

$$\frac{\partial \delta u_n}{\partial t} + u_n\frac{\partial \delta u_n}{\partial x} + \delta u_n\frac{\partial u_n}{\partial x} + \delta u_n\frac{\partial \delta u_n}{\partial x} - \frac{\partial}{\partial x}\Big[\nu_n\frac{\partial \delta u_n}{\partial x} + \delta\nu_n\frac{\partial u_n}{\partial x} + \delta\nu_n\frac{\partial \delta u_n}{\partial x}\Big] = 0.$$

Suppose that the magnitudes of terms $\delta\nu_n, \delta u_n$ and $\partial\delta u_n/\partial x$ are small and their products can be neglected; thus we obtain a system for δ-terms,

$$\begin{cases} \frac{\partial \delta u_n}{\partial t} + \frac{\partial}{\partial x}(u_n \cdot \delta u_n) - \frac{\partial}{\partial x}\left(\nu_n \frac{\partial \delta u_n}{\partial x}\right) = \frac{\partial}{\partial x}\left(\delta \nu_n \cdot \frac{\partial u_n}{\partial x}\right), \\ \delta u_n|_{t=0} = 0, \\ \delta u_n|_{x=0} = \delta u_n|_{x=1} = 0. \end{cases} \tag{2.6}$$

If u_{n+1} is approximately regarded as the exact solution to equation (2.1), then we derive an auxiliary condition for δu_n from (2.5) and condition (2.2) as

$$\frac{\partial \delta u_n(0,t)}{\partial x} = r(t) - \frac{\partial u_n(0,t)}{\partial x}. \tag{2.7}$$

Let $G_n(x,t;\xi,\tau)$ be Green's function of the differential operator on the left-hand side of equation (2.6). Then equation (2.6) can be transformed to an integral relation

$$\delta u_n(x,t) = \int_0^t \int_0^1 G_n(x,t;\xi,\tau)\frac{\partial}{\partial \xi}\left[\delta \nu_n(\xi)\frac{\partial u_n(\xi,\tau)}{\partial \xi}\right] d\xi d\tau. \tag{2.8}$$

For simplicity, we assume that $\nu(0)$ and $\nu(1)$ are known, therefore $\delta\nu_n(0,t) = \delta\nu_n(1,t) = 0$. Integrating the right-hand term of (2.8) by parts and using the auxiliary condition (2.7), the integral relation (2.8) can be changed to

$$\int_0^t \int_0^1 \frac{\partial^2}{\partial x \partial \xi} G_n(0,t;\xi,\tau)\frac{\partial u_n(\xi,\tau)}{\partial \xi}\delta\nu_n(\xi) d\xi d\tau = \frac{\partial u_n(0,t)}{\partial x} - r(t). \tag{2.9}$$

(2.9) is a Fredholm integral equation of the first kind for unknown function $\delta\nu_n(\xi)$. In general, it is ill-posed so some kind of regularization methods must be used to solve it.

Equations (2.3), (2.4) and (2.9) from the basic structure for each iteration in the numerical algorithm of the GPST.

§3. The Operator Equation

We shall now consider the same problem from a different view-point. Let ν_-, ν_+ and $\bar{\nu}$ be three positive constants with $\nu_- < \nu_+$ and Σ_0, Σ_1 be two sets of functions defined by

$$\Sigma_0 = \{\nu(x) |\ \nu(x) \in C[0,1],\ \nu_- < \nu(x) < \nu_+\}, \tag{3.1}$$

$$\Sigma_1 = \{\nu(x) \in \Sigma_0 \cap H^1[0,1],\ \|\nu'\| < \bar{\nu}\}. \tag{3.2}$$

Here and afterwards " , " denotes the derivative w.r.t. x and $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the norms in spaces $L^2[0,1]$ and $L^\infty[0,1]$, respectively. We consider the generalized solution to equation (2.1). Let

$$\begin{cases} a(x,t) = (1-x)r_0(t) + xr_1(t), \\ u(x,t) = v(x,t) + a(x,t) \end{cases} \tag{5.3}$$

then $v(x,t)$ satisfies the following equaiton

$$\begin{cases} \dfrac{\partial v}{\partial t} + vv' + (av)' - (\nu v')' = F + (\nu a')', \\ v(x,0) = U_0(x) - [\Gamma_0(0) + (r_1(0) - r_0(0))x], \\ v(0,t) = v(1,t) = 0 \end{cases} \tag{3.4}$$

where

$$F = F(x,t) = f(x,t) - \frac{\partial a}{\partial t} - aa'. \tag{3.5}$$

Definition. *If $v \in L^2[0,T;H_0^1]$ satisfies the following equation*

$$\frac{d}{dt}(v,w) + (vv',w) - (av,w') + (\nu v',w') = (F,w) - (\nu a',w'),$$

$$\forall\, w \in H_0^1[0,1], \tag{3.6}$$

$$v(\cdot,0) = U_0 - [r_0(0) - (r_1(0) - r_0(0))x] \tag{3.7}$$

then v is called a generalized solution to (3.4) and u given by (3.3) is called a generalized solution to (2.1), $(\cdot,\cdot)$ is the inner product in space $L^2[0,1]$.

Lemma 1. *Suppose the following conditions hold*

1° $\nu(x) \in \Sigma_0$,

2° $f \in L^2[0,T;L^2[0,1]]$, $U_0 \in L^2[0,1]$, r_0, $r_1 \in H^1[0,T]$.

Then there exists a unique generalized solution to equation (2.1). Moreover, u is almost everywhere equal to a function continuous from $[0,T]$ into $L^2[0,1]$ and that

$$u(\cdot,t) \to U_0 \text{ in } L^2[0,1] \text{ as } t \to 0.$$

Lemma 2. *Suppose the following conditions hold*

1° $\nu(x) \in \Sigma_1$,

2° $f \in H^1[0,T;L^2[0,1]]$, $U_0 \in H^2[0,1]$, $r_0, r_1 \in H^2[0,T]$.

Then the unique generalized solution to (2.1) satisfies

$$\frac{\partial u}{\partial t} \in L^2[0,T;H^1[0,1]] \cap L^\infty[0,T;L^2[0,1]]. \tag{3.8}$$

These two lemmas can be proved by using the methods developed in [7] or [8]. Due to a lemma in [7], (3.8) implies that

$$u \in L^\infty[0,T;H^1[0,1]] \tag{3.9.}$$

For fixed f, U_0, r_0 and r_1, equation (2.1) defines a map from Σ_0 or Σ_1 into $L^2[0,T;H^1]$ as

$$A(\nu) = u. \tag{3.10}$$

Hence the inverse problem discussed in Section 2 can be viewed as the following nonlinear operator equation

$$T(\nu) = B \cdot A(\nu) = r. \tag{3.11}$$

We show now that the derivative of operator T exists in some sense. We have following results.

Lemma 3. *Suppose the following conditions hold*

1° $\nu_1, \nu_2 \in \Sigma_1$,

2° *as* 2° *in Lemma* 2.

Then

$$\|\Delta u(t)\|^2 + \nu_- \int_0^t \|\Delta u'(\tau)\|^2 d\tau \le \frac{2}{\nu_-}\|\Delta\nu\|_\infty^2 \int_0^t \|u_2'(\tau)\|^2 \exp\left\{\frac{2C_1(t-\tau)}{\nu_-}\right\} d\tau \quad (3.12)$$

where $\Delta\nu = \nu_2 - \nu_1, \Delta u = u_2 - u_1$, $C_1 = \operatorname*{esssup}_{t\in[0,T]} \|u_1\|_\infty^2 < \infty$.

Lemma 4. *Suppose the conditions of Lemma* 3 *hold, then*

$$\operatorname*{esssup}_{t\in[0,T]} \|\Delta v(t)\|^2 \le C_2\|\Delta\nu\|_\infty^2, \quad \int_0^T \|\Delta v'(t)\|^2 dt \le C_2\|\Delta\nu\|_\infty^2 \quad (3.13)$$

where $\Delta v = \partial\Delta u/\partial t, C_2$ *is some positive constant.*

Lemma 5. *If the conditions of Lemma* 3 *are satisfied, then*

$$\operatorname*{esssup}_{t\in[0,T]} \|\Delta u'(t)\|^2 \le C_3\|\Delta\nu\|_\infty^2 \quad (3.14)$$

with some constant $C_3 > 0$.

From these lemmas we obtain the following continuity of map A.

Theorem 1. *Suppose the conditions of Lemma* 2 *hold, then*

$$\|A(\nu_1) - A(\nu_2)\|_{L^\infty[0,T;H^1]} = \|\Delta u\|_{L^\infty[0,T;H^1]} \le C_4\|\Delta\nu\|_\infty \quad (3.15)$$

where C_4 *is a constant independent of* ν_1 *and* ν_2.

We shall now consider the differentiability of map A. Let function $p(x,t)$ be the solution to the following linear equaiton

$$\begin{cases} \dfrac{\partial p}{\partial t} + (up)' - (\nu p')' = (\Delta\nu \cdot u')', \\ p(x,0) = 0, \\ p(0,t) = p(1,t) = 0 \end{cases} \quad (3.16)$$

where $\nu, \nu + \Delta\nu \in \Sigma_1$ and $u, u + \Delta u$ are the corresponding solution to (2.1), respectively. Set $q(x,t) = \Delta u(x,t) - p(x,t)$, then q satisfies the following equation

$$\frac{\partial q}{\partial t} + (uq)' - (\nu q')' = (\Delta\nu \cdot \Delta u')' - \Delta u \cdot \Delta u' \quad (3.17)$$

and the homogeneous initial and boundary conditions. By using the same argument we come to the conclusion that

$$\operatorname*{esssup}_{t\in[0,T]} \left(\|q(t)\| + \left\|\frac{\partial q(t)}{\partial t}\right\| + \|q'(t)\|\right) \le C_5\|\Delta\nu\|_\infty^2 \quad (3.18)$$

and thus we obtain the following result.

Theorem 2. *Suppose that the conditions of Lemma 2 hold, ν and $\nu + \Delta\nu \in \Sigma_1$, and that p is the solution to* (3.16). *Then we have*

$$\|A(\nu + \Delta\nu) - A(\nu) - p\|_{L^\infty[0,T;H^1]} = O(\|\Delta\nu\|_\infty^2), \quad as \ \|\Delta\nu\|_\infty \to 0.$$

Since Σ_1 is an open domain in space $H^1[0,1]$ Theorems 1 and 2 imply that $A(\nu)$ regarded as a map from $\Sigma_1 \subset H^1[0,1] \to L^\infty[0,T;H^1]$ is uniformly Lipschitz continuous and Fréchet differentiable on Σ_1. More precisely, we have

$$A'(\nu) \cdot h = p, \quad \forall h \in H^1[0,1] \tag{3.19}$$

where p is the solution to equation (3.16) with $\Delta\nu$ replaced by h. We can also regard $A(\nu)$ as a map from $\Sigma_1 \subset C[0,1] \to L^\infty[0,1;H^1]$. In this time $A(\nu)$ is still continuous and Fréchet differentiable on Σ_1. However, Σ_1 is now not an open domain in $C[0,1]$.

When applying the Newton-Kantorovich iteration method to the operator equation (3.11), we get

$$B \cdot A'(\nu_n)(\nu_{n+1} - \nu_n) = r - B \cdot A(\nu_n). \tag{3.20}$$

By comparing (3.20) with Section 2, one can see that this is just the algorithm of the GPST.

Equation (3.20) should be understood in the sense of generalized function. In order that it has the normal sense, one needs more regularity of u and p, e.g., u and $p \in C^1(\bar{Q}), \bar{Q} = [0,1] \times [0,T]$. These properties can be acheived by increasing the smoothness of $\nu(x), f(x,t)$ and other data[9].

§4. Numerical Implementation

In order to avoid the difficulties of calculating the Green funciton, the procedure in [6] is adopted. Let $h = 1/M, \tau = T/N$. We use the following notations

$$v_{x,i}^j = \frac{1}{h}(v_{i+1}^j - v_i^j), \qquad v_{\bar{x},i}^j = \frac{1}{h}(v_i^j - v_{i-1}^i),$$

$$v_{\hat{x},i}^j = \frac{1}{2h}(v_{i+1}^j - v_{i-1}^j), \qquad v_{t,i}^j = \frac{1}{\tau}(v_i^{j+1} - v_i^i), \tag{4.1}$$

$$J(u_i^j, v_i^j) = \frac{1}{3}[v_i^j u_{\hat{x},i}^j + (vu)_{\hat{x},i}^j]. \tag{4.2}$$

(4.2) is the difference approximation of the nonlinear term.

For solving (2.3), the following difference scheme is used (for simplicity, all subscripts n are omitted)

$$u_{t,i}^j + \frac{1}{4}J(u_i^j + u_i^{j+1}, u_i^j + u_i^{j+1}) - \frac{1}{4}[(\nu u_x^{j+1})_{\bar{x},i} + (\nu u_{\bar{x}}^{j+1})_{x,i} + (\nu u_x^j)_{\bar{x},i} + (\nu u_{\bar{x}}^j)_{x,i}]$$
$$= f_i^{j+1/2}, \quad i = 1,2,\cdots,M-1; \ \ j = 0,1,\cdots,N-1. \tag{4.3}$$

Scheme (4.3) is implicit, unconditionally stable and of the second order of accuracy.

For the δ-system (2.6), we use the following scheme

$$\delta u^j_{t,i} + \frac{1}{2}(\tilde{u}^{j+1}_i \cdot \delta u^{j+1}_i + \tilde{u}^j_i \cdot \delta u^j_i) + \frac{1}{2}(u^{j+1}_i \cdot \delta u^{j+1}_{\hat{x},i} + u^j_i \cdot \delta u^j_{\hat{x},i})$$

$$- \frac{1}{4}[(\nu \cdot \delta u^{j+1}_x)_{\bar{x},i} + (\nu \cdot \delta u^{j+1}_{\bar{x}})_{x,i} + (\nu \cdot \delta u^j_x)_{\bar{x},i} + (\nu \cdot \delta u^j_{\bar{x}})_{x,i}]$$

$$= \frac{1}{2}[(\delta\nu \cdot \tilde{u}^{j+1})_{\hat{x},i} + (\delta\nu \cdot \tilde{u}^j)_{\hat{x},i}], i = 1, 2, \cdots, M-1; j = 0, 1, \cdots, N-1 \quad (4.4)$$

where $\tilde{u}^j_i$ is the approximation of $\partial u(ih, j\tau)/\partial x$, which can be numerically calculated after solving (4.3). Equation (4.4) can be rewritten in the matrix form

$$A_j \delta u^{j+1} = B_j \delta u^j + C_j \delta\nu, \quad (4.5)$$

where

$$\delta u^j = (\delta u^j_1, \delta u^j_2, \cdots, \delta u^j_{M-1})^T, \quad \delta\nu = (\delta\nu_1, \delta\nu_2, \cdots, \delta\nu_{M-1})^T$$

and A_j, B_j and C_j are $(M-1) \times (M-1)$ matrices whose elements are known after solving the direct problem. Suppose that A_j for all j are invertible, then one has

$$\delta u^{j+1} = E_{j+1} \delta\nu, \quad (4.6)$$

where

$$E_{j+1} = A_j^{-1}(B_j E_j + C_j), \quad j = 0, 1, \cdots, \quad E_0 = 0. \quad (4.7)$$

For $\partial\delta u(0, j, \tau)/\partial x$, we use the following approximate formula $\partial\delta u(0, j\tau)/\partial x \approx \frac{1}{2h}(-3\delta u^j_0 + 4\delta u^j_1 - \delta u^j_2)$. Thus a linear system for $\delta\nu$ can be formed

$$\sum_{k=1}^{M-1} (4e^{(j)}_{1k} - e^{(j)}_{2k})\delta\nu_k = 2h[r(j\tau) - \tilde{u}^j_0], \quad j = 1, 2, \cdots, N \quad (4.8)$$

or, in the matrix form $Z\delta\nu = b, e^{(j)}_{lk}$ in (4.8) are the elements of E_j.

Since equation (4.8) usually is ill-posed, some regularization technique is necessary[10]. Instead of the original system (4.8), we try to find a vector $\delta\nu^*$ to minimize the following functional

$$J(\alpha', \beta') = \|Z\delta\nu - b\|^2 + \alpha'\|\delta\nu\|^2_{l_2} + \beta'\|\delta\nu_x\|^2_{l_2}, \quad (4.9)$$

where α', β' are two regularization parameters and $\|\cdot\|_{l_2}$ is the discrete L^2-norm: $\|v\|^2_{l_2} = h\sum_{i=1}^{M-1} v_i^2$. The above problem is equivalent to solving the following linear system

$$(Z^*Z + \alpha I + \beta D)\delta\nu = Z^*b \quad (4.10)$$

where $\alpha = \alpha', \beta = \beta'/h^2$, I is the identity matrix and D is a tridiagonal matrix with diagonal elements 2 and upper and lower diagonal elements -1. We shall consider three choices of α and β : $\alpha \neq 0$, $\beta = 0$; $\alpha = 0, \beta \neq 0; \alpha = \beta \neq 0$. They represent minimum

L^2-norm regularization, minimum H^1-seminorm regularization are minimum H^1-norm regularization, respectively.

§5. Numerical Examples

In the following numerical examples, we take $T = 1, M = 10, N = 20$, thus $h = 0.1, \tau = 0.05$. The nonlinear algebraic equation (4.3) is solved by Newton's iteration method with u^j, the solution of (4.3) at the jth level, as the initial guess of u^{j+1}. In order to compare the efficiency of the three choices of parameters, we bigin our algorithm with the same value of α are / or β, and for the next iteration α and / or β are always one tenth of the previous ones.

Example 1. The exact coefficient function, the right-hand side term and the solution are as follows

$$\nu(x) = 10 - x^2,$$

$$f(x,t) \triangleq 2(1+t)^2x^3 + (10+9t)x^2 + (23+20t)x - 10(1+2t),$$

$$u(x,t) = (1+t)x^2 + x + 10.$$

The other needed data can be reduced from the solution. Two initial guesses of $\nu(x)$ used in practical computation are

$$\text{I.} \quad \nu_0(x) = 10 - x^2 - 32x(1-x); \qquad \text{II.} \quad \nu_0(x) = 10 - x^2 + 32x(1-x).$$

The numerical results are tabulated in Table 1, where the values of $\alpha_1(\beta_1)$ indicate the starting values of these parameters.

Table 1

	I. $\alpha_1(\beta_1) = 0.01$				II. $\alpha_1(\beta_1) = 0.00001$			
		$\alpha \neq 0, \beta = 0$	$\alpha = 0, \beta \neq 0$	$\alpha = \beta \neq 0$		$\alpha \neq 0, \beta = 0$	$\alpha = 0, \beta \neq 0$	$\alpha = \beta \neq 0$
n	0	8	7	7	0	4	7	7
$\|\nu_n - \nu\|_{L^2}$	6.0249	1.2653	0.0207	0.0212	5.6598	0.1638	0.0354	0.0349
$\|\nu_n - \nu\|_{L^\infty}$	8.25	2.6379	0.0347	0.0352	7.75	0.3151	0.0638	0.0602

Example 2. The exact functions are as follows

$$\nu(x) = 10,$$

$$f(x,t) = 2(1-t)^2x^3 + (2-3t^2)x^2 + (2+2t+t^2)x - 20(1-t),$$

$$u(x,t) = (1-t)x^2 + tx + x.$$

The initial guesses in this case are

$$\text{I.} \quad \nu_0(x) = 10 + 32x(1-x); \qquad \text{II.} \quad \nu_0(x) = 10 - 132x(1-x)((1/2) - x).$$

The numerical results are tabulated in Table 2.

Table 2

	I. $\alpha_1(\beta_1) = 0.001$				II. $\alpha_1(\beta_1) = 0.1$			
		$\alpha \neq 0, \beta = 0$	$\alpha = 0, \beta \neq 0$	$\alpha = \beta \neq 0$		$\alpha \neq 0, \beta = 0$	$\alpha = 0, \beta \neq 0$	$\alpha = \beta \neq 0$
n	0	7	8	8	0	10	9	9
$\|\nu_n - \nu\|_{L^2}$	5.8424	0.1846	0.0160	0.0173	4.5544	0.3556	0.0321	0.0345
$\|\nu_n - \nu\|_{L^\infty}$	8	0.4097	0.0340	0.0368	6.3509	0.8019	0.0598	0.0644

Example 3. The exact functions are as follows

$$\nu(x) = \begin{cases} 10 + x, & 0 \leq x \leq 1/2, \\ 11 - x, & 1/2 < x \leq 1, \end{cases}$$

$$f(x,t) = 2(1-t)^2x^3 + (2 - 3t^2)x^2 + (3 + 2t - t^2)x + (1 + t + t^2) + g(x,t),$$

$$g(x,t) = \begin{cases} -4(1-t)x + (19t - 21), & 0 \leq x \leq 1/2, \\ 4(1-t)x + (23t - 21), & 1/2 < x \leq 1, \end{cases}$$

$$u(x,t) = (1-t)x^2 + (1+t)x + t.$$

The initial guesses in this case are

$$\text{I.} \quad \nu_0(x) = 10 - 32x(1-x); \quad \text{II.} \quad \nu_0(x) = \begin{cases} 10(1-x), & 0 \leq x \leq 1/2, \\ 10x, & 1/2 < x \leq 1. \end{cases}$$

The numerical results are tabulated in Table 3.

Table 3

	I. $\alpha_1(\beta_1) = 0.001$				II. $\alpha_1(\beta_1) = 0.01$			
		$\alpha \neq 0, \beta = 0$	$\alpha = 0, \beta \neq 0$	$\alpha = \beta \neq 0$		$\alpha \neq 0, \beta = 0$	$\alpha = 0, \beta \neq 0$	$\alpha = \beta \neq 0$
n	0	7	5	5	0	7	9	9
$\|\nu_n - \nu\|_{L^2}$	6.1278	0.1037	0.0638	0.0633	3.1754	0.1302	0.1689	0.1686
$\|\nu_n - \nu\|_{L^\infty}$	8.5	0.1442	0.1169	0.1172	5.5	0.2633	0.3230	0.3226

§6. Some Remarks

Though only a few examples are excuted, all the results obtained do illustrate that the GPST iterative algorithm works very well for solving the discussed inverse problem. In addition, the method seems to work for a large range of initial guesses, even tnough the initial guess is very far from the exact one. This property is important for practical applications.

These examples show that the results obtained by using H^1-norm or seminorm regularization methods are excellent and much better than those of using L^2-norm regularization method. This phenomenon is reasonable, considering the analysis in Section 3 and the general properties of regularization methods[10]. Another phenomenon observed from these examples is that the final results depend not only on the smoothness of the exact coefficients $\nu(x)$, but also on the smooth property of the initial guesses.

One important problem which remains to be investigated is the optimal choice of regularization parameters and we have not addressed here. In this paper, we restrict ourselves to the feasibility of the GPST for solving nonlinear model and to comparison between different regularization methods. However, we would like to point out that more suitable choices of α, β will give better results and need less computational effort. As an example, we give here one numerical result of example 1, case I using parameters $\alpha = 0$, $\beta_1 = 2 \times 10^{-3}$, $\beta_2 = 10^{-7}$ and $\beta_3 = 10^{-10}$. After only three iterations, the method gives an excellent result with $\|\nu_3 - \nu\|_{L^2} = 0.0467$ and $\|\nu_3 - \nu\|_{L^\infty} = 0.0858$.

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