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A GENERAL ALGORITHM AND SENSITIVITY ANALYSIS FOR VARIATIONAL INEQUALITIES*

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Abstract

The fixed point technique is used to prove the existence of a solution for a class of variational inequalities related with odd order boundary value problems and to suggest a general algorithm. We also make the sensitivity analysis for these variational inequalities and complementarity problems using the projection technique. Several special cases are discussed, which can be obtained from our results.

§1. Introduction

Variational inequality theory is a very useful and effective technique for studying a wide class of problems in a unified natural and general framework. This theory has been extended and generalized in several directions using new and powerful methods that have led to the solution of basic and fundamental problems thought to be inaccessible previously. Some of these developments have made mutually enriching contacts with other areas of mathematical and engineering sciences. We also remark that the theory so far developed upto now is only applicable to constrained boundary value problems of even order. On the other hand, little attention has been given to odd order boundary value problems. In recent years, the author has developed iterative type algorithms for a certain class of variational inequalities related with odd order boundary value problems having constrained conditions. We also study the qualitative behaviour of the solution of the variational inequalities when the given operator and the feasible convex set vary with a parameter. Such a study is known as sensitivity analysis, which is also important and meaningful. Sensitivity analysis provides useful information for designing, planning various equalibrium systems, predicting the future changes of the equilibria as a result of the changes in the governing systems. In addition, from a theoretical point of view, sensitivity properties of a mathematical programming problem can provide new insight concerning the problems being studied and can sometimes stimulate new ideas and techniques for solving them.

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Motivated and inspired by the recent research work going on in this area, we consider a new class of variational inequalities. Using the fixed point technique of Glowinski, Lions and Tremolieres [1], and Noor [2, 3], we prove the existence of a solution of these variational inequalities. This approach enables us to suggest and analyze a general algorithm for these variational inequalities. We also show that the variational inequality problem is equivalent to solving a fixed point problem using the projection method. This equivalence is used to analyze the sensitivity of the parametric variational inequality. This approach is due to Dafermos [4]. We also consider the sensitivity analysis for the general complementarity problems. Several special cases are also discussed.

In Section 2, we formulate the variational inequality problem and review some necessary basic results. The existence of the solution of the variational inequality problem is studied in Section 3 using the fixed point method along with a general algorithm. Sensitivity analysis is the subject of Section 4. The applications of the main results are considered in Section 5.

§2. Variational Inequality Formulation

Let H be a real Hilbert space with norm and inner product $\|.\|$ and $\langle\cdot,\cdot\rangle$ respectively. Let K be a nonempty closed convex set in H.

Given $T, g : H \to H$ continuous operators, consider the functional $I[v]$, defined by

$$I[v] = \frac{1}{2}\langle Tv, g(v)\rangle \ , \tag{2.1}$$

which is known as the general energy (cost) functional. Note that for $g = I$, the identity operator, then the functional $I[v]$, defined by (2.1) becomes

$$I_1[v]\frac{1}{2}\langle Tv,\ v\rangle \ ,$$

which is the classical energy functional.

If the operator T is linear, g-symmetric, that is

$$\langle T(u),\ g(v)\rangle = \langle g(u),\ T(v)\rangle, \ \text{for all } u,\ v \in H,$$

and g-positive definite, then we can show that the minimum of $I[v]$, defined by (2.1) on the convex set K in H, is equivalent to finding $u \in H$ such that $g(u) \in K$ and

$$\langle Tu, g(v) - g(u)\rangle \leq 0, \ \text{for all } g(v) \in K. \tag{2.2}$$

Inequality (2.2) is known as the general variational inequality, introduced and studied by Noor[5]. We remark that if $g = I$, the identity operator, then problem (2.2), is equivalent to finding $u \in K$ such that

$$\langle Tu,\ v - u\rangle \leq 0, \ \text{for all } v \in K, \tag{2.3}$$

which is known as the variational inequality problem considered and studied by Lions and Stampacchia[6].

If $K = H$, then problem (2.2) is equivalent to finding $u \in H$ such that

$$\langle Tu,\ g(v)\rangle = 0, \quad \text{for all } g(v) \in H. \tag{2.4}$$

Problem (2.4) is known as the weak formulation of the odd order boundary value problems and this appears to be a new one.

If $K^* = \{u \in H;\ \langle u, v\rangle \geq 0 \text{ for all } v \in K\}$ is a polar cone of the convex cone K in H and $K \subset g(K)$, then problem (2.2) is equivalent to finding $u \in H$ such that

$$g(u) \in K,\ Tu \in K^* \text{ and } \langle Tu,\ g(u)\rangle = 0, \tag{2.5}$$

a complementarity problem, which is essentially due to Oettli and Noor[7]. Note the symmetry role played by the operators T and g, since $K = K^* = R_+^N$. This problem includes many previously known problems as special cases.

It is clear that problems (2.3), (2.4) and (2.5) are special cases of the general variational inequality (2.2) introduced in this paper. In brief, we conclude that problem (2.2) is a most general and unifying one, which is one of the main motivation of this paper.

We now define the following concepts.

Definition 2.1. *An operator $T : H \to H$ is said to be*

(a) *strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv,\ u - v\rangle \geq \alpha\|u - v\|^2, \text{ for all } u, v \in H;$$

(b) *Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$\|Tu - Tv\| \leq \beta\|u - v\|, \text{ for all } u, v \in H.$$

It is clear that if β exists, then so does α, and $\alpha \leq \beta$.

§3. Existence Theory

In this section, using the fixed point technique of Glowinski, Lions and Tremolieres [1] and Noor [2, 3], we prove the existence of the solution of the general variational inequality (2.2).

Theorem 3.1. *Let the operators $T, g : H \xrightarrow{u} H$ be both strongly monotone and Lipschitz continuous respectively. If the operator g is one-to-one, then there exists a unique solution $u \in H$ such that $g(u) \in K$, a closed convex set in H and*

$$\langle Tu,\ g(v) - g(v)\rangle \geq 0, \text{ for all } g(v) \in K. \tag{3.1}$$

Proof. Uniqueness. Its proof is similar to that of [1]. Existence. We now use the fixed point technique to prove the existence of a solution of (3.1). For given $u \in H$, we consider the auxiliary problem of finding $w \in H$ such that $g(w) \in K$ satisfies the variational inequality

$$\langle w,\ v - w\rangle \geq \langle u, v - w\rangle - \rho\langle Tu, g(v) - g(w)\rangle, \text{ for all } g(v) \in K, \tag{3.2}$$

where $\rho > 0$ is a constant.

Let w_1, w_2 be two solutions of (3.2) related to u_1, $u_2 \in H$ respectively. It is enough to show that the mapping $u \to w$ has a fixed point belonging to H satisfying (3.1). In other words, we have to show that for $\rho > 0$ well chosen,

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = w_2$ (respectively w_1) in (3.2) related to u_1 (respectively u_2), we have

$$\langle w_1, w_2 - w_1\rangle \geq \rangle u_1, w_2 - w_1\rangle - \rho\langle Tu_1, g(w_2) - g(w_1)\rangle$$

and

$$\langle w_2, w_1 - w_2\rangle \geq \rangle u_2, w_1 - w_2\rangle - \rho\langle Tu_2, g(w_1) - g(w_2)\rangle.$$

Adding these inequalities, we obtain

$$\begin{aligned}\langle w_1 - w_2, w_1 - w_2\rangle &\leq \langle u_1 - u_2, w_1 - w_2\rangle - \rho\langle Tu_1 - Tu_2, g(w_1) - g(w_2)\rangle \\ &= \langle u_1 - u_2 - \rho(Tu_1 - Tu_2), w_1 - w_2\rangle \\ &\quad + \rho\langle Tu_1 - Tu_2, w_1 - w_2 - (g(w_1) - g(w_2))\rangle,\end{aligned}$$

from which it follows that

$$\begin{aligned}\|w_1 - w_2\|^2 &\leq \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\| \, \|w_1 - w_2\| \\ &\quad + \rho\|w_1 - w_2 - (g(w_1) - g(w_2))\| \, \|Tu_1 - Tu_2\|. \qquad (3.3)\end{aligned}$$

Since T, g are both strongly monotone and Lipschitz continuous, by using the techique of [8], we have

$$\|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 \leq (1 - 2\alpha\rho + \beta^2\rho^2)\|u_1 - u_2\|^2 \qquad (3.4)$$

and

$$\|w_1 - w_2 - (g(w_1) - g(w_2))\|^2 \leq (1 - 2\sigma + \delta^2)\|w_1 - w - 2\|^2. \qquad (3.5)$$

From (3.3), (3.4) and (3.5), we obtain, by using the Lipschitz continuity of T,

$$\begin{aligned}\|w_1 - w_2\| &\leq \{\beta\rho(\sqrt{1 - 2\sigma + \delta^2}) + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u_1 - u_2\| \\ &= \{\rho k + t(\rho)\}\|u_1 - u_2\| + \theta\|u_1 - u_2\|,\end{aligned}$$

where $\theta = \rho k + t(\rho)$,

$$k = \beta\sqrt{1 - 2\sigma + \delta^2} \text{ and } t(\rho) = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}.$$

We now have to show that $\theta < 1$. It is clear that $t(\rho)$ assumes its minimum value for $\bar{\rho} = \frac{\alpha}{\beta^2}$ with $t(\bar{\rho}) = \sqrt{1 - \frac{\alpha^2}{\beta^2}}$. For $\rho = \bar{\rho}$, $\rho k + t(\rho) \leq 1$ implies that $\rho k < 1$ and $k < \alpha$. Thus it follows that $\theta < 1$ for all ρ with

$$\rho < 2\frac{\alpha - k}{\beta^2 - k^2}, \quad \rho < \frac{1}{k} \quad \text{and} \quad k < \alpha.$$

Since $\theta < 1$, the mapping $u \to w$ defined by (3.2) has a fixed point, which is the solution of (3.1), the required result.

Remark 3.1. If $g = I$, the identity operator, then problem (3.2), is equivalent to finding $w \in H$, for given $u \in H$, such that

$$\langle w,\ v - w\rangle \geq \langle u,\ v - w\rangle - \rho\langle Tu,\ v - w\rangle, \quad \text{for all } v \in k, \tag{3.6}$$

and $\rho > 0$. From the proof of Theorem 3.1, we have $k = 0$ and $\theta = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2} < 1$ for $0 < \rho < \frac{2\alpha}{\beta^2}$, so the mapping $u \to w$ defined by (3.6) has a fixed point, which is the solution of the variational inequality (2.3), studied by Lions and Stampacchia [6]. Consequently, we have a new proof of the variational inequality problem (2.3).

Remark 3.2. We note that the solution of the variational inequality problem (3.2) is equivalent to finding the minimum of the function $F(w)$ on K, where

$$F(w) = 1/2\langle w,\ w\rangle - \langle u,\ w\rangle + \rho\langle Tu,\ g(w)\rangle, \tag{3.7}$$

Hence $\rho > 0$ is a constant. It turns out that this auxiliary problem is very useful in suggesting an iterative algorithm for computing the approximate solution of the variational inequality (3.1). Based on these observations, we extend the ideas of Cohen [9] and Noor [2] to propose a more general algorithm:

Ceneral Algorithm 3.1. For some $u \in K$, we introduce the following auxiliary problem

$$\min_{w \in K} F[w],$$

where

$$F[w] = E(w) + \rho\langle Tu,\ g(w)\rangle - \langle E'(u),\ w\rangle. \tag{3.8}$$

Here ρ is a constant and E is a convex differentiable functional.

It is clear that the solution w of (3.8) can be characterized by the variational inequality

$$\langle E'(w),\ u - w\rangle \geq \langle E'(u),\ v - w\rangle - \rho\langle Tu,\ g(v) - g(w)\rangle, \quad \text{for all } v \in K. \tag{3.9}$$

We note that if $w = u$, then clearly w is a solution of (3.1). This fact suggests the following iterative algorithm:

(i) Choose the initial vector w_o.

(ii) At step n, solve the auxiliary problem (3.8) with $u = w_n$. Let w_{n+1} denote the solution of this problem.

(iii) Calculate $\|w_{n+1} - w_n\|$. If $\|w_{n+1} - w_n\| \leq \in$, for given $\in > 0$, stop; otherwise repeat (ii).

It is obvious that if $E(w) = \langle w,\ w\rangle$, then the general auxiliary problem (3.8) is exactly the same as (3.7). Therefore, we may consider this algorithm as a generalization of the previous ideas of Noor [2]. We remark that the General Algorithm 3.1 is an interesting way of computing a solution of (3.1) as long as (3.8) or (3.9) is easier to solve than problem (3.1). This depends crucially on the choice of the auxiliary convex cost functional $F[w]$. Since the auxiliary problem (3.8) is essentially a minimization problem, a large number of algorithms are available to solve it.

§4. Sensitivity Analysis

In this section, we conduct a sensitivity analysis for the variational inequality of type (3.1). This problem has attracted considerable attention recently. The methodologies suggested so far vary with the problem settings being studied. Sensitivity analysis for variational inequalities has been studied by Tobin [10], Kyparisis [12, 11], Dafermos [4] and Qiu and Magnanti [13] using quite different techniques. We mainly follow the ideas and technique of Dafermos [4], as extended by Noor [14] for a class of quasi-variational inequalities, which is based on the projection technique. This approach has strong geometric flavour. Using this technique, one usually proves the equivalence between the variational inequality problem and the fixed point problem. Consequently, this technique implies that continuity, Lipschitz continuity and differentiability of the perturbed solution depend upon the continuity, Lipschitz continuity and differentiability of the projection operator on the family of feasible convex sets.

We now consider the parametric variational inequality version of problem (3.1). To formulate the problem, let M be an open subset of H in which the parameter λ takes values and assume that $\{K_\lambda : \lambda \in M\}$ is a family of closed convex subsets of H. The parametric general variational inequality problem is:

Find $u \in H$ such that $g(u) \in K_\lambda$ and

$$\langle T(u,\lambda),\ g(v) - g(u)\rangle \geq 0, \ \text{ for all } g(v) \in k_\lambda, \tag{4.1}$$

where $T(u,\lambda)$ is a given operator defined on the set of (u,λ) with $\lambda \in M$ and takes values in H. We also assume that for some $\bar{\lambda} \in M$, problem (4.1) admits a solution $\bar{u}$. We want to investigate those conditions under which, for each λ in a neighbourhood of $\bar{\lambda}$, problem (4.1) has a unique solution $u(\lambda)$ near $\bar{u}$ and the function $u(\lambda)$ is continuous, Lipschitz continuous or differentiable. We assume that X is the closure of a ball in H centered at $\bar{u}$.

We also need the following concepts.

Defintion 4.1. The operator $T(u,\lambda)$ defined on $X \times M$ is said to be locally, for all $\lambda \in M$, u, $v \in X$,

(a) strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T(u,\lambda) - T(v,\lambda),\ u - v\rangle \geq \alpha\|u - v\|^2; \tag{4.2}$$

(b) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|T(u,\lambda) - T(v,\lambda)\| \leq \beta\|u - v\|. \tag{4.3}$$

In particular, it follows that $\alpha \geq \beta$.

We need the following results to prove the main result of this section. The first one is due to Noor[5].

Lemma 4.1[5]. *The function $u \in K_\lambda$ is a solution of the parametric variational inequality (4.1) if and only if u is the fixed point of the map:*

$$F(u,\lambda) = u - g(u) + P_{K_\lambda}[g(u) - \rho T(u,\lambda)], \tag{4.4}$$

for all $\lambda \in M$ and some $\rho > 0$, where P_{K_λ} is the projection of H on the faimly of closed convex sets K_λ.

We are here only interested in the case that the solutions of the variational inequality (4.1) lie in the interior of X. For this purpose, we consider the map

$$F^*(u,\lambda) = u - g(u) + P_{K_\lambda \cap X}[g(u) - \rho T(u,\lambda)], \text{ for all } (u,\lambda) \in X \times M. \tag{4.5}$$

We have to show that the map $F^*(u,\lambda)$ has a fixed point, which by Lemma 4.1 is also a solution of the variational inequality (4.1). First of all, we prove that the map $F^*(u,\lambda)$ is a contraction map with respect to u, uniformly in $\lambda \in M$ by using the local strong monotonicity and Lipschitz continuity of the operator $T(u,\lambda)$ defined on $X \times M$.

Lemma 4.2. *For all $u, v \in X$, and $\lambda \in M$, we have*

$$\|F^*(u,\lambda) - F^*(v,\lambda)\| \le \theta \|u - v\|,$$

where $\theta = k + t(\rho) < 1$ for $|\rho - \frac{\alpha}{\beta} < \frac{\sqrt{\alpha^2 - \beta^2(2k - k^2)}}{\beta^2}|$, $\alpha > \beta\sqrt{k(k-2)}$ and $k < 1$.

Proof. For all $u, v \in X$, $\lambda \in M$, from (4.5), we have

$$\begin{aligned}\|F^*(u,\lambda) - F^*(v,\lambda)\| &\le \|u - v - (g(u) - g(v))\| + \|P_{K_\lambda \cap X}[g(u) - \rho T(u,\lambda)] \\ &- P_{K_\lambda \cap X}[g(v) - \rho T(v,\lambda)]\| \le 2\|u - v - (g(u) - g(v))\| \\ &+ \|u - v - \rho(T(u,\lambda) - T(v,\lambda))\|\end{aligned} \tag{4.6}$$

since the projection operator $P_{K_{lambda} \cap X}$ is nonexpansive; see [1].

Now the operators $T(u,\lambda)$ and g are both (locally) strongly monotone and Lipschitz continuous, so by the method of Noor [8],

$$\|u - v - \rho(T(u,\lambda) - T(v,\lambda))\|^2 \le (1 - 2\alpha\rho + \beta^2\rho^2)\|u - v\|^2, \tag{4.7}$$

and

$$\|u - v - (g(u) - g(v))\|^2 \le (1 - 2\delta + \sigma^2)\|u - v\|^2. \tag{4.8}$$

From (4.6), (4.7) and (4.8), we obtain

$$\begin{aligned}\|F^*(u,\lambda) - F^*(v,\lambda)\| &\le \{2(\sqrt{1 - 2\delta + \sigma^2}) + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u - v\| \\ &= (k + t(\rho))\|u - v\|\theta\|u - v\|,\end{aligned}$$

where $\theta = k + t(\rho)$, $k = 2(\sqrt{1 - 2\delta + \sigma^2})$, and $t(\rho) = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}$. Now, using the technique of Noor [5], we can show that $\theta < 1$ for

$$|\beta - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2(2k - k^2)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(k-2)},$$

and $k < 1$, from which it follows that the map $F^*(u,\lambda)$ defined by (4.5) is a contraction map, the required result,

Remark 4.1. From Lemma 4.2, we see that the map $F^*(u,\lambda)$ has a unique fixed point $u(\lambda)$, that is $u(\lambda) = F^*(u,\lambda)$. Also by assumption, the function $\bar{u}$ for $\lambda = \bar{\lambda}$ is a solution of the parametric variational inequality (4.1). Again using Lemma 4.2, we see that $\bar{u}$ is a fixed point of $F^*(u,\lambda)$ and it is also a fixed point of $F^*(u,\bar{\lambda})$. Consequently, we conclude that $u(\bar{\lambda}) = \bar{u} = F^*(u(\bar{\lambda}),\bar{\lambda})$.

Using Lemma 4.2 and the technique of Dafermos [4], we prove the continuity of the solution $u(\lambda)$ of the variational inequality (4.1), which is the motivation of our next result.

Lemma 4.3. *If the operators $T(\bar{u},\lambda)$, $g(\bar{u})$ and the map $\lambda \to P_{K_\lambda \cap X}[g(\bar{u}) - \rho T(\bar{u},\bar{\lambda})]$ are continuous (or Lipschitz continuous), then the function $u(\lambda)$ satisfying (4.1) is continous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$.*

Proof. For $\lambda \in M$, using Lemma 4.2 and the triangle inequality, we have

$$\begin{aligned}\|u(\lambda) - u(\bar{\lambda})\| &= \|F^*(u(\lambda),\lambda) - F^*(u(\bar{\lambda}),\bar{\lambda})\| \le \|F^*(u(\lambda),\lambda) - F^*(u(\bar{\lambda}),\lambda)\| \\ &\quad + \|F^*(u(\bar{\lambda}),\lambda) - F^*(u(\bar{\lambda}),\bar{\lambda})\| \le \theta\|u(\lambda) - u(\bar{\lambda})\| \\ &\quad + \|F^*(u(\bar{\lambda}),\lambda) - F^*(u(\bar{\lambda}),\bar{\lambda})\|. \end{aligned} \tag{4.9}$$

From (4.5) and the fact that the projection map is nonexpansive, we have

$$\begin{aligned}&\|F^*(u(\bar{\lambda}),\lambda) - F^*(u(\bar{\lambda}),\bar{\lambda})\| \\ &= \|P_{K_\lambda\cap X}[g(u(\bar{\lambda})) - \rho T(u(\bar{\lambda}),\lambda)] - P_{K_\lambda\cap X}[g(u(\bar{\lambda})) - \rho T(u(\bar{\lambda}),\bar{\lambda})]\| \\ &\le \rho\|T(u(\bar{\lambda}),\lambda) - T(u(\bar{\lambda}),\bar{\lambda})\| + \|P_{K_\lambda\cap X}[g(u(\bar{\lambda})) - \rho T(u(\bar{\lambda}),\bar{\lambda})] \\ &- P_{K_{\bar{\lambda}}\cap X}[g(u(\bar{\lambda})) - \rho T(u(\bar{\lambda}),\lambda)]\|. \end{aligned} \tag{4.10}$$

Now from Remark 4.1, and combining (4.9) with (4.10), we get

$$\begin{aligned}\|u(\lambda) - \bar{u}\| &\le \frac{\rho}{1-\theta}\|T(\bar{u},\lambda) - T(\bar{u},\lambda)\| + \frac{1}{1-\theta}\|P_{K_\lambda\cap X}[g(\bar{u}) - \rho T(\bar{u},\bar{\lambda})] \\ &\quad - P_{K_{\bar{\lambda}}\cap X}[g(\bar{u}) - \rho T(\bar{u},\bar{\lambda})]\|, \end{aligned} \tag{4.11}$$

from which the required result follows.

Lemma 4.4. *If the assumptions of Lemma 4.3 hold, then there exists a neighbourhood $N \subset M$ of $\bar{\lambda}$ such that for $\lambda \in N$, $u(\lambda)$ is the unique solution of the parametric variational inequality (4.1) in the interior of X.*

Proof. Its proof is similar to that of Lemma 2.5 in [4].

We now state and prove the main result of this section.

Theorem 4.1. *Let $\bar{u}$ be the solution of the parametric variational inequality (4.1) and $\lambda = \bar{\lambda}$ and $T(u,\lambda)$ be the locally strongly monotone Lipschitz continuous operator for all $u, r \in X$. If the operators $T(\bar{u},\lambda)$, $g(\bar{u})$ and the map $\lambda \to P_{K_\lambda\cap X}[g(\bar{u}) - \rho T(\bar{u},\bar{\lambda})]$ are continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$, then there exists a neighbourhood $N \subset M$ of $\bar{\lambda}$ such that for $\lambda \in N$, the parametric variational inequality (4.1) has* *a unique solution $u(\lambda)$ in the interior of X, $u(\bar{\lambda}) = \bar{u}$ and $u(\lambda)$ is cintinuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$.*

Proof. Its proof follows from Lemmas 4.2–4.4 and Remark 4.1.

Remark 4.2. The results obtained in this section can be extended when the operators T and g are both allowed to vary with the parameter λ along with the feasible convex sets. The variational inequality problem (4.1) becomes

Find $u \in H$ such that $g(u) \in K_\lambda$ and

$$\langle T(u,\lambda),\ g(v,\lambda) - g(u,\lambda)\rangle \geq 0 \text{ for all } g(v) \in K_\lambda \tag{4.12}$$

which is equivalent to finding $u \in H$ such that

$$F_1(u,\lambda) = P_{K_\lambda}[g(u,\lambda) - \rho T(u,\lambda)], \tag{4.13}$$

for $\lambda \in M$ and $\rho > 0$. This formulation allows us to obtain results similar to those in Theorem 4.1.

Remark 4.3. We note that for $g = I$, the identity operator, the parametric variational inequality (4.1) reduces to

Find $u \in K_\lambda$ such that

$$\langle T(u,\lambda),\ v - u\rangle \geq 0, \text{ for all } v \in K_\lambda,$$

the problem studied by Dafermos. Consequently, our results are exactly the same as proved in [4]. We also remark that the function $u(\lambda)$ as defined in Theorem 4.1 is continuously differentiable on some neighbourhood $\bar{N}$ of $\bar{\lambda}$. For this, see Dafermos [4].

§5. General Complementarity Problem

In this section, we illustrate that the results obtained in Sections 3 and 4 can be used to study the sensitivity for the general complementarity problem studied by Oettli and Noor [7]. To be more precise, given $T, g : H \to H$, find $u \in H$ such that

$$g(u) \in K,\ Tu \in K^* \text{ and } \langle g(u),\ Tu\rangle = 0 \tag{5.1}$$

where K^* is the polar cone of the convex cone K in H. Since problems (5.1) and (3.1) are equivalent, the results of Theorem 3.1 and Theorem 4.1 can be used to prove the existence of the solution and the sensitivity analysis of problem (5.1).

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