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CONVERGENCE ACCELERATION OF VECTOR SEQUENCES BY VECTOR PADÉ APPROXIMATION*1)

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Abstract

By making use of vector Padé approximants, a method for accelerating the convergence of vector sequences is derived. The method obtained includes the well known Henrici transformation as a special case. A main character of the method is that it can use partial vector components to accelerate convergence of the whole vector sequence. Results about the efficacy of the method are established. An algorithm and some numerical examples are given.

§1. Introduction

In applied mathematics, we sometimes need to compute vector sequences which, under certain assumptions, converge to the solution of the problem considered. However, it is often the case that the convergence is so slow that the computational approach is of no practical use. Hence, it is natural to employ some convergence acceleration methods. In this paper, we derive a method from the vector Padé approximation introduced in [5]. Differing from the existing acceleration methods (see [3]), our method does not use a fixed number of components of the vectors being accelerated. The number of components used varies with the degrees of Padé approximants.

In order to introduce the vector Padé approximation(VPA), we give firstly the following notations:

$$H_k := \{p(z) : p(z) = \sum_{i=0}^{k} a_i z^i, a_i \in C\},$$

$$E_k := \{e(z) : e(z) = \sum_{i=k+1}^{\infty} a_i z^i, a_i \in C\},$$

$$Z_+^p := \{\vec{n} : \vec{n} = (n_1, \cdots, n_p)^T, n_i \in Z_+, \ i = 1, \cdots, p\},$$

where p is a given positive integer, and Z_+ is the set of all nonnegative integers. $|\vec{n}| = \sum_{i=1}^{p} n_i$, for $\vec{n} \in Z_+^p$.

$$H_{\vec{n}}^p := (H_{n_1}, \cdots, H_{n_p})^T, \quad E_{\vec{\omega}}^p := (E_{\omega_1}, \cdots, E_{\omega_p})^T.$$

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If $g(z)=\sum_{k=0}^{\infty} c_k z^k, c_k \in C$, we denote

$$T_{m,n}^{l}(g)=\begin{bmatrix} c_l & c_{l-1} & \cdots & c_{l-n+1} \\ c_{l+1} & c_l & \cdots & c_{l-n+2} \\ \vdots & \vdots & & \vdots \\ c_{l+m-1} & c_{l+m-2} & \cdots & c_{l+m-n} \end{bmatrix} \tag{1.1}$$

with the convention $c_k=0$ if $k<0$.

Definition. *Let* $f(z)=\sum_{k=0}^{\infty} c_k z^k, c_k \in C^p$ *be a given power series,* $\vec{n} = (n_1,\cdots,n_p)^T \in Z_+^p$ *and* $\vec{w}=(w_1,\cdots,w_p)^T \in Z_+^p$ *be two given integer vectors such that*

$$\vec{e}:=\vec{w}-\vec{n}\in Z_+^p, \quad and \quad |\vec{w}|=|\vec{n}|+m ,$$

where m *is a given integer. If we can find a vector polynomial* $N(z)\in H_{\vec{n}}^p$ *and a scalar polynomial* $M(z)\in H_m$ *such that*

$$f(z)M(z)-N(z)\in E_{\vec{w}}^p, \quad \text{and} \quad M(0)=1 ,$$

then we call $N(z)M(z)^{-1}$ *the* $[\vec{n},m,\vec{w}]$ *vector Padé approximation of* f. *We denote it as* $[\vec{n},m,\vec{w}]_f$.

From (4.1) and (4.2) of [5], we have

Theorem 1 (Determinant expression for VPA). *If* $H(\vec{n},m,\vec{w})$ *is nonsingular, then*

$$M(z)=\frac{1}{H(\vec{n},m,\vec{w})}\det\begin{bmatrix} 1 & z & \cdots & z^m \\ B(\vec{n},m,\vec{w}) & & H(\vec{n},m,\vec{w}) & \end{bmatrix}, \tag{1.2}$$

$$N(z)=\frac{1}{H(\vec{n},m,\vec{w})}\operatorname{Det}\begin{bmatrix} f^{(\vec{n})} & zf^{(\vec{n}-1)} & \cdots & z^m f^{(\vec{n}-m)} \\ B(\vec{n},m,\vec{w}) & \cdot & H(\vec{n},m,\vec{w}) & \end{bmatrix}, \tag{1.3}$$

where

$$H(\vec{n},m,\vec{w})=\begin{bmatrix} T_{e_1,m}^{n_1}(f_1) \\ \vdots \\ T_{e_p,m}^{n_p}(f_p) \end{bmatrix}, \quad B(\vec{n},m,\vec{w})=\begin{bmatrix} T_{e_1,1}^{n_1+1}(f_1) \\ \vdots \\ T_{e_p,1}^{n_p+1}(f_p) \end{bmatrix},$$

$$f^{(\vec{n})}(z)=[f_1^{(n_1)}(z),\cdots,f_p^{(n_p)}(z)]^T, \quad f_i^{(k)}(z)=\begin{cases} \sum_{j=0}^{k}(c_j)_i z^j, & \text{for} \quad k\ge 0 , \\ 0, & \text{for} \quad k<0 , \end{cases}$$

for $i=1,\cdots,p$, *and* $\operatorname{Det}[\cdots]$ *denotes the vector obtained by expanding* $[\cdots]$ *with respect to its first "row". Here and in the following* $(\cdot)_i$ *denotes the* i*-th component of the vector* $(\cdot)$.

In Section 2, we will obtain an acceleration method from VPA and show that the Henrici transformation is a special case of the method. In Section 3 we will use the E-algorithm in [2] to implement it. Some numerical examples are given in Section 4, which show that the VPA method does accelerate the convergence of vector sequences, and is more powerful than the well known vector ε-algorithm [4] in some cases.

§2. Acceleration Method of Vector Sequences

Given a vector sequence $\{S_n\}_0^\infty \subset C^p$, define $f(z)$ by the formal power series:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_k \in C^p \tag{2.1}$$

where

$$c_0 = S_0, \quad c_k = \Delta S_{k-1} = S_k - S_{k-1}, \quad k = 1, 2, \cdots. \tag{2.2}$$

By (1.2) and (1.3) we can define the generalized Henrici transformation as follows:

$$h(\vec{n}, m, \vec{w}) = [\vec{n} + m, m, \vec{w}]_f(1) = \text{Det} \begin{bmatrix} S_{\vec{n}} & S_{\vec{n}+1} & \cdots & S_{\vec{n}+m} \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} & \cdots & \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix}$$

$$\div \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} & \cdots & \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix}, \tag{2.3}$$

where $\vec{n} + l = (n_1 + i, \cdots, n_p + l)^T$, $l = 1, \cdots, m$ $S_{\vec{n}} = [(S_{n_1})_1, \cdots, (S_{n_p})_p]^T$ $\vec{e} = \vec{w} - (\vec{n} + m) = (e_1, \cdots, e_p)^T$ $S_{\vec{n},\vec{e}} = [(S_{n_1})_1, \cdots, (S_{n_1+e_1-1})_1, \cdots, (S_{n_p})_p, \cdots, (S_{n_p+e_p-1})_p]^T$ $\Delta S_{\vec{n},\vec{e}} = S_{\vec{n}+1,\vec{e}} - S_{\vec{n},\vec{e}}$ and

$$\vec{w} - (\vec{n} + m) \geq 0, \quad |\vec{w}| = |\vec{n} + m| + m. \tag{2.4}$$

Remark. If we write $h(\vec{n}, m, \vec{w})$ in the form of $\sum_{i=0}^m a_i^{(\vec{n})} S_{\vec{n}+i}$, we can see that the coefficients $a_i^{(\vec{n})}$ $(i = 0, \cdots, m)$ depend on at most m components of the original vector when $m < p$.

Let

$$\Delta\, \mathbf{S}_{\vec{n},m,\vec{e}} = [\Delta S_{\vec{n},\vec{e}}, \Delta S_{\vec{n}+1,\vec{e}}, \cdots, \Delta S_{\vec{n}+m-1,\vec{e}}],$$

$$\Delta^2\, \mathbf{S}_{\vec{n},m,\vec{e}} = \Delta\, \mathbf{S}_{\vec{n}+1,m,\vec{e}} - \Delta\, \mathbf{S}_{\vec{n},m,\vec{e}},$$

$$\Delta\, \mathbf{S}_{\vec{n},m} = [\Delta S_{\vec{n}}, \Delta S_{\vec{n}+1}, \cdots, \Delta S_{\vec{n}+m-1}],$$

where $\Delta S_{\vec{n}} = S_{\vec{n}+1} - S_{\vec{n}}$. By using the knowledge of linear algebra and (2.3) we have

$$h(\vec{n}, m, \vec{w}) = S_{\vec{n}} - \Delta\, \mathbf{S}_{\vec{n},m} (\Delta^2\, \mathbf{S}_{\vec{n},m,\vec{e}})^{-1} \Delta S_{\vec{n},\vec{e}}.$$

Example 1. Set $n \in Z_+, p \in Z_+, \vec{n} = (n, \cdots, n)^T \in Z_+^p, m = p$, and $\vec{w} = \vec{n} + p + 1 \in Z_+^p$. Obviously, (2.4) holds, and (2.3) becomes the Henrici transformation defined in [2]. Hence we call $h(\vec{n}, m, \vec{w})$ generalized Henrici transformation (GHT).

Theorem 2. *Suppose that $\forall \vec{n} = \vec{n}_0 + k \in Z_+^p$ (i.e., $\forall k \in Z_+$, $\vec{n}_0 \geq \vec{0}$ fixed), det $(\Delta^i\, \mathbf{S}_{\vec{n},m,\vec{e}}) \neq 0$, for $i = 1, 2$. A necessary and sufficient condition that $\forall \vec{n} = \vec{n}_0 + k \in Z_+^p, h(\vec{n}, m, \vec{w}) = S$ is that the sequence $\{S_k\}$ satisfies*

$$\sum_{i=0}^{m} a_i (S_{\vec{n}+i} - S) = \vec{0}, \quad \text{and} \quad \sum_{i=0}^{m} a_i = 1, \quad a_i \in C. \tag{2.5}$$

Proof. The sufficiency is obvious.

Necessity. If $\forall \vec{n} = \vec{n}_0 + k (k \in Z_+) \in Z_+^p, h(\vec{n}, m, \vec{w}) = S$, then

$$\mathrm{Det}\begin{bmatrix} S_{\vec{n}} - S & S_{\vec{n}+1} - S & \cdots & S_{\vec{n}+m} - S \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} & \cdots & \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix} = \vec{0} \tag{2.6}$$

for $\forall \vec{n} = \vec{n}_0 + k (i.e., \forall k \in Z_+) \in Z_+^p$. Expanding this generalized determinant, we get

$$\sum_{i=0}^{m} x_i^{(\vec{n})} (S_{\vec{n}+i} - S) = \vec{0}, \quad \forall \vec{n} = \vec{n}_0 + k \in Z_+^p . \tag{2.7}$$

On the other hand, we have

$$\alpha_{\vec{n}} := \sum_{i=0}^{m} x_i^{(\vec{n})} = \det (\Delta^2 \mathbf{S}_{\vec{n},m,\vec{e}}) \neq 0 .$$

Set $a_i^{(\vec{n})} = x_i^{(\vec{n})} / \alpha_{\vec{n}}$. Then (2.6) becomes

$$\sum_{i=0}^{m} a_i^{(\vec{n})} (S_{\vec{n}+i} - S) = \vec{0}, \quad \text{with} \quad \sum_{i=0}^{m} a_i^{(\vec{n})} = 1 .$$

Now we prove that for a fixed $\vec{n}_0 \in Z_+^p, a_i^{(\vec{n})}$ is independent of k ($\vec{n} = \vec{n}_0 + k, k \in Z_+$). If $e_i > 0$, by row-adding, we have

$$\det\begin{bmatrix} (S_{n_i})_i & (S_{n_i+1})_i \cdots (S_{n_i+m})_i \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} \cdots \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix} = \det\begin{bmatrix} (S_{n_i+1})_i & (S_{n_i+2})_i \cdots (S_{n_i+m+1})_i \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} \cdots \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix}$$

$$= \cdots = \det\begin{bmatrix} (S_{n_i+e_i-1})_i & (S_{n_i+e_i})_i & \cdots & (S_{n_i+e_i+m-1})_i \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} & \cdots & \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix} .$$

Hence

$$\mathrm{Det}\begin{bmatrix} S_{\vec{n},\vec{e}} - \hat{S} & S_{\vec{n}+1,\vec{e}} - \hat{S} & \cdots & S_{\vec{n}+m,\vec{e}} - \hat{S} \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} & \cdots & \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix} = \vec{0} , \tag{2.8}$$

where

$$\hat{S} = [\overbrace{(S)_1, \cdots, (S)_1}^{e_1}, \cdots, \overbrace{(S)_p, \cdots, (S)_p}^{e_p}]^T .$$

Therefore

$$\sum_{i=0}^{m} a_i^{(\vec{n})} (S_{\vec{n}+i,\vec{e}} - \hat{S}) = \vec{0}, \quad \sum_{i=0}^{m} a_i^{(\vec{n})} = 1 . \tag{2.9}$$

By adding successively the rows $2, 3, \cdots, m$ to the 1st, 2nd, $\cdots$, m-th component of the vectors in the first row, we can write the relation (2.8) as

$$\mathrm{Det}\begin{bmatrix} S_{\vec{n}+1,\vec{e}} - \hat{S} & S_{\vec{n}+2,\vec{e}} - \hat{S} & \cdots & S_{\vec{n}+m+1,\vec{e}} - \hat{S} \\ \Delta S_{\vec{n},\vec{e}} & \Delta S_{\vec{n}+1,\vec{e}} & \cdots & \Delta S_{\vec{n}+m,\vec{e}} \end{bmatrix} = \vec{0} . \tag{2.10}$$

Hence

$$\sum_{i=0}^{m} a_i^{(\vec{n})} (S_{\vec{n}+i+1,\vec{e}} - \hat{S}) = \vec{0}, \quad \text{with} \quad \sum_{i=0}^{m} a_i^{(\vec{n})} = 1 . \tag{2.11}$$

If we replace $\vec{n}$ by $\vec{n}+1$ in (2.9) and combine with (2.8), then we have

$$\sum_{i=0}^{m} \Delta a_i^{(\vec{n})} = 0, \sum_{i=0}^{m} \Delta a_i^{(\vec{n})} (S_{\vec{n}+i+1,\vec{e}} - \hat{S}) = \vec{0} .$$

Since $\det \Delta \mathrm{S}_{\vec{n}+1,m,\vec{e}+1} \neq 0$, $\Delta a_i^{(\vec{n})} = 0$, for any $\vec{n} = \vec{n}_0 + k \in Z_+^p$ and $i = 0, 1, \cdots, m$, the necessity is proved.

Corollary 3. If $S_k = S + \sum_{j=1}^{J} A_j^k x_j$, $x_j \in C^p$, and none of the eigenvalues of $A_j (j = 1, \cdots, J)$ equals 1, and $\Delta^i \mathrm{S}_{\vec{n},m,\vec{e}} \neq 0, i = 1, 2$, then $[\vec{n} + m, m, \vec{w}]_f$ exists and $[\vec{n} + m, m, \vec{w}]_f(1) = S$. Here $f(z)$ is given by (2.1) and (2.2), $\vec{n} = (n, \cdots, n)^T, m, \vec{w}$ satisfy (2.4), and $m = \partial P(\lambda)$ is the degree of $P(\lambda)$, while $P(\lambda)$ is the least common multiple of all the minimal polynomials of A_j with respect to $x_j (j = 1, \cdots, J)$.

Proof. Let $P(\lambda) = \sum_{i=0}^{m} a_i' \lambda^i$ be the least common multiple of all the minimal polynomials of A_j with respect to x_j. Then $P(A_j) x_j = 0$, for $j = 1, \cdots, J$. So

$$\sum_{i=0}^{m} a_i' (S_{\vec{n}+i} - S) = \sum_{j=1}^{J} A_j^n P(A_j) x_j = \vec{0} .$$

By the assumption, $P(1) = \sum_{i=0}^{m} a_i' \neq 0$. Set $a_i = a_i' / \sum_{k=0}^{m} a_k'$. We have $\sum_{i=0}^{m} a_i (S_{\vec{n}+i} - S) = \vec{0}$ with $\sum_{i=0}^{m} a_i = 1, a_i \in C$. From Theorem 2, we can easily see that the corollary is true.

Example 2. Let

$$S_{k+1} = A S_k + b, \quad k = 0, 1, \cdots \tag{2.12}$$

be a linear iterative scheme, and $I - A$ be nonsingular. Then $S_k - S = A^k (S_0 - S)$. It follows from Corollary 3 that $h(\vec{n}, m, \vec{w}) = S$, where m is the degree of the minimal polynomial of A with respect to $(S_0 - S)$.

Let c be a real number, $c > 0$. We denote the set $K(c)$ of square matrices by

$$K(c) = \left\{ A = [a^1, \cdots, a^p] : a^i \in C^p \setminus \{\vec{0}\},\ i = 1, \cdots, p, \quad \text{and} \quad |\det A| \geq c \prod_{i=1}^{p} \|a^i\|_2 \right\} .$$

Definition (see [2]). *A sequence of matrices $\{B_n\}$ is said to be uniformly invertible if and only if $\exists d > 0$, $N \in Z_+$, such that $B_n \in K(d)$, $\forall n > N$.*

Theorem 4. *Let $\{S_n\}$ be a sequence of complex vectors converging to $S \in C^p$, and $(S_{\vec{n}+1,\vec{e}} - \hat{S}) = (B - B_{\vec{n}})(S_{\vec{n},\vec{e}} - \hat{S})$ with $\rho(B) < 1$, and $\lim_{\vec{n}\to\infty} B_{\vec{n}} = 0$ ($\vec{n} = \vec{n}_0 + k \to \infty$ means $k \to \infty$, $\vec{n}_0$ fixed), where $\rho(B)$ denotes the spectral radius of B. If the matrix sequence $\{\Delta \mathrm{S}_{\vec{n},m,\vec{e}}\}$ is uniformly invertible, then*

(1) *the sequence $\{h(\vec{n}, m, \vec{w})\}$ is defined for $\vec{n}$ sufficiently large;*

(2)

$$\lim_{\vec{n}\to\infty} \frac{\|\hat{h}(\vec{n}, m, \vec{w}) - \hat{S}\|_2}{\|S_{\vec{n},\vec{e}} - \hat{S}\|_2} = 0 ,$$

where $\hat{h}(\vec{n}, m, \vec{w}) = S_{\vec{n},\vec{e}} - \Delta \mathrm{S}_{\vec{n},m,\vec{e}} (\Delta^2 \mathrm{S}_{\vec{n},m,\vec{e}})^{-1} \Delta S_{\vec{n},\vec{e}}$, $\vec{n}, m, \vec{w}$ satisfy (2.4).

Remark. The proof of this theorem is similar to that of Theorem 8 of [2].

§3. An Algorithm

In this section we use the E-algorithm ([2]) to implement our generalized Henrici transformation. Suppose $\{g_i(\vec{n})\}$ are auxiliary scalar sequences. Let us consider the following ratio of determinants:

$$H_k^{(\vec{n})} = \mathrm{Det}\begin{bmatrix} S_{\vec{n}} & S_{\vec{n}+1} & \cdots & S_{\vec{n}+k} \\ g_1(\vec{n}) & g_1(\vec{n}+1) & \cdots & g_1(\vec{n}+k) \\ g_2(\vec{n}) & g_2(\vec{n}+1) & \cdots & g_2(\vec{n}+k) \\ \vdots & \vdots & & \vdots \\ g_k(\vec{n}) & g_k(\vec{n}+1) & \cdots & g_k(\vec{n}+k) \end{bmatrix}$$

$$\div \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ g_1(\vec{n}) & g_1(\vec{n}+1) & \cdots & g_1(\vec{n}+k) \\ g_2(\vec{n}) & g_2(\vec{n}+1) & \cdots & g_2(\vec{n}+k) \\ \vdots & \vdots & & \vdots \\ g_k(\vec{n}) & g_k(\vec{n}+1) & \cdots & g_k(\vec{n}+k) \end{bmatrix}$$

Using the vector Sylvester's identity for the numerator and the scalar one for the denominator, we obtain a recursive algorithm, the E-algorithm, for computing $H_k^{(\vec{n})}$'s:

$$H_0^{(\vec{n})} = S_{(\vec{n})}, \quad g_{0,i}^{(\vec{n})} = g_i(\vec{n}), \quad \vec{n} \in Z_+^p; \quad i = 1, 2, \cdots,$$

$$H_k^{(\vec{n})} = H_{k-1}^{(\vec{n})} - g_{k-1,k}^{(\vec{n})} \frac{\Delta H_{k-1}^{(\vec{n})}}{\Delta g_{k-1,k}^{(\vec{n})}}, \quad \vec{n} \in Z_+^p; \quad k = 1, 2, \cdots,$$

$$g_{k,i}^{(\vec{n})} = g_{k-1,i}^{(\vec{n})} - g_{k-1,k}^{(\vec{n})} \frac{\Delta g_{k-1,i}^{(\vec{n})}}{\Delta g_{k-1,k}^{(\vec{n})}}, \quad \vec{n} \in Z_+^p; \quad k = 1, 2, \cdots, \quad i > k.$$

where

$$\Delta H_{k-1}^{(\vec{n})} = H_{k-1}^{(\vec{n}+1)} - H_{k-1}^{(\vec{n})}, \quad \Delta g_{k-1,i}^{(\vec{n})} = g_{k-1,i}^{(\vec{n}+1)} - g_{k-1,i}^{(\vec{n})}.$$

Let $m \in Z_+$, $\vec{n}_0$ and $\vec{e} = (e_1, \cdots, e_p)^T$ be given; moreover, $m = |\vec{e}|$. Suppose $\vec{e}_i (i = 1, \cdots, p)$ is the ith column of the unity matrix $I_{p\times p}$. Define $\vec{w}_0, \vec{w}_1, \cdots, \vec{w}_m \in Z_+^p$ and the initial values $g_j(\vec{n})$ by the following steps:

(1) Set $\vec{w}_0 = \vec{n}_0 = (n_1, \cdots, n_p)^T$; $j = 0$; $i = 1$; and $e_{0k} = e_k$, for $k = 1, \cdots, p$.

(2) If $e_{0i} \geq 1$, set $j := j + 1$, $g_j(\vec{n}_0 + k) = (\Delta S_{n_i+e_i-e_{0i}+k})_i, k = 0, 1, \cdots, m$, $e_{0i} := e_{0i} - 1$, $\vec{w}_j := \vec{w}_{j-1} + \vec{e}_i$.

(3) Set $i := i + 1$ if $i \leq p$, go to (2); otherwise go to (4).

(4) If $j < m$, set $i = 1$, go to (2).

(5) Stop.

Then the E–algorithm gives

$$H_l^{(\vec{n}_0+k)} = h(\vec{n}_0 + k, l, \vec{w}_l + k), \quad l \in Z_+.$$

Hence we obtain the H-Table

$$\begin{array}{cccc} H_0^{(\vec{n}_0)} & H_1^{(\vec{n}_0)} & \cdots & H_m^{(\vec{n}_0)} \\ H_0^{(\vec{n}_0+1)} & H_1^{(\vec{n}_0+1)} & \cdots & H_m^{(\vec{n}_0+1)} \\ H_0^{(\vec{n}_0+2)} & H_1^{(\vec{n}_0+2)} & \cdots & H_m^{(\vec{n}_0+2)} \\ \vdots & \vdots & & \vdots \end{array}$$

Taking $\vec{n}_0 = \vec{0}$ and $\vec{e} = \alpha(1,\cdots,1)^T$ for a positive integer α, we will give some numerical examples according to this scheme in the next section.

§4. Numerical Examples

In the field of numerical analysis, vector sequences are often generated by some numerical methods for finding the solution of a given problem, such as various iterative methods for solving systems of linear or nonlinear equations and systems of differential equations. These sequences may converge slowly or even diverge. In this section, we will use four examples to compare the efficacy of the generalized Henrici transformation(GHT) with that of vector ε−algorithm(VEA) given by Wynn [4].

Vector ε−algorithm:

$$\varepsilon_{-1}^{(\vec{n})} = 0, \varepsilon_0^{(\vec{n})} = S_{\vec{n}}, \quad \vec{n} \in Z_+^p;$$

$$\varepsilon_{k+1}^{(\vec{n})} = \varepsilon_{k-1}^{(\vec{n}+1)} + (\varepsilon_k^{(\vec{n}+1)} - \varepsilon_k^{(\vec{n})})^{-1},$$

in which the Samelson inverse y^{-1} of $y \in C^p$ is defined by $y^{-1} = \bar{y}/\|y\|_2^2$. The results $\varepsilon_k^{(\vec{n})}$ can be arranged in a table, i.e., the ε −table (see [4]).

When we compare two entries coming from H−Table and ε-Table respectively, the following two principles should be followed.

Principle 1. The largest subscripts of the initial vector sequence used in the computation of the two entries are the same, e.g., we compare $H_j^{(\vec{n}_0+j+i-1)}$ with $\varepsilon_{2j}^{(i)}$, for $i = 0,1,\cdots; j = 1,2,\cdots$; and $\vec{n}_0 = \vec{0}$ (the largest subscript is $2j+i$).

Principle 2. The two entries use the same number of vectors of the initial sequence, e.g., we compare H_{k-1}^0 with $\varepsilon_{\Delta_k}^{\delta_k}$(they use k original vectors), where

$$\delta_k = \begin{cases} 1, & \text{if k is odd,} \\ 0, & \text{otherwise;} \end{cases} \quad \Delta_k = \begin{cases} k-1, & \text{if k is odd,} \\ k, & \text{otherwise.} \end{cases}$$

Tables 1–4 are arrayed according to Principle 1, and Table 5 according to Principle 2. The entries given in these tables denote the numbers of significant digits SD defined by

$$\text{SD} = -\log_{10}\|\text{True Solution} - \text{Approximate Solution}\|_\infty .$$

The first row of Table 1–Table 4 and the second column of Table 5 are notations of the approximate solutions.

Example 3. We consider the solution of the system of linear equations[4]

$$\begin{bmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 23 \\ 32 \\ 33 \\ 31 \end{bmatrix}.$$

The true solution of the equation is $(1,1,1,1)^T$. We obtain the initial vector sequence, which diverges, by the using Jacobi iterative method as in [3]; the starting iterative point is $(0,0,0,0)^T$.

Table 1 (Example 3)

k	S_k	H_1^{k-2}	ε_2^{k-2}	H_2^{k-3}	ε_4^{k-4}	H_3^{k-4}	ε_6^{k-6}	H_4^{k-5}	ε_8^{k-8}	H_5^{k-6}	ε_{10}^{k-10}
4	−1.699	0.601	0.601	0.527	0.654						
6	−2.488	0.637	0.637	0.448	0.656	0.685	0.683				
8	−3.726	0.652	0.652	0.762	0.657	0.687	0.685	6.491	4.253		
10	−4.064	0.658	0.658	0.714	0.657	0.688	0.686	5.627	3.861	5.341	4.420

Example 4. Now we consider the solution of the system of linear equations[4]

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 8 \\ 6 \end{bmatrix}.$$

The true solution of the equation is $(1,1,1,1)^T$. We obtain the initial vector sequence, which diverges more rapidly, by using the Seidel iterative method as in [3]; the starting iterative point is $(0,0,0,0)^T$.

Table 2 (Example 4)

k	S_k	H_1^{k-2}	ε_2^{k-2}	H_2^{k-3}	ε_4^{k-4}	H_3^{k-4}	ε_6^{k-7}
2	−1.847	−0.110	−0.289				
4	−2.291	−1.064	−1.504	2.102	0.057		
6	−3.808	−2.302	−2.268	5.384	3.837	13.04	3.962
8	−4.401	−2.978	−3.358	8.667	7.119	13.13	12.62

Example 5. In this example, we want to find the numerical solution $\vec{y}(1)$ of the ordinary differential equations

$$\begin{cases} y_1' = y_2, \\ y_2' = -y_1, \\ y_3' = -y_3, \\ y_1(0) = -1, \quad y_2(0) = 0, \quad y_3(0) = 1 \end{cases}$$

by the well known Euler method.

The initial sequence is generated by taking step lengths 2^{-k} $(k = 0, 1, \cdots)$ successively. The true solution is $\vec{y}(1) = (-cos(1), \sin(1), e^{-1})^T$.

Table 3 (Example 5)

k	S_k	H_1^{k-2}	ε_2^{k-2}	H_2^{k-3}	ε_4^{k-4}	H_3^{k-4}	ε_6^{k-6}
2	0.983	1.185	1.377				
4	1.582	2.386	2.733	3.120	2.961		
6	2.183	3.598	3.906	4.931	4.863	5.697	5.000
8	2.784	4.806	5.108	6.738	6.643	8.143	7.551
10	3.386	6.011	6.312	8.544	8.425	10.57	10.04
12	3.988	7.215	7.516	10.35	10.23	13.15	12.48

Example 6[2]**.** Here we produce the initial vector sequence by the quadratic iteration $S_{n+1} = G(S_n)$, where $S_0 = (1.5, 1.6, 1.7, 1.8)^T$,

$$G(S) = b + AS + Q(S)$$

with

$$A = \begin{bmatrix} 2.25 & 0.01 & 0.05 & 0.50 \\ 0.01 & 1.75 & 0.00 & 0.05 \\ 0.05 & 0.00 & 1.75 & 0.01 \\ 0.50 & 0.05 & 0.01 & 2.25 \end{bmatrix}, \quad b = \begin{bmatrix} -0.81 \\ -0.31 \\ -0.31 \\ -0.81 \end{bmatrix}, \quad Q(x) = -0.5 \begin{bmatrix} x_1^2 + x_1 x_4 \\ x_2^2 \\ x_3^2 \\ x_1 x_4 + x_4^2 \end{bmatrix}.$$

The sequence $\{S_n\}$ converges to the vector $S = (1, 1, 1, 1)^T$.

Table 4 (Example 6)

k	S_k	H_1^{k-2}	ε_2^{k-2}	H_2^{k-3}	ε_4^{k-4}	H_3^{k-4}	ε_6^{k-6}	H_4^{k-5}	ε_8^{k-8}
6	1.278	1.547	1.838	2.507	2.107	2.654	2.113		
8	1.506	1.990	2.227	2.836	2.651	2.918	2.877	2.863	2.878
10	1.720	2.325	2.586	3.123	3.154	3.110	3.544	2.621	3.754
12	1.925	2.625	2.924	3.474	3.641	2.932	4.187	2.932	4.550
14	2.123	2.914	3.250	3.857	4.117	3.302	4.797	3.518	5.272

Table 5

	k	2	3	4	5	6	7	8	9
	S_k	−0.900	−1.313	−1.699	−2.906	−2.488	−2.882	−3.276	−3.670
Example 3	H_{k-1}^0	0.507	0.532	0.683	7.346	7.419	7.871	7.628	7.119
	$\varepsilon_{\Delta_k}^{\delta_k}$	0.507	0.564	0.654	0.655	0.683	0.684	4.253	3.762
	S_k	−1.847	−2.215	−2.291	−3.074	−3.808	−4.271	−4.401	−4.800
Example 4	H_{k-1}^0	−0.110	−0.126	2.128	13.83	13.92	13.96	13.60	13.81
	$\varepsilon_{\Delta_k}^{\delta_k}$	−0.289	−0.774	0.057	2.196	3.962	13.04	12.62	12.91
	S_k	0.983	1.283	1.582	1.882	2.183	2.483	2.784	3.085
Example 5	H_{k-1}^0	1.185	2.178	3.227	5.324	7.523	9.751	12.81	14.09
	$\varepsilon_{\Delta_k}^{\delta_k}$	1.377	2.123	2.961	3.945	5.000	6.325	7.668	9.286
	S_k	0.714	0.880	1.024	1.156	1.278	1.395	1.506	1.615
Example 6	H_{k-1}^0	0.515	1.151	2.046	2.621	2.650	3.113	3.360	3.096
	$\varepsilon_{\Delta_k}^{\delta_k}$	0.750	1.134	1.406	1.800	2.113	2.523	2.878	3.332

Remark. From these tables we can see that according to Principle 1, VPA method is better than vector $\varepsilon-$algorithm except for Example 6. But according to Principle 2, the former is much better than the latter for all the above examples.

§5. Conclusion

It is well known that most acceleration formulas for vector sequences can be expressed as a linear combination of the original vectors (see [3]). A common feature of these methods is that the coefficients of the combination depend upon each component of the vectors. This seems quite natual, for we accelerate every component of the vectors. However, in many cases, the coefficients constructed from partial components can accelerate vector sequences as well. The reason lies in the fact that all components of the vectors are related to each other in general. The full information of the vectors may be got from partial components. With Theorem 2 in mind, we take the linear iterative scheme (2.12), which is commonly used in linear algebra computation, as an example. If the degree m of the minimal polynomial of the matrix A corresponding to $(S - S_0)$ is less than p, then by using m components we can get the true solution.

The examples given in the last section show that the partial components mothod (GHT) does accelerate the convergence for the given sequences in many cases. Furthermore, it sometimes behaves better than the vector $\varepsilon-$algorithm, which is now widely used. Another property of the Padé method is that it may cost fewer computations since it uses fewer components.

Finally, it should be pointed out that the Padé method indeed includes the case that all the components are used (i.e., $\vec{e} \geq 1$). Therefore the Henrici transformation is a special case of the GHT method ($\vec{e} = (1, \cdots, 1)$).

References

[1] P.R. Graves-Morris, Vector valued rational interpolation *I*, *Numer. Math.*, 42 (1983), 331–348.

[2] H. Sadok, About Henrici's transformation for accelerating vector sequences, *J. Comp. Appl. Math.*, 29 (1990), 101–110.

[3] D.A. Smith, W.F.Ford, and A.Sidi, Extrapolation methods for vector sequences, *SIAM Rev.*, 29 (1987), 199–233.

[4] P. Wynn, Acceleration techniques for iterative vector and matrix problems, *Math. Comp.*, 16 (1962), 301–322.

[5] Xu Guo-liang and A.Bultheel, Matrix Padé approximation: Definitions and properties, *Lin. Alg. Appl.*, 137/138(1990), 67–136.