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THE SPECTRAL-DIFFERENCE METHOD FOR COMPRESSIBLE FLOW*

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Abstract

A spectral-difference scheme is proposed for semi-periodic compressible flow with strict estimation.

§1. Introduction

Tani[1] proved the existence of the local smooth solution of compressible viscous flow. In [2,3], a difference method and a spectral method were given. We consider the semi-periodic problem and use the spectral-difference method which has been successfully applied to fluid flow (see [4-7]).

Let n_1 and n_2 be positive integers. Let $x' = (x_1, \cdots, x_{n_1})^*, x'' = (x_{n_1+1}, \cdots, x_n)^*$ and $x = (x_1, \cdots, x_n)^*$. Let $\Omega = \Omega_1 \times \Omega_2$ where

$$\Omega_1 = \{x' | 0 < x_j < 1, 1 \le j \le n_1\}, \qquad \Omega_2 = \{x'' | 0 < x_j < 2\pi, n_1 + 1 \le j \le n\}.$$

We denote by Γ the boundary of Ω_1. The closures of Ω and Ω_l are denoted by $\overline{\Omega}$ and $\overline{\Omega_l}$. Let u be the velocity and $u = (u^{(1)}, \cdots, u^{(n)})^*$. p is the pressure. T is the absolute temperature. ρ is the density. f is the external force and $f = (f^{(1)}, \cdots, f^{(n)})^*$. $\nu(T, \rho) > 0$ is the viscosity, $\nu'(T, \rho)$ is the second viscosity and $\kappa(T, \rho) = \nu'(T, \rho) - \frac{2}{3}\nu(T, \rho)$. $\mu(T, \rho) > 0$ is the heat conduction coefficient. $S(T, \rho)$ is the entropy, $S_T = \frac{\partial S}{\partial T}$ and $S_\rho = \frac{\partial S}{\partial \rho}$. In order to avoid the instability of computation, we put $\varphi = \ln \rho$ as in [2]. For simplicity, we suppose that $p = R_0 \rho T$, R_0 being a positive constant. Then we have

$$\begin{cases} \dfrac{\partial u^{(l)}}{\partial t} + (u \cdot \nabla) u^{(l)} - e^{-\varphi} \dfrac{\partial}{\partial x_l}(\kappa \nabla \cdot u) - e^{-\varphi} \displaystyle\sum_{j=1}^{n} \dfrac{\partial}{\partial x_j}[\nu(\dfrac{\partial u^{(l)}}{\partial x_j} + \dfrac{\partial u^{(j)}}{\partial x_l})] \\ \quad + R_0 \dfrac{\partial T}{\partial x_l} + R_0 T \dfrac{\partial \varphi}{\partial x_l} = f^{(l)}, \quad l = 1, \cdots, n, \quad (x,t) \in \Omega \times (0, t_0], \\ \dfrac{\partial T}{\partial t} + (u \cdot \nabla) T - e^{-\varphi} T^{-1} S_T^{-1} (\nabla \cdot \mu \nabla) T - \dfrac{1}{2} \nu e^{-\varphi} T^{-1} S_T^{-1} \displaystyle\sum_{j,l=1}^{n} (\dfrac{\partial u^{(l)}}{\partial x_j} + \dfrac{\partial u^{(j)}}{\partial x_l})^2 \\ \quad - \kappa e^{-\varphi} T^{-1} S_T^{-1} (\nabla \cdot u)^2 - S_\varphi S_T^{-1} (\nabla \cdot u) = 0, \quad (x,t) \in \Omega \times (0, t_0], \\ \dfrac{\partial \varphi}{\partial t} + (u \cdot \nabla) \varphi + \nabla \cdot u = 0, \quad (x,t) \in \Omega \times (0, t_0]. \end{cases} \tag{1.1}$$

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Suppose that all functions have the period 2π for the variable $x_j (n_1+1\le j\le n)$ and that there exist positive constants $B_0, B_1, B_2, \nu_0, \nu_1, \kappa_1, p_0, \mu_0, \mu_1, S_0, S_1, S_2, \Phi_0$ and Φ_1 such that if

$$(T,\varphi)\in Q=\{(T,\varphi)|B_0<T<B_1, |\varphi|<B_2\},$$

then $\left|\frac{\partial\eta}{\partial\varphi}\right|$ is bounded where $\eta=\nu,\kappa,\mu,S_T,S_\varphi$ and $q=T,\varphi$, and

$$\begin{aligned}&\nu_0<\nu<\nu_1, \quad |\kappa|<\kappa_1, \quad \min(n\kappa+(n+1)\nu,\nu)>p_0,\\ &\mu_0<\mu<\mu_1, \quad S_0<S_T<S_1, \quad |S_\varphi|<S_2, \quad \Phi_0<e^{-\varphi}<\Phi_1.\end{aligned} \tag{1.2}$$

§2. The Scheme and Error Estimation

Let J be a positive integer and $h=\frac{1}{J}$. The mesh domain is defined by

$$\Omega_{1,h}=\{x'|x_j=h,2h,\cdots,1-h, \text{ and } 1\le j\le n_1\},$$

$$\overline{\Omega}_{1,h}=\Omega_{1,h}\cup\Gamma_h, \qquad \Omega_h=\Omega_{1,h}\times\Omega_2, \qquad \overline{\Omega}_h=\overline{\Omega}_{1,h}\times\overline{\Omega}_2,$$

where Γ_h is the boundary of $\Omega_{1,h}$, in the following form:

$$\Gamma_h=\bigcup_{j=1}^{n_1}\Gamma_j, \qquad \Gamma_j=\Gamma_{+j}\bigcup\Gamma_{-j},$$

$$\Gamma_{+j}=\{x'|x_j=1 \text{ and } x_{j'}=h,2h,\cdots,1-h, \text{ for } j'\ne j\},$$

$$\Gamma_{-j}=\{x'|x_j=0 \text{ and } x_{j'}=h,2h,\cdots,1-h, \text{ for } j'\ne j\}.$$

In addition, let $l''=(l_{n_1+1},\cdots,l_n), l''x''=\sum_{j=n_1+1}^{n} l_jx_j$ and $|l''|_\infty=\max_{n_1+1\le j\le n}|l_j|$, where $l_j(n_1+1\le j\le n)$ is an integer. For any positive integer N, define

$$\overline{V}_N=\text{span}\,\{e^{il''x''}|\ |l''|_\infty\le N\}.$$

Let V_N be the subset of $\overline{V}_N$ involving all real valued functions. Let P_N be L^2-orthogonal projection from $L^2(\Omega_2)$ onto V_N. Let $\tau>0$ be the mesh size of time t, and

$$\lambda=\max(2n_1\tau h^{-2}, n_2\tau N^2), \qquad Z_\tau=\left\{t=k\tau|k=0,1,\cdots,\left[\frac{t_0}{\tau}\right]\right\}.$$

Denote the value of the function η at point x and time $k\tau$ by $\eta(x,k)$, or simply by $\eta(k)$ or η. Assume that η,ξ,a,b are scalar functions and $w=(w^{(1)},\cdots,w^{(n)})^*$. Define

$$\eta_{x_j}(x,k)=\frac{1}{h}[\eta(x+he_j,k)-\eta(x,k)], \qquad \eta_{\bar{x}_j}(x,k)=\eta_{x_j}(x-he_j,k),$$

$$\eta_{\hat{x}_j}(x,k)=\frac{1}{2}[\eta_{x_j}(x,k)+\eta_{\bar{x}_j}(x,k)], \qquad \eta_t(x,k)=\frac{1}{\tau}[\eta(x,k+1)-\eta(x,k)],$$

where e_j is the n-dimensional vector whose jth component equals 1 and the others are zero. Also define

$$D_j^+(\eta)=\begin{cases}\eta_{x_j}, & \text{if } 1\le j\le n_1,\\ \dfrac{\partial\eta}{\partial x_j}, & \text{if } n_1+1\le j\le n\end{cases},\quad D_j^-(\eta)=\begin{cases}\eta_{\bar{x}_j}, & \text{if } 1\le j\le n_1,\\ \dfrac{\partial\eta}{\partial x_j}, & \text{if } n_1+1\le j\le n\end{cases},$$

$$D^+(w)=\sum_{j=1}^n D_j^+(w^{(j)}),\quad D^-(w)=\sum_{j=1}^n D_j^-(w^{(j)}),\quad D(w)=\frac{1}{2}[D^+(w)+D^-(w)],$$

$$\Delta_j^\nu\eta=\frac{1}{2}D_j^+[\nu D_j^-(\eta)]+\frac{1}{2}D_j^-[\nu D_j^+(\eta)],\ \Delta^\nu\eta=\sum_{j=1}^n\Delta_j^\nu\eta,\ D_j(\eta)=\frac{1}{2}[D_j^+(\eta)+D_j^-(\eta)],$$

$$d^{(\alpha)}(\eta,w)=\sum_{j=1}^n[\alpha w^{(j)}D_j(\eta)+(1-\alpha)D_j(\eta w^{(j)})],\quad 0\le\alpha\le 1.$$

Now let u_N, T_N and φ_N be the approximations to u, T and φ, where $u_N\in(V_N)^n, T_N$, $\varphi_N\in V_N$ for all $x'\in\overline{\Omega}_{1,h}$ and $t\in Z_\tau$. The spectral-difference scheme for solving (1.1) is as follows:

$$\begin{cases} u_{Nt}^{(l)}+P_N[d^{(\alpha_1)}(u_N^{(l)},u_N)-(1-\alpha_1)u_N^{(l)}D(u_N)-e^{-\varphi_N}H_1^{(l)}(\kappa(T_N,\varphi_N),u_N)\\ \quad -e^{-\varphi_N}\Delta^{\nu(T_N,\varphi_N)}u_N^{(l)}-e^{-\varphi_N}H_2^{(l)}(\nu(T_N,\varphi_N),u_N)+R_0D_l(T_N)\\ \quad +R_0T_ND_l(\varphi_N)]=P_Nf^{(l)},\quad l=1,\cdots,n,\\ T_{Nt}+P_N[d^{(\alpha_2)}(T_N,u_N)-(1-\alpha_2)T_ND(u_N)-e^{-\varphi_N}T_N^{-1}S_T^{-1}(T_N,\varphi_N)\Delta^{\mu(T_N,\varphi_N)}T_N\\ \quad -e^{-\varphi_N}H_3(u_N,T_N,\varphi_N)]=0,\\ \varphi_{Nt}+P_N[d^{(\alpha_3)}(\varphi_N,u_N)-(1-\alpha_3)\varphi_ND(u_N)]+D(u_N)=0, \end{cases}\tag{2.1}$$

where $(x,t)\in\Omega_h\times Z_\tau, 0\le\alpha_q\le 1 (q=1,2,3)$, and

$$H_1^{(l)}(\kappa(T_N,\varphi_N),u_N)=\frac{1}{2}D_l^+[\kappa(T_N,\varphi_N)D^-(u_N)]+\frac{1}{2}D_l^-[\kappa(T_N,\varphi_N)D^+(u_N)],$$

$$H_2^{(l)}(\nu(T_N,\varphi_N),u_N)=\frac{1}{2}\sum_{j=1}^n\{D_j^+[\nu(T_N,\varphi_N)D_l^-(u^{(j)})]+D_j^-[\nu(T_N,\varphi_N)D_l^+(u^{(j)})]\},$$

$$H_3(u_N,T_N,\varphi_N)=\frac{1}{2}\nu(T_N,\varphi_N)T_N^{-1}S_T^{-1}(T_N,\varphi_N)\sum_{j,l=1}^n[D_j(u^{(l)})+D_l(u^{(j)})]^2$$
$$+\kappa(T_N,\varphi_N)T_N^{-1}S_T^{-1}(T_N,\varphi_N)[D(u_N)]^2+e^{\varphi_N}S_\varphi(T_N,\varphi_N)S_T^{-1}(T_N,\varphi_N)D(u_N).$$

For error estimation, we need the following semi-discrete inner products and norms:

$$(\eta(x',k),\xi(x',k))_{\Omega_2}=\frac{1}{(2\pi)^{n_2}}\int_{\Omega_2}\eta(x,k)\xi(x,k)dx'',$$

$$\|\eta(x',k)\|_{\Omega_2}^2=(\eta(x',k),\eta(x',k))_{\Omega_2},$$

$$(\eta(k),\xi(k))=\sum_{x'\in\Omega_{1,h}}h^{n_1}(\eta(x',k),\xi(x',k))_{\Omega_2},\qquad \|\eta(k)\|^2=(\eta(k),\eta(k)),$$

$$(\eta(k),\xi(k))_{\Gamma_{+j}} = \sum_{x'\in\Gamma_{+j}} h^{n_1-1}(\eta(x',k),\xi(x',k))_{\Omega_2},$$

$$(\eta(k),\xi(k))_{\Gamma_{-j}} = \sum_{x'\in\Gamma_{-j}} h^{n_1-1}(\eta(x',k),\xi(x',k))_{\Omega_2},$$

$$\|\eta(k)\|^2_{\Gamma_j} = (\eta(k),\eta(k))_{\Gamma_{+j}} + (\eta(k),\eta(k))_{\Gamma_{-j}}, \qquad \|\eta(k)\|^2_{\Gamma_h} = \sum_{j=1}^{n_1}\|\eta(k)\|^2_{\Gamma_j},$$

$$|\eta(k)|_1^2 = \frac{1}{2}\sum_{j=1}^{n}[\|D_j^+(\eta(k))\|^2 + \|D_j^-(\eta(k))\|^2],$$

$$|\eta(k)|^2_{\Gamma_h,1} = \frac{1}{2}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq j}}^{n}[\|D_l^+(\eta(k))\|^2_{\Gamma_j} + \|D_l^-(\eta(k))\|^2_{\Gamma_j}],$$

$$\|\eta(k)\|_\infty = \sup_{x\in\Omega_h}|\eta(x,k)|, \qquad \|w(k)\|^2 = \sum_{j=1}^{n}\|w^{(j)}(k)\|^2,$$

$$\|w(k)\|_\infty = \max_{1\le j\le n}\|w^{(j)}(k)\|_\infty, \text{ etc.}$$

We also introduce some summations as follows:

$$Y(w,\eta) = \frac{1}{2}\sum_{j=1}^{n_1}\sum_{\substack{l=1\\ l\neq j}}^{n}[\|\eta D_l^+(w)\|^2_{\Gamma_j} + \|\eta D_l^-(w)\|^2_{\Gamma_j}],$$

$$A_{\Gamma_h}(\eta,\xi,w) = \frac{1}{2}\sum_{j=1}^{n_1}[(\eta,(\xi w^{(j)})^{-x_j})_{\Gamma_{+j}} + (\eta^{-x_j},\xi w^{(j)})_{\Gamma_{+j}} - (\eta,(\xi w^{(j)})^{+x_j})_{\Gamma_{-j}}$$
$$- (\eta^{+x_j},\xi w^{(j)})_{\Gamma_{-j}}],$$

$$D^{(j)}_{\Gamma_l}(a,b,\eta,\xi) = (aD_j^-(\eta),(b\xi)^{-x_l})_{\Gamma_{+l}} + ([aD_j^+(\eta)]^{-x_l},b\xi)_{\Gamma_{+l}}$$
$$- (aD_j^+(\eta),(b\xi)^{+x_l})_{\Gamma_{-l}} - ([aD_j^-(\eta)]^{+x_l},b\xi)_{\Gamma_{-l}},$$

where $\eta^{+x_j}(x,k)$ and $\eta^{-x_j}(x,k)$ denote $\eta(x+he_j,k)$ and $\eta(x-he_j,k)$ respectively. Furthermore, let $B(\Omega_2)$ be a Banach space of functions defined on Ω_2,

$$C(\Omega_1,B(\Omega_2)) = \left\{\eta \mid \eta:\bar{\Omega}_1 \to B(\Omega_2), \|\eta\|_{C(\Omega_1,B(\Omega_2))} = \max_{x'\in\bar{\Omega}_1}\|\eta(x')\|_{B(\Omega_2)} < \infty\right\}.$$

We can define the spaces $C^l(\Omega_1,B(\Omega_2))(l=1,2,\cdots)$ and $C(0,t_0;C^l(\Omega_1,B(\Omega_2)))$. For simplicity, $\|\eta\|_{C(0,t_0;C^l(\Omega_1,B(\Omega_2)))}$ and $C^l(\Omega_1,H^\sigma_p(\Omega_2))$ are denoted by $|||\eta|||_{C^l(\Omega_1,B(\Omega_2))}$ and $C^l(H^\sigma)$ where $H^\sigma_p(\Omega_2)$ is the subset of $H^\sigma(\Omega_2)$, the elements of which have the period 2π for $x_j(n_1+1\le j\le n)$.

Now let $\tilde{u} = u_N - P_N u$, $\tilde{T} = T_N - P_N T$ and $\tilde{\varphi} = \varphi_N - P_N\varphi$. Then the errors satisfy

$$\begin{cases} L_1^{(l)}(\tilde{u},\tilde{T},\tilde{\varphi}) \equiv \tilde{u}_t^{(l)} + P_N\Big[\sum_{j=1}^{7}\tilde{F}_j^{(l)} - \sum_{j=1}^{3}\tilde{E}_j^{(l)}\Big] = P_N\tilde{f}_1^{(l)}, \quad l=1,\cdots,n, \\ L_2(\tilde{u},\tilde{T},\tilde{\varphi}) \equiv \tilde{T}_t + P_N\Big[\tilde{F}_8 + \tilde{F}_9 - \sum_{j=4}^{8}\tilde{E}_j\Big] = P_N\tilde{f}_2, \\ L_3(\tilde{u},\tilde{\varphi}) \equiv \tilde{\varphi}_t + P_N[\tilde{F}_{10} - \tilde{E}_9] = P_N\tilde{f}_3 \end{cases} \tag{2.2}$$

where

$$\tilde{F}_1^{(l)} = -e^{-P_N\varphi-\tilde{\varphi}} H_1^{(l)}(\kappa(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi}), \tilde{u}),$$

$$\tilde{F}_2^{(l)} = -e^{-P_N\varphi-\tilde{\varphi}} H_1^{(l)}(\kappa(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi}) - \kappa(P_NT, P_N\varphi), P_Nu),$$

$$\tilde{F}_3^{(l)} = -e^{-P_N\varphi-\tilde{\varphi}} \Delta^{\nu(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi})} \tilde{u}^{(l)},$$

$$\tilde{F}_4^{(l)} = -e^{-P_N\varphi-\tilde{\varphi}} (\Delta^{\nu(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi})} P_Nu^{(l)} - \Delta^{\nu(P_NT, P_N\varphi)} P_Nu^{(l)}),$$

$$\tilde{F}_5^{(l)} = -e^{-P_N\varphi-\tilde{\varphi}} H_2^{(l)}(\nu(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi}), \tilde{u}),$$

$$\tilde{F}_6^{(l)} = -e^{-P_N\varphi-\tilde{\varphi}} H_2^{(l)}(\nu(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi}) - \nu(P_NT, P_N\varphi), P_Nu),$$

$$\tilde{F}_7^{(l)} = R_0 P_NT D_l(\tilde{\varphi}),$$

$$\tilde{F}_8 = -e^{-P_N\varphi-\tilde{\varphi}} (P_NT+\tilde{T})^{-1} S_T^{-1}(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi}) \Delta^{\mu(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi})} \tilde{T},$$

$$\tilde{F}_9 = -e^{-P_N\varphi-\tilde{\varphi}} (P_NT+\tilde{T})^{-1} S_T^{-1}(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi}) (\Delta^{\mu(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi})} P_NT$$
$$- \Delta^{\mu(P_NT, P_N\varphi)} P_NT),$$

$$\tilde{F}_{10} = d^{(\alpha_3)}(\tilde{\varphi}, P_Nu),$$

$$\tilde{E}_1^{(l)} = -d^{(\alpha_1)}(\tilde{u}^{(l)}, P_Nu+\tilde{u}) - d^{(\alpha_1)}(P_Nu^{(l)}, \tilde{u}) + (1-\alpha_1)(P_Nu^{(l)}+\tilde{u}^{(l)})D(\tilde{u})$$
$$+ (1-\alpha_1)\tilde{u}^{(l)} D(P_Nu),$$

$$\tilde{E}_2^{(l)} = (e^{-P_N\varphi-\tilde{\varphi}} - e^{-P_N\varphi})[H_1^{(l)}(\kappa(P_NT, P_N\varphi), P_Nu) + \Delta^{\nu(P_NT, P_N\varphi)} P_Nu^{(l)}$$
$$+ H_2^{(l)}(\nu(P_NT, P_N\varphi), P_Nu)],$$

$$\tilde{E}_3^{(l)} = -R_0 D_l(\tilde{T}) - R_0\tilde{T} D_l(\tilde{\varphi}) - R_0\tilde{T} D_l(P_N\varphi),$$

$$\tilde{E}_4 = -d^{(\alpha_2)}(\tilde{T}, P_Nu+\tilde{u}) - d^{(\alpha_2)}(P_NT, \tilde{u}) + (1-\alpha_2)(P_NT+\tilde{T})D(\tilde{u})$$
$$+ (1-\alpha_2)\tilde{T} D(P_Nu),$$

$$\tilde{E}_5 = e^{-P_N\varphi-\tilde{\varphi}} (P_NT+\tilde{T})^{-1} [S_T^{-1}(P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi})$$
$$- S_T^{-1}(P_NT, P_N\varphi)] \Delta^{\mu(P_NT, P_N\varphi)} P_NT,$$

$$\tilde{E}_6 = e^{-P_N\varphi-\tilde{\varphi}} [(P_NT+\tilde{T})^{-1} - (P_NT)^{-1}] S_T^{-1}(P_NT, P_N\varphi) \Delta^{\mu(P_NT, P_N\varphi)} P_NT,$$

$$\tilde{E}_7 = (e^{-P_N\varphi-\tilde{\varphi}} - e^{-P_N\varphi})(P_NT)^{-1} S_T^{-1}(P_NT, P_N\varphi) \Delta^{\mu(P_NT, P_N\varphi)} P_NT,$$

$$\tilde{E}_8 = e^{-P_N\varphi-\tilde{\varphi}} H_3(P_Nu+\tilde{u}, P_NT+\tilde{T}, P_N\varphi+\tilde{\varphi}) - e^{-P_N\varphi} H_3(P_Nu, P_NT, P_N\varphi),$$

$$\tilde{E}_9 = -d^{(\alpha_3)}(\tilde{\varphi}, \tilde{u}) - d^{(\alpha_3)}(P_N\varphi, \tilde{u}) + [(1-\alpha_3)(P_N\varphi+\tilde{\varphi}) - 1]D(\tilde{u})$$
$$+ (1-\alpha_3)\tilde{\varphi} D(P_Nu),$$

$$\tilde{f}_1^{(l)} = \frac{\partial u^{(l)}}{\partial t} - u_t^{(l)} + (u\cdot\nabla)u^{(l)} - d^{(\alpha_1)}(P_N u^{(l)}, P_N u) + (1-\alpha_1)P_N u^{(l)} D(P_N u)$$

$$+ e^{-P_N\varphi} H_1^{(l)}(\kappa(P_N T, P_N\varphi), P_N u) - e^{-\varphi}\frac{\partial}{\partial x_l}(\kappa(T,\varphi)\nabla\cdot u)$$

$$+ e^{-P_N\varphi} H_2^{(l)}(\nu(P_N T, P_N\varphi), P_N u) - e^{-\varphi}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(\nu(T,\varphi)\frac{\partial u^{(j)}}{\partial x_l})$$

$$+ e^{-P_N\varphi}\Delta^{\nu(P_N T, P_N\varphi)} P_N u^{(l)} - e^{-\varphi}(\nabla\cdot\nu(T,\varphi)\nabla)u^{(l)} + R_0\frac{\partial T}{\partial x_l}$$

$$- R_0 D_l(T) + R_0 T\frac{\partial\varphi}{\partial x_l} - R_0 P_N T D_l(P_N\varphi),$$

$$\tilde{f}_2 = \frac{\partial T}{\partial t} - T_t + (u\cdot\nabla)T - d^{(\alpha_2)}(P_N T, P_N u) + (1-\alpha_2)P_N T D(P_N u)$$

$$+ e^{-P_N\varphi}(P_N T)^{-1} S_T^{-1}(P_N T, P_N\varphi)\Delta^{\mu(P_N T, P_N\varphi)} P_N T$$

$$- e^{-\varphi} T^{-1} S_T^{-1}(T,\varphi)(\nabla\cdot\mu(T,\varphi)\nabla)T + e^{-P_N\varphi} H_3(P_N u, P_N T, P_N\varphi)$$

$$- e^{-\varphi}[\frac{1}{2}\nu(T,\varphi)T^{-1} S_T^{-1}(T,\varphi)\sum_{j,l=1}^{n}\Big(\frac{\partial u^{(l)}}{\partial x_j} + \frac{\partial u^{(j)}}{\partial x_l}\Big)^2$$

$$+ \kappa(T,\varphi)T^{-1} S_T^{-1}(T,\varphi)(\nabla\cdot u)^2 + e^{\varphi} S_\varphi(T,\varphi) S_T^{-1}(T,\varphi)(\nabla\cdot u)],$$

$$\tilde{f}_3 = \frac{\partial\varphi}{\partial t} - \varphi_t + (u\cdot\nabla)\varphi - d^{(\alpha_3)}(P_N\varphi, P_N u) + (1-\alpha_3)P_N\varphi D(P_N u)$$

$$+ \nabla\cdot u - D(u).$$

Hereafter, let $\tilde{f}_1 = (\tilde{f}_1^{(1)}, \cdots, \tilde{f}_1^{(n)})^*$ and C_0 denote a positive constant independent of any functions and coefficients, which may be different in different cases. Suppose that

$$h\|\tilde{\varphi}\|_{\Gamma_h}^2 \le C_0\|\tilde{\varphi}\|^2 + \|\tilde{f}_4\|^2, \quad \|\tilde{f}_4\|^2 \le C_1 N^{-n_2} h^{n_1+2}, \tag{2.3}$$

$$|u| \le C_0 h, \quad (x,t)\in\Gamma_h\times\Omega_2\times Z_\tau, \tag{2.4}$$

$$\lambda \le \frac{1}{32n}\min\Big(\frac{p_0\Phi_0}{\Phi_1^2(\kappa_1^2+\nu_1^2)}, \frac{4\mu_0\Phi_0 B_0^2 S_0^2}{\mu_1^2\Phi_1^2 B_1 S_1}\Big), \tag{2.5}$$

where C_1 is a positive constant which may depend on φ. Moreover,

$$\tilde{E}(\tilde{u},\tilde{T},\tilde{\varphi};k) = \|\tilde{u}(k)\|^2 + \|\tilde{T}(k)\|^2 + \|\tilde{\varphi}(k)\|^2 + \frac{\tau}{2}\sum_{j=0}^{k-1}[p_0\Phi_0|\tilde{u}(j)|_1^2$$

$$+ \mu_0\Phi_0 B_1^{-1} S_1^{-1}|\tilde{T}(j)|_1^2] + \frac{\tau^2}{2}\sum_{j=0}^{k-1}[\|\tilde{u}_t(j)\|^2 + \|\tilde{T}_t(j)\|^2 + \|\tilde{\varphi}_t(j)\|^2],$$

$$\rho^*(k) = \|\tilde{u}(0)\|^2 + \|\tilde{T}(0)\|^2 + \|\tilde{\varphi}(0)\|^2 + \tau\sum_{j=0}^{k-1}\sum_{l=1}^{4}\|\tilde{f}_l(j)\|^2 + \frac{\tau}{h}\sum_{j=0}^{k-1}[\|\tilde{u}(j)\|^2_{\Gamma_h}$$
$$+ \|\tilde{T}(j)\|^2_{\Gamma_h} + h|\tilde{u}(j)|^2_{\Gamma_h,1} + \|(\tilde{T}(j))^2\|^2_{\Gamma_h} + \|\tilde{T}(j)\tilde{u}(j)\|^2_{\Gamma_h} + \|\tilde{\varphi}(j)\tilde{u}(j)\|^2_{\Gamma_h}$$
$$+ \|\tilde{\varphi}(j)\tilde{T}(j)\|^2_{\Gamma_h} + N^{n_2}h^{1-n_1}\|\tilde{u}(j)\|^4_{\Gamma_h} + Y(\tilde{u}(j),\tilde{T}(j)) + Y(\tilde{u}(j),\tilde{\varphi}(j))].$$

Theorem 1. *Assume that the following conditions are fulfilled:*

(i) (1.2) *holds and* $(T,\varphi) \in Q$,

(ii) *for* $\beta > n_2/2$, $\varphi \in C(0,t_0;C^1(H^\beta)\bigcap C(H^{\beta+1}))$, $T, u^{(l)} \in C(0,t_0;C^2(H^\beta)\bigcap C^1(H^{\beta+1})\bigcap C(H^{\beta+2}))$; *and for some* $\sigma \geq n_2/2+1$, $u^{(l)} \in C(0,t_0;C(H^{\beta+\delta}))$, $l=1,\cdots,n$, $\delta = \sigma - n_2/2$,

(iii) $h^2 = O(N^{n_2-2\sigma})$, $\alpha_3 = 1/2$, *and* (2.3), (2.4), (2.5) *hold,*

(iv) *for* $k\tau \leq t_0^*(\rho^*)$, $\quad \rho^*(k) \leq M_1N^{-n_2}h^{n_1+2}$.

Then

$$\tilde{E}(\tilde{u},\tilde{T},\tilde{\varphi};k) \leq M_2\rho^*(k)e^{M_3k\tau},$$

where $t_0^*(\rho^*)$ *is a positive constant depending on* ρ^*, C_1, M_1, M_2 *and* M_3 *with* M_l *being positive constants depending on* u, T, φ *and the functions mentioned in* (1.2).

Now suppose that $g_1(x,t), g_2(x,t), u_0(x), T_0(x)$ and $\varphi_0(x)$ are sufficiently smooth, $g_1 = (g_1^{(1)},\cdots,g_1^{(n)})^*$, $u_0 = (u_0^{(1)},\cdots,u_0^{(n)})^*$ and

$$\begin{cases} u(x,t) = g_1(x,t), \quad T(x,t) = g_2(x,t), \quad (x,t) \in \Gamma\times\Omega_2\times(0,t_0], \\ u(x,0) = u_0(x), \quad T(x,0) = T_0(x), \quad \varphi(x,0) = \varphi_0(x), \quad x \in \overline{\Omega}. \end{cases} \tag{2.6}$$

Because $\Gamma\times\Omega_2$ is a characteristic surface of the last equation in (1.1), we could not give the boundary value of φ. We take

$$\begin{cases} u_N(x,t) = P_Ng_1(x,t), \quad T_N(x,t) = P_Ng_2(x,t), \quad (x,t) \in \Gamma_h\times\Omega_2\times Z_\tau, \\ u_N(x,0) = P_Nu_0(x), \quad T_N(x,0) = P_NT_0(x), \quad \varphi_N(x,0) = P_N\varphi_0(x), \quad x \in \overline{\Omega_h}. \end{cases} \tag{2.7}$$

The value $\varphi_N(x,t)$ on $\Gamma_h\times\Omega_2\times Z_\tau$ is calculated by extrapolation.

Theorem 2. *Assume that the following conditions are fulfilled:*

(i) (2.6) *and* (2.7) *hold,*

(ii) *the conditions* (i) *and* (iii) *of Theorem* 1 *hold,*

(iii) $n_1 = 1$ *and* $h = O(N^{-n_2})$,

(iv) $\beta > \frac{n_2}{2}, r \geq \max(\beta, 2n_2)$,

$u^{(l)} \in C(0,t_0;C^4(L^2)\bigcap C^3(H^1)\bigcap C^2(H^r)\bigcap C^1(H^{r+1})\bigcap C(H^{r+2})\bigcap C(H^{\beta+n_2}))$,

$T \in C(0,t_0;C^4(L^2)\bigcap C^2(H^r)\bigcap C^1(H^{\beta+1})\bigcap C(H^{r+2}))$,

$\varphi \in C(0,t_0;C^3(L^2)\bigcap C^2(H^\beta)\bigcap C^1(H^r)\bigcap C(H^{r+1}))$,

$\dfrac{\partial^2\eta}{\partial t^2} \in C(0,t_0;C(L^2))$, $\quad \xi \in C^3(Q)$,

where $\eta = u^{(l)}, T$ *and* φ, $\xi = \nu, \kappa$ *and* μ, $l = 1,\cdots,n$. *Then for* $k\tau \leq t_0$,

$$\tilde{E}(\tilde{u},\tilde{T},\tilde{\varphi};k) \leq M_4e^{M_5k\tau}(\tau^2 + h^4 + N^{-2r}).$$

where M_4, M_5 are positive constants depending on $C_1, \|S\|_{C^2(Q)}$ and the norms of $u, T, \varphi, \nu, \kappa, \mu$ in the spaces mentioned.

Remark 1. The condition $(T,\varphi)\in Q$ is satisfied (see [1]).

Remark 2. For the problem with fixed wall, $u=0$ on $\Gamma\times\Omega_2$ and thus (2.4) holds. If the value of φ_N on $\Gamma_h\times\Omega_2\times Z_\tau$ is computed by linear extrapolation, then (2.3) also holds.

§3. The Proof of Theorems

Lemma 1. *If $\eta(x)$ is smooth enough on $\overline{\Omega}$ and $\beta > n_2/2$, then*

$$\|P_N\eta\|_\infty \le C_0\|\eta\|_{C(H^\beta)},$$
$$\|D_j^\pm(P_N\eta)\|_\infty \le C_0\|\eta\|_{C^1(H^\beta)}, \quad \text{for } 1\le j\le n_1,$$
$$\|\frac{\partial}{\partial x_j}(P_N\eta)\|_\infty \le C_0\|\eta\|_{C(H^{\beta+1})}, \quad \text{for } n_1+1\le j\le n, \text{ etc.}$$

Lemma 2. *If $\eta\in C(H^{\beta+\delta})$ for $\beta>n_2/2$ and $\delta\ge 0$, then*

$$\|P_N\eta-\eta\|_\infty \le C_0N^{-\delta}\|\eta\|_{C(H^{\beta+\delta})}.$$

Lemma 3. *If a, b, η, ξ have the period 2π for $x_j (n_1+1\le j\le n)$ and are suitably smooth, then for $1\le i\le n$,*

$$(D_m^+[aD_i^-(\eta)]+D_m^-[aD_i^+(\eta)], b\xi)+(aD_i^+(\eta), bD_m^+(\xi))+(aD_i^-(\eta), bD_m^-(\xi))$$
$$=\begin{cases} -(aD_i^+(\eta), \xi^{+x_m}b_{x_m})-(aD_i^-(\eta),\xi^{-x_m}b_{\bar{x}_m})+2D_{\Gamma_m}^{(i)}(a,b,\eta,\xi), \\ \qquad\qquad \text{if } 1\le m\le n_1, \\ -2\left(aD_i(\eta), \xi\dfrac{\partial b}{\partial x_m}\right), \quad \text{if } n_1+1\le m\le n. \end{cases}$$

Lemma 4. *If η, ξ, w have the period 2π for $x_j (n_1+1\le j\le n)$ and are suitably smooth, then*

$$(\eta, d^{(\frac12)}(\xi,w))+(\xi, d^{(\frac12)}(\eta,w))=\frac12 A_{\Gamma_h}(\eta,\xi,w)+\frac12 A_{\Gamma_h}(\xi,\eta,w).$$

We only prove Theorem 1. We take the following inner product:

$$2\sum_{l=1}^n(\tilde{u}^{(l)}+\tau\tilde{u}_t^{(l)}, L_1^{(l)}(\tilde{u},\tilde{T},\tilde{\varphi})-P_N\tilde{f}_1^{(l)})+2(\tilde{T}+\tau\tilde{T}_t, L_2(\tilde{u},\tilde{T},\tilde{\varphi})-P_N\tilde{f}_2)$$
$$+2(\tilde{\varphi}+\tau\tilde{\varphi}_t, L_3(\tilde{u},\tilde{\varphi})-P_N\tilde{f}_3)=0. \tag{3.1}$$

It follows from conditions (i), (ii) and Lemma 2 that if N is suitably large, then $(P_NT, P_N\varphi)\in Q$. Let ε be a small positive constant to be fixed later, and M be a positive constant depending on C_1, the bounds in (1.2) and the norms of u, T, φ in the spaces mentioned in condition (ii). Let $M^*=M^*(\tilde{T},\tilde{\varphi})$ and $M^{**}=M^{**}(\tilde{T},\tilde{\varphi})$ be positive functions such that if

$$\begin{cases} \|\tilde{T}\|^2\le \tilde{B}^2(2N+1)^{-n_2}h^{n_1}, & \|\tilde{T}\|_{\Gamma_h}^2\le\tilde{B}^2(2N+1)^{-n_2}h^{n_1-1}, \\ \|\tilde{\varphi}\|^2\le\tilde{B}^2(2N+1)^{-n_2}h^{n_1}, & \|\tilde{\varphi}\|_{\Gamma_h}^2\le\tilde{B}^2(2N+1)^{-n_2}h^{n_1-1}, \end{cases} \tag{3.2}$$

where $\tilde{B}$ is a sufficiently small positive constant, then $M^{**} \le \tilde{B}M^*$ and $M^* \le M$. Clearly, (3.2) implies $(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}) \in Q$. Now we suppose that (3.2) is satisfied which can be ensured later.

Firstly, we put $m = l, i = j, a = \kappa(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}), b = e^{-P_N\varphi - \tilde{\varphi}}, \eta = \tilde{u}^{(j)}, \xi = \tilde{u}^{(l)}$ in Lemma 3 and sum it up for l, j. Then

$$2\sum_{l=1}^{n}(\tilde{u}^{(l)}, \tilde{F}_1^{(l)}) = \mathcal{D}_1 + (e^{-P_N\varphi-\tilde{\varphi}}\kappa(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}), [D^+(\tilde{u})]^2 + [D^-(\tilde{u})]^2) + \mathcal{D}_1'$$

where

$$\mathcal{D}_1 = -2\sum_{l=1}^{n_1}\sum_{j=1}^{n} D_{\Gamma_l}^{(j)}(\kappa(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}), e^{-P_N\varphi-\tilde{\varphi}}, \tilde{u}^{(j)}, \tilde{u}^{(l)}),$$

$$|\mathcal{D}_1'| \le \varepsilon|\tilde{u}|_1^2 + M^*[(1 + N^{n_2}h^{-n_1-2}\|\tilde{\varphi}\|^2)(\|\tilde{u}\|^2 + h\|\tilde{u}\|_{\Gamma_h}^2) + h^{-1}\|\tilde{\varphi}\tilde{u}\|_{\Gamma_h}^2]. \qquad (3.3)$$

Similarly,

$$2\sum_{l=1}^{n}(\tilde{u}^{(l)}, \tilde{F}_q^{(l)}) = \mathcal{D}_q + \mathcal{D}_q', \quad q = 2, 4, 6,$$

$$2\sum_{l=1}^{n}(\tilde{u}^{(l)}, \tilde{F}_3^{(l)}) = \mathcal{D}_3 + \sum_{l=1}^{n}\sum_{j=1}^{n}(e^{-P_N\varphi-\tilde{\varphi}}\nu(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}), [D_j^+(\tilde{u}^{(l)})]^2 + [D_j^-(\tilde{u}^{(l)})]^2) + \mathcal{D}_3',$$

$$2\sum_{l=1}^{n}(\tilde{u}^{(l)}, \tilde{F}_5^{(l)}) = \mathcal{D}_5 + (e^{-P_N\varphi-\tilde{\varphi}}\nu(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}), [D^+(\tilde{u})]^2 + [D^-(\tilde{u})]^2) + \mathcal{D}_5'$$

where

$$\mathcal{D}_2 = -2\sum_{l=1}^{n_1}\sum_{j=1}^{n} D_{\Gamma_l}^{(j)}(\kappa(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}) - \kappa(P_N T, P_N\varphi), e^{-P_N\varphi-\tilde{\varphi}}, P_N u^{(j)}, \tilde{u}^{(l)}),$$

$$\mathcal{D}_3 = -2\sum_{l=1}^{n}\sum_{j=1}^{n_1} D_{\Gamma_j}^{(j)}(\nu(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}), e^{-P_N\varphi-\tilde{\varphi}}, \tilde{u}^{(l)}, \tilde{u}^{(l)}),$$

$$\mathcal{D}_4 = -2\sum_{l=1}^{n}\sum_{j=1}^{n_1} D_{\Gamma_j}^{(j)}(\nu(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}) - \nu(P_N T, P_N\varphi), e^{-P_N\varphi-\tilde{\varphi}}, P_N u^{(l)}, \tilde{u}^{(l)}),$$

$$\mathcal{D}_5 = -2\sum_{l=1}^{n_1}\sum_{j=1}^{n} D_{\Gamma_l}^{(j)}(1, \nu(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi})e^{-P_N\varphi-\tilde{\varphi}}, \tilde{u}^{(j)}, \tilde{u}^{(l)}),$$

$$\mathcal{D}_6 = -2\sum_{l=1}^{n}\sum_{j=1}^{n_1} D_{\Gamma_j}^{(l)}(\nu(P_N T + \tilde{T}, P_N\varphi + \tilde{\varphi}) - \nu(P_N T, P_N\varphi), e^{-P_N\varphi-\tilde{\varphi}}, P_N u^{(j)}, \tilde{u}^{(l)}).$$

Moreover, $|\mathcal{D}_3'|$ is bounded by (3.3), and

$$|\mathcal{D}_q'| \le \varepsilon|\tilde{u}|_1^2 + M^*[\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2 + N^{n_2}h^{-n_1-2}\|\tilde{\varphi}\|^2(\|\tilde{u}\|^2 + h\|\tilde{u}\|_{\Gamma_h}^2) + h^{-1}\|\tilde{\varphi}\tilde{u}\|_{\Gamma_h}^2], \quad q = 2, 4, 6,$$

$$|\mathcal{D}_5'| \le \varepsilon|\tilde{u}|_1^2 + M^*\{[1 + N^{n_2}h^{-n_1-2}(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)](\|\tilde{u}\|^2 + h\|\tilde{u}\|_{\Gamma_h}^2)$$
$$+ N^{n_2}h^{-n_1-2}\|\tilde{u}\|^2(\|\tilde{T}\|^2 + h\|\tilde{T}\|_{\Gamma_h}^2 + \|\tilde{\varphi}\|^2 + h\|\tilde{\varphi}\|_{\Gamma_h}^2)$$
$$+ h|\tilde{u}|_{\Gamma_h,1}^2 + h^{-1}[\|\tilde{T}\tilde{u}\|_{\Gamma_h}^2 + \|\tilde{\varphi}\tilde{u}\|_{\Gamma_h}^2 + Y(\tilde{u},\tilde{T}) + Y(\tilde{u},\tilde{\varphi})]\}.$$

We have

$$2\sum_{l=1}^{n}(\tilde{u}^{(l)}, \tilde{F}_7^{(l)}) = 2R_0A_{\Gamma_h}(\tilde{\varphi}, P_NT, \tilde{u}) - 2R_0(D(\tilde{u}P_NT), \tilde{\varphi}).$$

The absolute value of the last term is bounded by

$$\varepsilon|\tilde{u}|_1^2 + M(\|\tilde{u}\|^2 + \|\tilde{\varphi}\|^2).$$

We have

$$2(\tilde{T}, \tilde{F}_8) = \mathcal{D}_7 + \sum_{j=1}^{n}(e^{-P_N\varphi-\tilde{\varphi}}\mu(P_NT + \tilde{T}, P_N\varphi + \tilde{\varphi})(P_NT + \tilde{T})^{-1}S_T^{-1}(P_NT$$
$$+ \tilde{T}, P_N\varphi + \tilde{\varphi}), [D_j^+(\tilde{T})]^2 + [D_j^-(\tilde{T})]^2) + \mathcal{D}_7',$$

$$2(\tilde{T}, \tilde{F}_9) = \mathcal{D}_8 + \mathcal{D}_8'$$

where

$$\mathcal{D}_7 = -2\sum_{j=1}^{n_1} D_{\Gamma_j}^{(j)}(\mu(P_NT + \tilde{T}, P_N\varphi + \tilde{\varphi}), e^{-P_N\varphi-\tilde{\varphi}}(P_NT + \tilde{T})^{-1}S_T^{-1}(P_NT$$
$$+ \tilde{T}, P_N\varphi + \tilde{\varphi}), \tilde{T}, \tilde{T}),$$
$$\mathcal{D}_8 = -2\sum_{j=1}^{n_1} D_{\Gamma_j}^{(j)}(\mu(P_NT + \tilde{T}, P_N\varphi + \tilde{\varphi}) - \mu(P_NT, P_N\varphi), e^{-P_N\varphi-\tilde{\varphi}}(P_NT$$
$$+ \tilde{T})^{-1}S_T^{-1}(P_NT + \tilde{T}, P_N\varphi + \tilde{\varphi}), P_NT, \tilde{T}),$$
$$|\mathcal{D}_7'| \le \varepsilon|\tilde{T}|_1^2 + M^*[1 + N^{n_2}h^{-n_1-2}(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)](\|\tilde{T}\|^2 + h\|\tilde{T}\|_{\Gamma_h}^2)$$
$$+ M^*h^{-1}(\|\tilde{\varphi}\tilde{T}\|_{\Gamma_h}^2 + \|\tilde{T}^2\|_{\Gamma_h}^2),$$
$$|\mathcal{D}_8'| \le \varepsilon|\tilde{T}|_1^2 + M^*[1 + N^{n_2}h^{-n_1-2}(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)](\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2$$
$$+ M^*h^{-1}(\|\tilde{\varphi}\tilde{T}\|_{\Gamma_h}^2 + \|\tilde{T}^2\|_{\Gamma_h}^2).$$

As a result of $\alpha_3 = 1/2$ and Lemma 4, we obtain

$$2(\tilde{\varphi}, \tilde{F}_{10}) = A_{\Gamma_h}(\tilde{\varphi}, \tilde{\varphi}, P_Nu).$$

On the other hand, $|\kappa(P_NT + \tilde{T}, P_N\varphi + \tilde{\varphi})| \le \kappa_1(1 + M^{**})$, etc. Thus

$$\Big|2\tau\sum_{q=1,3,5}\sum_{l=1}^{n}(\tilde{u}_t^{(l)}, \tilde{F}_q^{(l)})\Big| \le \frac{3}{8}\tau\|\tilde{u}_t\|^2 + 32\lambda\Phi_1^2(n\kappa_1^2 + 2\nu_1^2)(1 + M^{**})^2(|\tilde{u}|_1^2 + h|\tilde{u}|_{\Gamma_h,1}^2)$$
$$+ M^*(\|\tilde{u}\|^2 + h\|\tilde{u}\|_{\Gamma_h}^2),$$
$$|2\tau(\tilde{T}_t, \tilde{F}_8)| \le \frac{\tau}{4}\|\tilde{T}_t\|^2 + 16\lambda\mu_1^2\Phi_1^2B_0^{-2}S_0^{-2}(1 + M^{**})^2|\tilde{T}|_1^2 + M^{**}(\|\tilde{T}\|^2 + h\|\tilde{T}\|_{\Gamma_h}^2).$$

We also have

$$\left|2\tau\sum_{q=2,4,6}\sum_{l=1}^{n}(\tilde u_t^{(l)},\tilde F_q^{(l)})\right|\le\varepsilon\tau\|\tilde u_t\|^2+M^*(1+N^{n_2}h^{-n_1}\|\tilde\varphi\|^2)(\|\tilde T\|^2+\|\tilde\varphi\|^2)$$
$$+M^*h(\|\tilde T\|^2_{\Gamma_h}+\|\tilde\varphi\|^2_{\Gamma_h}),$$
$$|2\tau(\tilde T_t,\tilde F_q)|\le\varepsilon\tau\|\tilde T_t\|^2+M^*[1+N^{n_2}h^{-n_1}(\|\tilde T\|^2+\|\tilde\varphi\|^2)](\|\tilde T\|^2+\|\tilde\varphi\|^2)$$
$$+M^*h(\|\tilde T\|^2_{\Gamma_h}+\|\tilde\varphi\|^2_{\Gamma_h}).$$

Furthermore, from condition (ii), it is obvious that

$$\left|2\tau\sum_{l=1}^{n}(\tilde u_t^{(l)},\tilde F_7^{(l)})\right|\le\varepsilon\tau\|\tilde u_t\|^2+M(\|\tilde\varphi\|^2+h\|\tilde\varphi\|^2_{\Gamma_h}),$$
$$|2\tau(\tilde\varphi_t,\tilde F_{10})|\le\varepsilon\tau\|\tilde\varphi_t\|^2+M(\|\tilde\varphi\|^2+h\|\tilde\varphi\|^2_{\Gamma_h}),$$
$$\left|2\sum_{l=1}^{n}(\tilde u^{(l)}+\tau\tilde u_t^{(l)},\tilde E_2^{(l)}+\tilde f_1^{(l)})\right|\le\varepsilon\tau\|\tilde u_t\|^2+M^*\|\tilde\varphi\|^2+M(\|\tilde u\|^2+\|\tilde f_1\|^2),$$
$$|2(\tilde T+\tau\tilde T_t,\tilde E_5+\tilde E_6+\tilde E_7+\tilde f_2)|\le\varepsilon\tau\|\tilde T_t\|^2+M^*(\|\tilde T\|^2+\|\tilde\varphi\|^2)+M\|\tilde f_2\|^2,$$
$$|2(\tilde\varphi+\tau\tilde\varphi_t,\tilde f_3)|\le\varepsilon\tau\|\tilde\varphi_t\|^2+M(\|\tilde\varphi\|^2+\|\tilde f_3\|^2).$$

It could be verified that

$$\left|2\sum_{l=1}^{n}(\tilde u^{(l)}+\tau\tilde u_t^{(l)},\tilde E_1^{(l)}+\tilde E_3^{(l)})\right|\le 2\varepsilon\tau\|\tilde u_t\|^2+\varepsilon(|\tilde u|_1^2+|\tilde T|_1^2)+M[1$$
$$+N^{n_2}h^{-n_1}(\|\tilde u\|^2+h\|\tilde u\|^2_{\Gamma_h})](\|\tilde u\|^2+h\|\tilde u\|^2_{\Gamma_h})+M(\|\tilde T\|^2+h\|\tilde T\|^2_{\Gamma_h})$$
$$+MN^{n_2}h^{-n_1-2}\|\tilde T\|^2(\|\tilde\varphi\|^2+h\|\tilde\varphi\|^2_{\Gamma_h}),$$
$$|2(\tilde T+\tau\tilde T_t,\tilde E_4+\tilde E_8)|\le 2\varepsilon\tau\|\tilde T_t\|^2+\varepsilon(2|\tilde u|_1^2+|\tilde T|_1^2)+M[1+N^{n_2}h^{-n_1}(\|\tilde u\|^2$$
$$+\|\tilde T\|^2)](\|\tilde T\|^2+h\|\tilde T\|^2_{\Gamma_h})+Mh^{-1}\|\tilde T\tilde u\|^2_{\Gamma_h}+M^*(\|\tilde T\|^2+\|\tilde\varphi\|^2)$$
$$+(M+M^*)[1+N^{n_2}h^{-n_1-2}(\|\tilde u\|^2+h\|\tilde u\|^2_{\Gamma_h}+\|\tilde T\|^2)](\|\tilde u\|^2+h\|\tilde u\|^2_{\Gamma_h}),$$
$$|2(\tilde\varphi+\tau\tilde\varphi_t,\tilde E_9)|\le\varepsilon\tau\|\tilde\varphi_t\|^2+\varepsilon|\tilde u|_1^2+M(1+N^{n_2}h^{-n_1-2}\|\tilde u\|^2)(\|\tilde\varphi\|^2+h\|\tilde\varphi\|^2_{\Gamma_h})$$
$$+M(1+N^{n_2}h^{-n_1-2}\|\tilde\varphi\|^2)(\|\tilde u\|^2+h\|\tilde u\|^2_{\Gamma_h})+Mh^{-1}\|\tilde\varphi\tilde u\|^2_{\Gamma_h}.$$

Next, we estimate the terms $\mathcal{D}_j$ and A_{Γ_h}. By an argument as in [2,3], we obtain

$$\mathcal{D}_1\ge h^{-1}\sum_{l=1}^{n_1}\{((1-\varepsilon\cdot\operatorname{sign}\ \kappa)\kappa(P_NT+\tilde T,P_N\varphi+\tilde\varphi)(e^{-P_N\varphi-\tilde\varphi})^{-x_l},[(\tilde u^{(l)})^2]^{-x_l})_{\Gamma_{+l}}$$
$$+((1-\varepsilon\cdot\operatorname{sign}\kappa)\kappa(P_NT+\tilde T,P_N\varphi+\tilde\varphi)(e^{-P_N\varphi-\tilde\varphi})^{+x_l},[(\tilde u^{(l)})^2]^{+x_l})_{\Gamma_{-l}}\}$$
$$-\varepsilon|\tilde u|_1^2-M^*(\|\tilde u\|^2+h^{-1}\|\tilde u\|^2_{\Gamma_h}+|\tilde u|^2_{\Gamma_h,1}),$$
$$\mathcal{D}_3\ge(1-\varepsilon)h^{-1}\sum_{l=1}^{n}\sum_{j=1}^{n_1}\{(\nu(P_NT+\tilde T,P_N\varphi+\tilde\varphi)(e^{-P_N\varphi-\tilde\varphi})^{-x_j},[(\tilde u^{(l)})^2]^{-x_j})_{\Gamma_{+j}}$$
$$+(\nu(P_NT+\tilde T,P_N\varphi+\tilde\varphi)(e^{-P_N\varphi-\tilde\varphi})^{+x_j},[(\tilde u^{(l)})^2]^{+x_j})_{\Gamma_{-j}}\}-\varepsilon|\tilde u|_1^2-M^*h^{-1}\|\tilde u\|^2_{\Gamma_h},$$

$$\mathcal{D}_5 \ge (1-\varepsilon)h^{-1}\sum_{l=1}^{n_1}\{(([\nu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})e^{-P_N\varphi-\tilde{\varphi}}]^{-x_l},[(\tilde{u}^{(l)})^2]^{-x_l})_{\Gamma_{+l}}$$
$$+([\nu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})e^{-P_N\varphi-\tilde{\varphi}}]^{+x_l},[(\tilde{u}^{(l)})^2]^{+x_l})_{\Gamma_{-l}}\}-\varepsilon|\tilde{u}|_1^2-M^*(\|\tilde{u}\|^2$$
$$+h^{-1}\|\tilde{u}\|^2_{\Gamma_h}+|\tilde{u}|^2_{\Gamma_h,1}),$$
$$\mathcal{D}_7 \ge (1-\varepsilon)h^{-1}\sum_{j=1}^{n_1}\{(\mu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})[e^{-P_N\varphi-\tilde{\varphi}}(P_NT+\tilde{T})^{-1}S_T^{-1}(P_NT$$
$$+\tilde{T},P_N\varphi+\tilde{\varphi})]^{-x_j},(\tilde{T}^2)^{-x_j})_{\Gamma_{+j}}+(\mu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})[e^{-P_N\varphi-\tilde{\varphi}}(P_NT$$
$$+\tilde{T})^{-1}S_T^{-1}(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})]^{+x_j},(\tilde{T}^2)^{+x_j})_{\Gamma_{-j}}\}-\varepsilon|\tilde{T}|_1^2-M^*h^{-1}\|\tilde{T}\|^2_{\Gamma_h},$$
$$|\mathcal{D}_q| \le \varepsilon|\tilde{u}|_1^2+M^*(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2+h\|\tilde{T}\|^2_{\Gamma_h}+h\|\tilde{\varphi}\|^2_{\Gamma_h}+h^{-1}\|\tilde{u}\|^2_{\Gamma_h}),\quad q=2,4,6,$$
$$|\mathcal{D}_8| \le \varepsilon|\tilde{T}|_1^2+M^*(\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2+h\|\tilde{\varphi}\|^2_{\Gamma_h}+h^{-1}\|\tilde{T}\|^2_{\Gamma_h}).$$

We also have

$$|2R_0A_{\Gamma_h}(\tilde{\varphi},P_NT,\tilde{u})| \le \varepsilon|\tilde{u}|_1^2+M(\|\tilde{\varphi}\|^2+h\|\tilde{\varphi}\|^2_{\Gamma_h}+h^{-1}\|\tilde{u}\|^2_{\Gamma_h}),$$
$$|A_{\Gamma_h}(\tilde{\varphi},\tilde{\varphi},P_Nu)| \le M(\|\tilde{\varphi}\|^2+h\|\tilde{\varphi}\|^2_{\Gamma_h}+h^{-1}\|\tilde{\varphi}P_Nu\|^2_{\Gamma_h}) \le M(\|\tilde{\varphi}\|^2+h\|\tilde{\varphi}\|^2_{\Gamma_h}).$$

If $\nu+\kappa>0$, then the estimation is simple. If $\nu+\kappa\le 0$, then

$$e^{-P_N\varphi-\tilde{\varphi}}[\nu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})+\kappa(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})][(D^+(\tilde{u}))^2+(D^-(\tilde{u}))^2]$$
$$\ge n\sum_{j=1}^{n}e^{-P_N\varphi-\tilde{\varphi}}[\nu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})+\kappa(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})][(D_j^+(\tilde{u}^{(j)}))^2$$
$$+(D_j^-(\tilde{u}^{(j)}))^2].$$

Let

$$\tilde{P}(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})=\min(n\kappa(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})+(n+1)\nu(P_NT$$
$$+\tilde{T},P_N\varphi+\tilde{\varphi}),\nu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})),$$
$$\varepsilon\le\frac{1}{72}\min(1,p_0\Phi_0,3\mu_0\Phi_0B_1^{-1}S_1^{-1}).$$

It is easy to see that

$$(1-\varepsilon\cdot\text{sign}\ \kappa)\kappa+2(1-\varepsilon)\nu\ge\frac{1}{2}\tilde{P}.$$

By substituting the above estimate into (3.1), we get

$$(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)_t+\frac{\tau}{2}(\|\tilde{u}_t\|^2+\|\tilde{T}_t\|^2+\|\tilde{\varphi}_t\|^2)+\frac{1}{2}p_0\Phi_0|\tilde{u}|_1^2+\frac{1}{2}\mu_0\Phi_0B_1^{-1}S_1^{-1}|\tilde{T}|_1^2$$
$$\le R(\tilde{u},\tilde{T},\tilde{\varphi})+G(\tilde{u},\tilde{T},\tilde{\varphi})+\sum_{j=1}^{3}f_j^*(\tilde{u},\tilde{T},\tilde{\varphi})$$

where

$$R(\tilde{u},\tilde{T},\tilde{\varphi})=M^*(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)[1+N^{n_2}h^{-n_1-2}(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2)],$$
$$G(\tilde{u},\tilde{T},\tilde{\varphi})=M\sum_{j=1}^{4}\|\tilde{f}_j\|^2+M^*h^{-1}[\|\tilde{u}\|^2_{\Gamma_h}+\|\tilde{T}\|^2_{\Gamma_h}+h|\tilde{u}|^2_{\Gamma_h,1}+\|\tilde{T}^2\|^2_{\Gamma_h}+\|\tilde{T}\tilde{u}\|^2_{\Gamma_h}$$

$$+\|\tilde{\varphi}\tilde{u}\|^2_{\Gamma_h}+\|\tilde{\varphi}\tilde{T}\|^2_{\Gamma_h}+N^{n_2}h^{1-n_1}\|\tilde{u}\|^4_{\Gamma_h}+Y(\tilde{u},\tilde{T})+Y(\tilde{u},\tilde{\varphi})],$$

$$f_1^*(\tilde{u},\tilde{T},\tilde{\varphi})=-\sum_{l=1}^{n}\sum_{j=1}^{n}(e^{-P_N\varphi-\tilde{\varphi}}\tilde{P}(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})-\frac{3}{8}p_0\Phi_0$$
$$-16n\lambda\Phi_1^2(\nu_1^2+\kappa_1^2)(1+M^{**})^2-M^*N^{n_2}h^{-n_1}(\|\tilde{u}\|^2+\|\tilde{T}\|^2$$
$$+\|\tilde{\varphi}\|^2),[D_j^+(\tilde{u}^{(l)})]^2+[D_j^-(\tilde{u}^{(l)})]^2),$$

$$f_2^*(\tilde{u},\tilde{T},\tilde{\varphi})=-\sum_{j=1}^{n}(e^{-P_N\varphi-\tilde{\varphi}}\mu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})(P_NT+\tilde{T})^{-1}S_T^{-1}(P_NT$$
$$+\tilde{T},P_N\varphi+\tilde{\varphi})-\frac{3}{8}\mu_0\Phi_0B_1^{-1}S_1^{-1}-4n\lambda\mu_1^2\Phi_1^2B_0^{-2}S_0^{-2}(1+M^{**})^2$$
$$-M^*N^{n_2}h^{-n_1}(\|\tilde{u}\|^2+\|\tilde{T}\|^2+\|\tilde{\varphi}\|^2),[D_j^+(\tilde{T})]^2+[D_j^-(\tilde{T})]^2),$$

$$f_3^*(\tilde{u},\tilde{T},\tilde{\varphi})=-h^{-1}\sum_{l=1}^{n_1}\{(\frac{1}{2}\tilde{P}(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})-|h[\nu(P_NT+\tilde{T},P_N\varphi$$
$$+\tilde{\varphi})]_{\bar{x}_l}|,[e^{-P_N\varphi-\tilde{\varphi}}(\tilde{u}^{(l)})^2]^{-x_l})_{\Gamma_{+l}}+(\frac{1}{2}\tilde{P}(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})$$
$$-|h[\nu(P_NT+\tilde{T},P_N\varphi+\tilde{\varphi})]_{x_l}|,[e^{-P_N\varphi-\tilde{\varphi}}(\tilde{u}^{(l)})^2]^{+x_l})_{\Gamma_{-l}}\}.$$

The rest of proof is the same as in [3].

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