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A FAMILY OF VISCOSITY SPLITTING SCHEME FOR THE NAVIER-STOKES EQUATIONS*

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Abstract

In the paper, a family of viscosity splitting method is introduced for solving the initial boundary value problems of Navier-Stokes equation. Some stability and convergence estimates of the method are proved.

§1. Introduction

Since the publication of Chorin's work in 1973, the convergence problem of viscous splitting for the Navier-Stokes equation has been considered by several authors. Beale and Majda proved a convergence theorem for the Cauchy problems. Chorin, Hughes, McCracken and Marsden suggested a product formula for the initial boundary value problem, without convergence proof, follows:

$$u_n(t) = (H(\frac{t}{n}) \circ \phi \circ E(\frac{t}{n}))^n u_0 \tag{1.1}$$

where $H(\cdot)$ is the Stokes solver, $E(\cdot)$ is the Euler solver and ϕ is a so called "vorticity creation operator", the capacity of which is to maintain the no-slip condition at the surface. Ying Long-an considered this scheme and proved that (1.1) does not converge; he also proved that if a nonhomogeneous term is added to the Stokes equation to neutralize the error arising from the operator ϕ, then this scheme converges, the rate of convergence is $O(k)$ in $L^\infty(0,T;(H^1(\Omega))^2)$ for the two dimensional case, and $O(k)$ in $L^\infty(0,T;(L^2(\Omega))^3)$ for the three dimensional case, where k is the length of time step. Alessandrini, Douglis and Fabes also considered the initial boundary value problems and proved the convergence of the scheme

$$u_n(t) = (H(\frac{t}{n}) \circ E_M(\frac{t}{n}))^n u_0 \tag{1.2}$$

where $E_M(\cdot)$ is an approximate Euler solver with the solutions of the Euler equation replaced by polynomials. Zheng and Huang considered a scheme similar to (1.2), where there is also no operator ϕ, but $E_M(\cdot)$ is replaced by $E(\cdot)$; they proved that the rate

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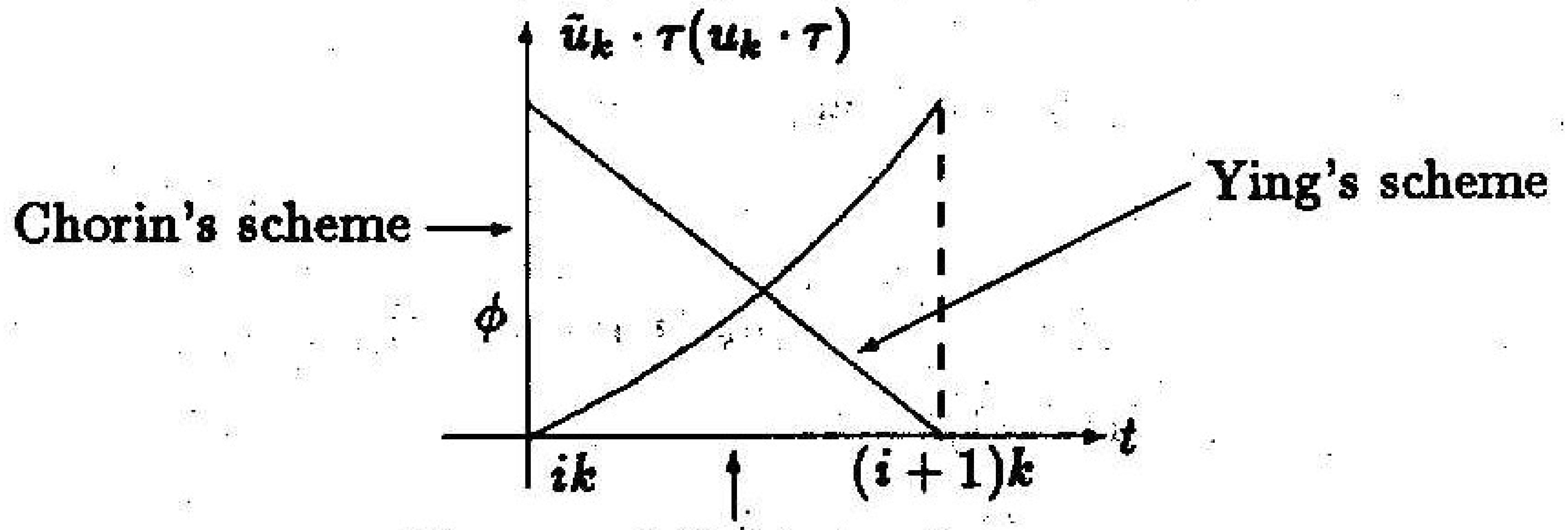
of convergence for the two dimensional case is $O(k^{\frac{3}{4}-\varepsilon})$ in $L^\infty(0,T;(L^2(\Omega))^2)$, where $0<\varepsilon<1/4$. Recently, Ying Long-an considered a scheme

$$u_n(t)=(\widehat{H}(\frac{t}{n})\circ E(\frac{t}{n}))^n u_0 \tag{1.3}$$

where $\widehat{H}(\cdot)$ is the Stokes solver with nonhomogeneous on boundary conditions; he proved that the rate of convergence for the two dimensional case is $O(k)$ in $L^\infty(0,T;(H^1(\Omega))^2)$.

To understand those schemes clearly, let us give a chart.



In the chart, $\tilde{u}_k$ are the solutions of the Euler equations, u_k are the solutions of the Stokes equations, and τ is the tagent vector.

The purpose of this paper is to study a family of viscosity splitting scheme similar to (1.3). We will prove a convergence theorem where the rate of convergence for the two-dimensional case is $O(k^{\frac{1}{4}-\varepsilon})$ in $L^\infty(0,T;(H^1(\Omega))^2)$, where $0<\varepsilon<1/4$. For simplicity, we only consider simply connected bounded domains in R^2.

§2. The Scheme and the Main Theorem

Let $x=(x_1,x_2)$ and $y=(y_1,y_2)$ be points in R^2 and Ω be a simply connected domain in R^2 with sufficiently smooth boundary $\partial\Omega$. The initial boundary value problem of the Navier-Stokes equation is given as

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\frac{1}{\rho}\nabla P=\nu\Delta u+f,\quad x\in\Omega,t>0, \tag{2.1}$$

$$\nabla\cdot u=0,\qquad x\in\Omega,t>0, \tag{2.2}$$

$$u|_{x\in\partial\Omega}=0, \tag{2.3}$$

$$u|_{t=0}=u_0(x) \tag{2.4}$$

where $u=(u_1,u_2)$ is the velocity, P is the pressure, and ν,ρ are positive constants. Throughout this paper we assume that the solution (u,P) of the above problem is sufficiently smooth on $\overline{\Omega}\times[0,T]$, and the usual notations $H^s(\Omega)$ and $W^{m,p}(\Omega)$ for Sobolev spaces and $\|\cdot\|_s$ and $\|\cdot\|_{m,p}$ for norms in Sobolev spaces are applied.

We divide the interval $[0,T]$ into equal subintervals with length k. Then we solve $\widetilde{u}_k(t), \widetilde{P}_k(t), u_k(t), P_k(t)$ on each interval $(ik,(i+1)k), i=0,1,\cdots$, according to the following procedure:

$$\frac{\partial \widetilde{u}_k}{\partial t} + (\widetilde{u}_k\cdot\nabla)\widetilde{u}_k + \frac{1}{\rho}\nabla \widetilde{P}_k = f, \tag{2.5}$$

$$\nabla\cdot\widetilde{u}_k = 0, \tag{2.6}$$

$$\widetilde{u}_k\cdot n|_{x\in\partial\Omega} = 0, \tag{2.7}$$

$$\widetilde{u}_k(ik) = u_k(ik-0) \tag{2.8}$$

where n is the unit outward normal vector and $u_k(-0)=u_0$,

$$\frac{\partial u_k}{\partial t} + \frac{1}{\rho}\nabla P_k = \nu \Delta u_k, \tag{2.9}$$

$$\nabla\cdot u_k = 0, \tag{2.10}$$

$$u_k|_{x\in\partial\Omega} = g\left(\frac{(i+1)k-t}{k}\right)\widetilde{u}_k((i+1)k-0)|_{x\in\partial\Omega}, \tag{2.11}$$

$$u_k(ik) = \widetilde{u}_k(i+1)k-0) \tag{2.12}$$

where $g(t)\in C^2[0,1], g(0)=0, g(1)=1$, and

$$|g(t)|\le\alpha_0, |g'(t)|\le\alpha_1, |g''(t)|\le\alpha_2 \quad \forall t\in[0,1];$$

$g(t)$ exists, for example, $g(t)=t^n$, where $\alpha_0=1, \alpha_1=n, \alpha_2=n(n-1)$.

Our main result is the following

Theorem. *If $u_0\in(H^4(\Omega))^2\cap(H_0^1(\Omega))^2, \nabla\cdot u_0=0$; $f\in L^\infty(0,T;(H^4(\Omega))^2)$ $\cap W^{2,\infty}(0,T;(H^{\frac{1}{2}}(\Omega))^2)$, u is the solution of problem (2.1)–(2.4), $\widetilde{u}_k, u_k$ is the solution of problem (2.5)-(2.12), $0\le s<\frac{3}{2}$, then*

$$\sup_{0\le t\le T}(\|u_k(t)\|_{s+1}, \|\widetilde{u}_k(t)\|_{s+1}) \le M, \tag{2.13}$$

$$\sup_{0\le t\le T}(\|u(t)-u_k(t)\|_1, \|u(t)-\widetilde{u}_k(t)\|_1) \le M'k^{\frac{1}{4}-\varepsilon} \tag{2.14}$$

for any $0<\varepsilon<1/4$, where the constants M, M' depend only on the domain Ω, constants $\varepsilon,\nu,s,T,\alpha_0,\alpha_1,\alpha_2$, functions f, u_0 and u.

§3. Solution of the Stokes Equation

We consider the linear Stokes equation

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\nabla P = \nu\Delta u + f \tag{3.1}$$

coupled with equation (2.2), initial condition (2.4) and a boundary condition (2.3) and a nonhomogeneous boundary condition

$$u|_{x\in\partial\Omega} = u_1(x,t) \tag{3.2}$$

where u_1 satisfies

$$\int_{\partial\Omega} u_1 \cdot n ds = 0, \quad u_1(x,0) = u_0(x)|_{x\in\partial\Omega}.$$

We extend function u_1 continuously to the domain Ω at every time t, such that u_1 is the solution of the stationary Stokes problem

$$\frac{1}{\rho} \nabla P = \nu \triangle u_1, \quad \nabla u_1 = 0. \tag{3.3}$$

Then by the estimate of the Stokes problems,

$$\|u_1\|_{1,\Omega} \le C\|u_1\|_{\frac{1}{2},\partial\Omega}. \tag{3.4}$$

Here and hereafter we always denote by C a generic constant which depends only on the domain Ω and constants ν, s, T; by C_0 a generic constant which depends only on the domain Ω, constants ν, s, T, the known function f, u_0 and the solution u of problem (2.1)–(2.4); by $C_1, C_2, \cdot, \cdot, \cdot, M_0, M_1 \cdot, \cdot, \cdot$ some other constants which are determined according to special requirements.

$\dfrac{\partial u_1}{\partial t}$ is also a solution of equation (3.3), so we have

$$\Big\|\frac{\partial u_1}{\partial t}\Big\|_{1,\Omega} \le C\Big\|\frac{\partial u_1}{\partial t}\Big\|_{\frac{1}{2},\partial\Omega}. \tag{3.5}$$

Similarly to Lemma 1 of [1], we have

Lemma 1. *Let $v = u - u_1$; then*

$$\frac{d}{dt}\|\nabla \wedge v\|_0^2 \le C\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\frac{1}{2},\partial\Omega} + \|f\|_0^2\Big) \tag{3.6}$$

where $\nabla\wedge = \Big(\dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_1}\Big)$.

We will use the Helmholtz operator P and the Stokes operator A frequently. It is known that

$$(L^2(\Omega))^2 = X \oplus G$$

where

$$X = \text{Closure in } (L^2(\Omega))^2 \text{ of } \{u \in (C_0^\infty(\Omega))^2; \nabla \cdot u = 0\},$$
$$G = \{\nabla p; p \in H^1(\Omega)\}.$$

P is the orthogonal projection $P : (L^2(\Omega))^2 \to X$, which is a bounded operator from $H^s(\Omega))^2$ to $(H^s(\Omega))^2$ for any nonnegative s. A is defined as $A = -P\triangle$ with domain $D(A) = X \cap \{u \in (H^2(\Omega))^2; u|_{\partial\Omega} = 0\}$ which admits the following properties:

$$\|A^\alpha e^{-tA}\| \le Ct^{-\alpha}, \quad \alpha \ge 0, t > 0, \tag{3.7}$$

$$\frac{1}{C}\|u\|_{2\alpha} \le \|A^\alpha u\|_0 \le C\|u\|_{2\alpha}, \quad \forall u \in D(A^\alpha), \quad \alpha \ge 0 \tag{3.8}$$

and if $0 \le s < 1/2$, $u \in X \cap (H^s(\Omega))^2$, then $u \in D(A^{\frac{s}{2}})$; if $1 \le s < \frac{3}{2}, u \in D(A) \cap (H^{s+1}(\Omega))^2$, then $u \in D(A^{\frac{s+1}{2}})$.

Similarly to Lemma 2 of [1], we have

Lemma 2. *If* $u_0 \in D(A) \cap (H^{s+1}(\Omega))^2, 0 \le s < 3/2$, $f \in L^\infty(0,T;(H^r(\Omega))^2), s-1 < r < 1/2$, *$u$ is a solution of problem* (3.1),(2.2)–(2.4), *then*

$$\|u(t)\|_{s+1} \le C\Big(\|u_0\|_{s+1} + \sup_{0\le\tau\le t} \|f(t)\|_r\Big). \tag{3.9}$$

Now we apply scheme (2.5)–(2.12) to problem (3.1), (2.2)–(2.4). Equation (2.5) is reduced to

$$\frac{\partial \widetilde{u}_k}{\partial t} + \frac{1}{\rho}\nabla \widetilde{P}_k = f.$$

We apply the operator P to it and obtain $\partial\widetilde{u}_k/\partial t = Pf$. Thus

$$\widetilde{u}_k(t) = \widetilde{u}_k(ik) + \int_{ik}^{t} Pf(\tau)d\tau \quad ik \le t < (i+1)k. \tag{3.10}$$

Let

$$v(t) = u_k(t) - g(\frac{(i+1)k-t}{k})(\widetilde{u}_k((i+1)k-0) - \widetilde{u}_k(ik)). \tag{3.11}$$

Then by (2.8)–(2.12), v is the solution of

$$\begin{aligned}\frac{\partial v}{\partial t} + \frac{1}{\rho}\nabla P_k = \nu\triangle v + \nu g(\frac{(i+1)k-t}{k})\triangle(\widetilde{u}_k((i+1)k-0) - \widetilde{u}_k(ik))&\\ + \frac{1}{k}g'(\frac{(i+1)k-t}{k})(\widetilde{u}_k((i+1)k-0) - \widetilde{u}_k(ik)),& \quad (3.12)\end{aligned}$$

$$\nabla\cdot v = 0, \tag{3.13}$$

$$v|_{x\in\partial\Omega} = 0, \tag{3.14}$$

$$v(ik) = \widetilde{u}_k(ik) = u_k(ik-0) = v(ik-0). \tag{3.15}$$

Therefore v is a continuous function on $\overline{\Omega}\times[0,T]$.

By (3.8), (3.10), we have

$$\begin{aligned}u_k(t) =& e^{-\nu tA}u_0 + \sum_{i=0}^{[\frac{t}{k}]-1}\int_{ik}^{(i+1)k} e^{-\nu(t-\tau)A}\{\nu g(\frac{(i+1)k-\tau}{k})\int_{ik}^{(i+1)k} P\triangle Pf(\xi)d\xi\\ &+ \frac{1}{k}g'(\frac{(i+1)k-\tau}{k})\int_{ik}^{(i+1)k} Pf(\xi)d\xi\}d\tau\\ &+ \int_{[\frac{t}{k}]k}^{t} e^{-\nu(t-\tau)A}\{\nu g(\frac{([\frac{t}{k}]+1)k-\tau}{k})\int_{[\frac{t}{k}]k}^{([\frac{t}{k}]+1)k} P\triangle Pf(\xi)d\xi\\ &+ \frac{1}{k}g'(\frac{([\frac{t}{k}]+1)k-\tau}{k})\int_{[\frac{t}{k}]k}^{(2[\frac{t}{k}]+1)k-t} Pf(\xi)d\xi\}d\tau\\ &+ g(\frac{([\frac{t}{k}]+1)k-\tau}{k})\int_{[\frac{t}{k}]k}^{(2[\frac{t}{k}]+1)k-t} Pf(\tau)d\tau\end{aligned} \tag{3.16}$$

where [ ] denotes the integral part of a number.

Lemma 3. *If* $0 \le s < 3/2, s-1 < r < 1/2$, $u_0 \in D(A) \cap (H^{s+1}(\Omega))^2$, $f \in L^\infty(0,T;(H^{2+r}(\Omega))^2 \cap W^{1,\infty}(0,T;(H^r(\Omega))^2)$, *then*

$$\|u_k(t)\|_{s+1} \le C(\|u_0\|_{s+1} + k \sup_{0<\zeta<([\frac{t}{k}]+1)k} \|f(\zeta)\|_{2+r} + \sup_{0<\zeta<([\frac{t}{k}]+1)k} \|f(\zeta)\|_r$$
$$+ C(([\frac{t}{k}]+1)k - t) \sup_{[\frac{t}{k}]k \le \zeta < ([\frac{t}{k}]+1)k} \|f(\zeta)\|_{s+1}. \tag{3.17}$$

Proof. The estimate of the first term is obvious. Besides, we have

$$\left\|e^{-\nu(t-\tau)A}\nu g(\frac{(i+1)k-\tau}{k}) \int_{ik}^{(i+1)k} P\triangle Pf(\xi)d\xi\right\|_{s+1}$$
$$\le C\left\|A^{\frac{s+1-r}{2}} e^{-\nu(t-\tau)A} \int_{ik}^{(i+1)k} A^{\frac{r}{2}} P\triangle Pf(\xi)d\xi\right\|_0$$
$$\le Ck \sup_{ik\le\zeta<(i+1)k} \|f(\zeta)\|_{2+r}(t-\tau)^{-\frac{s+1-r}{2}}$$

and

$$\left\|e^{-\nu(t-\tau)A}\frac{1}{k}g'(\frac{(i+1)k-\tau}{k}) \int_{ik}^{(i+1)k} P\triangle Pf(\xi)d\xi\right\|_{s+1}$$
$$\le C\left\|A^{\frac{s+1-r}{2}} e^{-\nu(t-\tau)A} \frac{1}{k}\int_{ik}^{(i+1)k} A^{\frac{r}{2}} P\triangle Pf(\xi)d\xi\right\|_0$$
$$\le Ck \sup_{ik\le\zeta<(i+1)k} \|f(\zeta)\|_{r}(t-\tau)^{-\frac{s+1-r}{2}}.$$

By the mean value theorem,

$$g(\frac{(i+1)k-\tau}{k}) = g'(\zeta)\cdot(\frac{(i+1)k-\tau}{k})$$

where $0<\zeta<\frac{(i+1)k-\tau}{k}$, $ik<\tau<(i+1)k$,

$$\|g(\frac{([\frac{t}{k}]+1)k-t}{k}) \int_{[\frac{t}{k}]k}^{([\frac{t}{k}]+1)k} Pf(\tau)\|_{s+1} \le C\Big(\Big(\Big[\frac{t}{k}\Big]+1\Big)k-t\Big)\|f\|_{s+1}.$$

Then (3.17) follows.

Lemma 4. *If* $0 \le s < 3/2, s-1 < r < 1/2$, $u_0 \in D(A) \cap (H^{s+1}(\Omega))^2$, $f \in L^\infty(0,T;(H^{2+r}(\Omega))^2 \cap W^{1,\infty}(0,T;(H^r(\Omega))^2)$, *and the solution* u *of problem* (3.1),(2.2)–(2.4) *is sufficiently smooth, then for any* $0<\varepsilon<(3-2s)/4$,

$$\sup_{0\le t\le T}(\|u(t)-u_k(t)\|_{s+1}, \|u(t)-\tilde{u}_k(t)\|_{s+1}) \le C_0 k^{\frac{3-2s}{4}-\varepsilon}. \tag{3.18}$$

Proof. By (3.6) and

$$u(t) = e^{-\nu tA}u_0 + \int_0^t e^{-\nu(t-\tau)A}Pf(\tau)d\tau, \tag{3.19}$$

we have

$$u(t)-u_k(t)=\sum_{i=0}^{[\frac{t}{k}]-1}\int_{ik}^{(i+1)k}e^{-\nu(t-\tau)A}\{-\nu g(\frac{(i+1)k-\tau}{k})\int_{ik}^{(i+1)k}P\triangle Pf(\xi)d\xi$$
$$+Pf(\tau)-\frac{1}{k}g'(\frac{(i+1)k-\tau}{k})\int_{ik}^{(i+1)k}Pf(\xi)d\xi\}d\tau$$
$$+\int_{[\frac{t}{k}]k}^{t}e^{-\nu(t-\tau)A}\{-\nu g(\frac{([\frac{t}{k}]+1)k-\tau}{k})\int_{[\frac{t}{k}]k}^{([\frac{t}{k}]+1)k}P\triangle Pf(\xi)d\xi$$
$$+Pf(\tau)-\frac{1}{k}g'(\frac{([\frac{t}{k}]+1)k-\tau}{k})\int_{[\frac{t}{k}]k}^{([\frac{t}{k}]+1)k}Pf(\xi)d\xi\{d\tau$$
$$-g(\frac{([\frac{t}{k}]+1)k-t}{k})\int_{[\frac{t}{k}]k}^{([\frac{t}{k}]+1)k}Pf(\tau)d\tau.$$

Some estimates are the same as in the proof of Lemma 3. And we have

$$\int_{ik}^{(i+1)k}e^{-\nu(t-\tau)A}\Big(Pf(\tau)-\frac{1}{k}g'\Big(\frac{(i+1)k-\tau}{k}\Big)\int_{ik}^{(i+1)k}Pf(\xi)d\xi d\tau\Big\|_{s+1}$$
$$=\Big\|\int_{ik}^{(i+1)k}e^{-\nu(t-\zeta)A}Pf(\zeta)d\zeta-\int_{ik}^{(i+1)k}\frac{1}{k}g'(\frac{(i+1)k-\tau}{k})\cdot\int_{ik}^{(i+1)k}Pf(\zeta)d\zeta d\tau\Big\|_{s+1}$$
$$=\Big\|\int_{ik}^{(i+1)k}-\frac{1}{k}g'(\frac{(i+1)k-\tau}{k})\int_{ik}^{(i+1)k}(e^{-\nu(t-\tau)A}-e^{-\nu(t-\zeta)A})Pf(\zeta)d\zeta d\tau\Big\|_{s+1}$$
$$=\Big\|\int_{ik}^{(i+1)k}-e^{-\nu(t-\tau)A}\frac{1}{k}g'\Big(\frac{(i+1)k-\tau}{k}\Big)\int_{ik}^{(i+1)k}(I-e^{-(\tau-\zeta)A})Pf(\zeta)d\zeta d\tau\Big\|_{s+1}$$
$$=\Big\|\int_{ik}^{(i+1)k}-e^{-\nu(t-\tau)A}\frac{1}{k}g'\Big(\frac{(i+1)k-\tau}{k}\Big)\int_{ik}^{(i+1)k}A\int_0^{\tau-\zeta}e^{-\nu\xi A}d\xi Pf(\zeta)d\zeta d\tau\Big\|_{s+1}$$
$$\le C\Big\|\int_{ik}^{(i+1)k}A^{1-\frac{\epsilon}{2}}-e^{-\nu(t-\tau)A}\frac{1}{k}g'(\frac{(i+1)k-\tau}{k})\int_{ik}^{(i+1)k}\int_0^{\tau-\zeta}A^{\frac{2s+1}{4}+\epsilon}$$
$$\times e^{-\nu\xi A}d\xi A^{\frac{1}{4}-\frac{\epsilon}{2}}Pf(\zeta)d\zeta d\tau\Big\|_0$$
$$\le C\int_{ik}^{(i+1)k}(t-\tau)^{-1+\frac{\epsilon}{2}}\frac{1}{k}\int_{ik}^{(i+1)k}(\tau-\zeta)^{\frac{3-2s}{4}-\epsilon}d\zeta\|f\|_{\frac{1}{2}-\epsilon}d\tau$$
$$\le C\sup_{ik\le\zeta<(i+1)k}\|f(\zeta)\|_{\frac{1}{2}-\epsilon}k^{\frac{3-2s}{4}-\epsilon}\int_{ik}^{(i+1)k}(t-\tau)^{-1+\frac{\epsilon}{2}}d\tau.$$

Then the estimate for $\|u(t)-u_k(t)\|_{s+1}$ is obtained.

Now we estimate $\|u(t)-\widetilde{u}_k(t)\|_{s+1}$. Since u is sufficiently smooth, we have

$$\|u(t)-u(ik)\|_{s+1}\le C_0k,\qquad t\in[ik,(i+1)k).$$

By (3.10) and the initial condition (2.8),

$$\|\widetilde{u}_k(t)-u_k(ik-0)\|_{s+1}\le Ck\sup_{ik\le\tau\le t}\|f(\tau)\|_{s+1}.$$

Therefore

$$\|u(t)-\widetilde{u}_k(t)\|_{s+1}\le C_0k+\|u(ik)-u_k(ik-0)\|_{s+1}\le C_0k^{\frac{3-2s}{4}-\epsilon}.$$

Now, we state some results about the Euler equation:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla P = f, \tag{3.20}$$

$$\nabla\cdot u = 0, \tag{3.21}$$

$$u\cdot n|_{x\in\partial\Omega} = 0, \tag{3.22}$$

$$u|_{t=0} = u_0(x). \tag{3.23}$$

Lemma 5. *If integer $m \geq 3, \|u_0\|_m \leq M_1, u_0 \in X$, then there is a constant $C > 0$ such that*

$$k_0 = \frac{1}{C(M_1 + \sup_{0\leq t\leq T}\|f(t)\|_m + 1)} \tag{3.24}$$

where $0 \leq t \leq k_0$; solution u of (3.20)–(3.23) satisfies

$$\|u\|_\sigma \leq C_1(\|u_0\|_\sigma + 1) \tag{3.25}$$

where $\sigma \leq m$, the constant C_1 depends only on the domain Ω, constants m, σ, T and $\sup_{0\leq t\leq T}\|f(t)\|_m$.

Now, u is assumed to be an arbitrary smooth function, $u(\cdot,t) \in X, \xi(y,\tau;t)$ is the characteristic which satisfies

$$\frac{\partial}{\partial t}\xi(y,\tau;t) = u(\xi(y,\tau;t),t), \quad \xi(y,\tau;\tau) = y$$

$u_0 \in H^1(\Omega)$, ω is the solution of

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = -\nabla\wedge f = F, \quad \omega|_{t=0} = -\nabla\wedge u_0 = \omega_0$$

and ψ is the stream function which satisfies

$$-\Delta\psi = \omega, \quad \psi|_{x\in\partial\Omega} = 0.$$

Let

$$\Psi(y) = \psi(\xi(y,t;0)), \quad \theta = -\Delta\Psi.$$

Then we have

Lemma 6. *If $u_0 \in D(A)$, then*

$$\|\theta(t) - \omega(t)\|_0 \leq C_2 t\|u_0\|_1 + \int_0^t \|F(\tau)\|_0 d\tau \tag{3.26}$$

where the constant C_2 depends only on the domain Ω and function u.

The above results were proved in [1].

§4. Some Estimates for the Viscosity Splitting Scheme

In this section, we give some estimates for the solutions of scheme (2.5)–(2.12). We always denote by u and ω the solution of problem (2.1)–(2.4) and associated vorticity, and by ω_k and $\widetilde{\omega_k}$ the associated vorticity of u_k and $\widetilde{u}_k$.

Lemma 7. *If* $0 < r < 1$, *then*

$$\|(w \cdot \nabla)w\|_r \le C\|w\|_1\|w\|_2 \tag{4.1}$$

for $w \in (H^2(\Omega))^2$ *and*

$$\|(w \cdot \nabla)w\|_{2+r} \le C\|w\|_1\|w\|_4 \tag{4.2}$$

for $w \in (H^4(\Omega))^2$.

Proof. By the imbedding theorem, the interpolation inequality and the Hölder inequality, letting $p = \dfrac{2}{2-r}$, we have

$$\|(w\cdot\nabla)w\|_r \le C\|(w\cdot\nabla)w\|_{1,p} \le C(\|w\|_{0,\frac{2}{1-r}}\|w\|_2 + \|w\|^2_{1,2p})$$
$$\le C(\|w\|_1\|w\|_2 + \|w\|^2_{1+\frac{r}{2}}) \le C(\|w\|_1\|w\|_2 + \|w\|_1\|w\|_{1+r}) \le C\|w\|_1\|w\|_2.$$

The proof of (4.2) is similar.

We have assumed in the main theorem that $u_0 \in D(A) \cap (H^4(\Omega))^2$. It was proved in [6] that the solution of the Euler equation belongs to $L^\infty(0,T;(C^{1,\lambda}(\Omega))^2)$ globally provided $u_0 \in X \cap (C^{1,\lambda}(\Omega))^2$. By the equation

$$\frac{\partial}{\partial t}\xi(y,\tau;t) = u(\xi(y,\tau;t),t), \quad \xi(y,\tau;\tau) = y,$$
$$\omega(x,t) = \omega_0(\xi(x,t;0)) + \int_0^t F(\xi(x,t;\zeta),\zeta)d\zeta,$$
$$u = \nabla\wedge\psi, \quad -\Delta\psi = \omega, \quad \psi|_{x\in\partial\Omega} = 0$$

it is easy to see that $u \in L^\infty(0,T;(H^4(\Omega))^2)$. Applying this result to scheme (2.5)–(2.12), we get $\tilde{u}_k \in L^\infty(0,k;(H^4(\Omega))^2)$. We will prove in the next lemma that $u_k(jk-0) \in (H^4(\Omega))^2$ provided $\tilde{u}_k(t) \in (H^{s+1}(\Omega))^2$ on $[0,jk)$ by induction $\tilde{u}_k(t) \in (H^4(\Omega))^2$ for all $t \in [0,T]$.

Lemma 8. *If* $\sup\limits_{0\le t\le jk}\|\tilde{u}_k(t)\|_1 \le M_0$, *where j is a positive integer, then*

$$\|u_k(jk-0)\|_\sigma \le C_3 k^{-\frac{\sigma-s-1}{2}}\Big(\sup_{0\le\tau<jk}\|\tilde{u}_k(\tau)\|_{s+1} + 1\Big) \tag{4.3}$$

where $s+1 \le \sigma \le 4, 1 < s < \dfrac{3}{2}$, *the constant C_3 depends only on domain Ω, constants* $\sigma, s, \nu, T, M_0, \alpha_0, \alpha_1, \alpha_2$ *and the function f.*

Proof. We denote by C_3 a generic constant which possesses the above property. Let $w = \dfrac{\partial v}{\partial t}, \pi = \dfrac{\partial P_k}{\partial t}$, where (v, P_k) is the solution of (3.11)–(3.14). Then we differentiate those equations formally with respect to t, and obtain

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\nabla\pi = \nu\Delta w - \frac{\nu}{k}g'(\frac{jk-t}{k})\Delta(\tilde{u}_k(jk-0) - \tilde{u}_k((j-1)k))$$
$$-\frac{1}{k^2}g''(\frac{jk-t}{k})(\tilde{u}_k(jk-0) - \tilde{u}_k((j-1)k))$$

$$\nabla\cdot w = 0,$$
$$w|_{x\in\partial\Omega} = 0,$$
$$w((j-1)k) = \nu P\,\triangle\,\tilde{u}_k(jk-0) + \frac{1}{k}g'(1)(\tilde{u}_k(jk-0) - \tilde{u}_k((j-1)k)).$$

$\partial v/\partial t$ is its weak solution, but the above problem possesses a strong solution

$$\begin{aligned} w(t) =& e^{-\nu(t-(j-1)k)A}w((j-1)k) - \nu\int_{(j-1)k}^{t} e^{-\nu(t-\tau)A}\Big(-\frac{\nu}{k}g'\Big(\frac{jk-t}{k}\Big)\triangle(\tilde{u}_k(jk-0) \\ & - \tilde{u}_k((j-1)k)\Big) - \frac{1}{k^2}g''\Big(\frac{jk-t}{k}\Big)(\tilde{u}_k(jk-0) - \tilde{u}_k((j-1)k)))d\tau. \end{aligned} \tag{4.4}$$

Hence (4.4) is the expression of $\partial v/\partial t$.

We estimate $\|w((j-1)k)\|_{s-1}$ by equation (2.5) and Lemma 7. Because

$$\begin{aligned} \Big\|\frac{\partial}{\partial t}\tilde{u}_k(jk-0)\Big\|_{s-1} &= \|P(f - (\tilde{u}_k\cdot\nabla)\tilde{u}_k\|_{s-1} \\ &\le C_3\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k(\tau)\|_1\|\tilde{u}_k(\tau)\|_2 + 1\Big) \\ &\le C_3\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k(\tau)\|_2 + 1\Big) \end{aligned} \tag{4.5}$$

therefore

$$\|w((j-1)k)\|_{s-1} \le C_3\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k(\tau)\|_{s+1} + 1\Big).$$

By (4.4),

$$\begin{aligned} \|w(t)\|_2 \le\ & C(\|A^{1-\frac{s-1}{2}}e^{-\nu(t-(j-1)k)A}A^{\frac{s-1}{2}}w((j-1)k)\|_0 \\ & + \frac{1}{k}\int_{(j-1)k}^{t}\|A^{1-\frac{s-1}{2}}e^{-\nu(t-\tau)A}A^{\frac{s-1}{2}}P\triangle(\tilde{u}_k((jk-0) - \tilde{u}_k((j-1)k))\|_0 d\tau \\ & + \frac{1}{k^2}\int_{(j-1)k}^{t}\|A^{1-\frac{s-1}{2}}e^{-\nu(t-\tau)A}A^{\frac{s-1}{2}}(\tilde{u}_k(jk-0) - \tilde{u}_k((j-1)k)\|_0 d\tau \\ \le\ & C(t-(j-1)k)^{-1+\frac{s-1}{2}}\|A^{\frac{s-1}{2}}w((j-1)k)\|_0 \\ & + \frac{C}{k}\int_{(j-1)k}^{t}(t-\tau)^{-1+\frac{s-1}{2}}\|A^{\frac{s-1}{2}}P\triangle(\tilde{u}_k((jk-0) - \tilde{u}_k((j-1)k))\|_0 d\tau \\ & + \frac{C}{k^2}\int_{(j-1)k}^{t}(t-\tau)^{-1+\frac{s-1}{2}}\|A^{\frac{s-1}{2}}(\tilde{u}_k(jk-0) - \tilde{u}_k((j-1)k)\|_0 d\tau \\ \le\ & C_3\Big((t-(j-1)k)^{-1+\frac{s-1}{2}} + \frac{(t-(j-1)k)^{\frac{s-1}{2}}}{k}\Big)\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k\|_{s+1} + 1\Big). \end{aligned}$$

As $t = jk-0$, we have

$$\|w(jk-0)\|_2 \le C_3 k^{-1+\frac{s-1}{2}}\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k\|_{s+1} + 1\Big). \tag{4.6}$$

We apply the operator P to equation (3.12) for $t = jk-0$ and obtain

$$\|Av(jk-0)\|_0 \le \|w(jk-0)\|_2 + \frac{C}{k}\|\tilde{u}_k(jk-0) - \tilde{u}_k((j-1)k)\|_2.$$

By the interpolation inequality and (4.5),

$$\frac{1}{k}\|\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k)\|_2 \le \frac{C}{k}\|\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k)\|_{s+1}^{1-\frac{s-1}{2}}$$
$$\times\|\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k)\|_{s-1}^{\frac{s-1}{2}} \le C_3 k^{-1+\frac{s-1}{2}}\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k\|_{s+1}+1\Big).$$

By (4.6),

$$\|Av(jk-0)\|_2 \le C_3 k^{-1+\frac{s-1}{2}}\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k\|_{s+1}+1\Big).$$

By (3.11),

$$\|u_k(jk-0)\|_4 = \|v(jk-0)\|_4 \le C\|Av(jk-0)\|_2.$$

Therefore

$$\|u_k(jk-0)\|_4 \le Ck^{-1+\frac{s-1}{2}}\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k\|_{s+1}+1\Big). \tag{4.7}$$

Applying (3.19) to problem (3.12)-(3.15), we get

$$v(jk-0)=e^{-\nu kA}\tilde{u}_k((j-1)k)+\int_{(j-1)k}^{jk}e^{-\nu(jk-\tau)A}\Big(\nu g\big(\frac{jk-\tau}{k}\big)\triangle(\tilde{u}_k(jk-0)$$
$$-\tilde{u}_k((j-1)k))+\frac{1}{k}g'\big(\frac{jk-\tau}{k}\big)(\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k))\Big)d\tau.$$

Therefore

$$\|v(jk-0)\|_{s+1}\le C\|e^{-\nu kA}A^{s+1}\tilde{u}_k((j-1)k)\|_0$$
$$+C\int_{(j-1)k}^{jk}\Big(\|Ae^{-\nu(jk-\tau)A}g'(\zeta)\frac{jk-\tau}{k}A^{\frac{s-1}{2}}P\triangle(\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k))$$
$$+\|A^{\frac{s+1-r}{2}}e^{-\nu(jk-\tau)A}\frac{1}{k}g'\big(\frac{jk-\tau}{k}\big)A^{\frac{r}{2}}(\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k)))\|_0 d\tau$$
$$\le C\|\tilde{u}_k((j-1)k))\|_{s+1}+C\int_{(j-1)k}^{jk}\frac{1}{k}\|\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k)\|_{s+1}d\tau$$
$$+C\int_{(j-1)k}^{jk}(jk-\tau)^{-\frac{s+1-r}{2}}\frac{1}{k}\|\tilde{u}_k(jk-0)-\tilde{u}_k((j-1)k)\|_r d\tau$$

where $s-1<r<1/2$. Then by (4.5) we get

$$\|v(jk-0)\|_{s+1}\le C_3\Big(\sup_{(j-1)k\le\tau<jk}\|\tilde{u}_k\|_{s+1}+1\Big) \tag{4.8}$$

by (3.15) and $v(jk-0)=u_k(jk-0)$. Applying the interpolation inequality to (4.7) and (4.8), we obtain (4.3).

Similarly to Lemma 9 of [1], we have

Lemma 9. *If* $\sup_{0\le\tau<jk}\|\tilde{u}_k(\tau)\|_1\le M_0$, *where j is a positive integer, and there are constants C_1, k_0, such that*

$$\|\tilde{u}_k(t)\|_\sigma \le C_1(\|\tilde{u}_k(ik)\|_\sigma+1),\qquad \sigma=s+1\quad or\quad 4 \tag{4.9}$$

for $ik < t < (i+1)k$, *where* i *is any integer satisfying* $0 \le i \le j$, *and* $1 < s < 3/2$, $0 < k \le k_0$, *then*

$$\sup_{0 \le t < (j+1)k} \|\tilde{u}_k(t)\|_{s+1} \le M_2 \tag{4.10}$$

for $0 < k \le k_1 \le k_0$, *where the constants* M_2, k_1 *depend only on the domain* Ω, *constants* $C_1, \nu, s, T, M_0, \alpha_0, \alpha_1, \alpha_2$ *and functions* f, u_0.

If we replace $(\tilde{u}_k \cdot \nabla)\tilde{u}_k$ in equation (2.5) by $(u \cdot \nabla)u$, then (2.5) becomes a linear equation

$$\frac{\partial \tilde{u}_k}{\partial t} + \frac{1}{\nu} \nabla \widetilde{P_k} = f - (u \cdot \nabla)u. \tag{4.11}$$

The solution of problem (4.15), (2.6)–(2.12) is denoted $\tilde{u}^*, \tilde{P}^*, u^*, P^*$. Let $\tilde{\omega}^*, \omega^*$ be the associated vorticities. By Lemma 4,

$$\sup_{0 \le t \le T} (\|u(t) - u^*(t)\|_{s'+1}, \|u(t) - \tilde{u}^*(t)\|_{s'+1}) \le C_0 k^{\frac{3-2s'}{4} - \varepsilon} \tag{4.12}$$

for any s', $0 \le s' < 3/2$, $0 < \varepsilon < (3 - 2s')/4$.

Lemma 10. *If* $1 < s < 3/2, \|\tilde{u}_k(t)\|_{s+1} \le M_2$, *for* $ik \le t < (i+1)k$, *then*

$$\|(\tilde{u}^* - \tilde{u}_k((i+1)k - 0)\|_{\frac{1}{2},\partial\Omega} \le C_5 k \Big(\sup_{ik \le \tau < (i+1)k} \|\tilde{u}^* - \tilde{u}_k\|_{1,\Omega} + k^{\frac{1}{4} - \varepsilon} \Big) \tag{4.13}$$

for any $\varepsilon, 0 < \varepsilon < \frac{1}{4}$, *where the constant* C_5 *depends only on the domain* Ω, *the constants* ε, s, ν, M_2, *the functions* f, u_0, *and the solution* u *of* (2.1)–(2.4).

Proof. We denote by C_5 a generic constant with the above property. By (4.11) and (2.5)

$$\frac{\partial \tilde{\omega}^*}{\partial t} + u \cdot \nabla \omega = - \nabla \wedge f, \quad \frac{\partial \tilde{\omega}_k}{\partial t} + \tilde{u}_k \cdot \nabla \tilde{\omega}_k = - \nabla \wedge f.$$

Subtracting one equation from the other we obtain

$$\frac{\partial \tilde{\omega}^* - \tilde{\omega}_k}{\partial t} + u \cdot \nabla(\tilde{\omega}^* - \tilde{\omega}_k) = u \cdot \nabla(\tilde{\omega}^* - \omega) - (u - \tilde{u}_k) \cdot \nabla \tilde{\omega}_k. \tag{4.14}$$

By Lemma 6,

$$\|\theta - (\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k - 0)\|_0 \le C_5 k \|(\tilde{u}^* - \tilde{u}_k)(ik)\|_1 + \int_{ik}^{(i+1)k} \|u \cdot \nabla(\tilde{\omega}^* - \omega) - (u - \tilde{u}_k) \cdot \nabla \tilde{\omega}_k\|_0 d\tau \tag{4.15}$$

where $\theta = - \triangle \Psi, \Psi(y) = \psi(\xi(y, (i+1)k; ik)), \psi$ is the stream function corresponding to $(\tilde{u}^* - \tilde{u}_k)(ik)$.

We estimate the integrand by (4.12),

$$\|u \cdot \nabla(\tilde{\omega}^* - \omega)\|_0 \le C_5 \|\tilde{u}^* - u\|_2 \le C_5 k^{\frac{1}{4} - \varepsilon},$$

for any $0 < \varepsilon \le 1/4$. Let $p = 2/(2 - s), q = 2/(s - 1)$. Then

$$\|(u - \tilde{u}_k) \cdot \nabla \tilde{\omega}_k\|_0 = \Big(\int_\Omega |(u - \tilde{u}_k) \cdot \nabla \tilde{\omega}_k|^2 dx \Big)^{\frac{1}{2}}$$

$$\le \Big(\int_\Omega |\nabla \tilde{\omega}_k|^p dx \Big)^{\frac{1}{p}} \Big(\int_\Omega |u - \tilde{u}_k|^q dx \Big)^{\frac{1}{q}} \le \|\tilde{\omega}_k\|_{1,p} \|u - \tilde{u}_k\|_{0,q}.$$

By the imbedding theorem and (4.12),

$$\|\widetilde{\omega}_k\|_{1,p} \le \|\widetilde{\omega}_k\|_s \le C\|\widetilde{u}_k\|_{s+1},$$

$$\|u-\widetilde{u}_k\|_{0,q} \le C\|u-\widetilde{u}_k\|_1 \le C_0(\|\widetilde{u}^*-\widetilde{u}_k\|_1 + k^{\frac{3}{4}-\varepsilon}).$$

Therefore

$$\|u\cdot\nabla(\widetilde{\omega}^*-\omega)-(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k\|_0 \le C_5(\|\widetilde{u}^*-\widetilde{u}_k\|_1 + k^{\frac{1}{4}-\varepsilon}). \tag{4.16}$$

Substituting (4.16) into (4.15), we obtain

$$\|\theta-(\widetilde{\omega}^*-\widetilde{\omega}_k)((i+1)k-0)\|_0 \le C_5 k(\sup_{ik\le\tau<(i+1)k}\|\widetilde{u}^*-\widetilde{u}_k\|_1 + k^{\frac{1}{4}-\varepsilon}). \tag{4.17}$$

Since $\psi|_{x\in\partial\Omega}$ and $\left.\frac{\partial\psi}{\partial n}\right|_{x\in\partial\Omega}$ are zero, we have

$$\begin{aligned}\|(\widetilde{u}^*-\widetilde{u}_k)((i+1)k-0)\|_{\frac{1}{2},\partial\Omega} &= \|\nabla\wedge\psi-(\widetilde{u}^*-\widetilde{u}_k)((i+1)k-0)\|_{\frac{1}{2},\partial\Omega}\\ &\le C\|\nabla\wedge\psi-(\widetilde{u}^*-\widetilde{u}_k)((i+1)k-0)\|_{1,\Omega}.\end{aligned} \tag{4.18}$$

Let ϕ be the stream function corresponding to the velocity $\nabla\wedge\phi-(\widetilde{u}^*-\widetilde{u}_k)((i+1)k-0)$. Then it is the solution of

$$-\triangle\phi = \theta-(\widetilde{\omega}^*-\widetilde{\omega}_k)((i+1)k-0), \quad \phi|_{\partial\Omega}=0.$$

By the estimate for the elliptic problems,

$$\|\phi\|_2 \le C\|\theta-(\widetilde{\omega}^*-\widetilde{\omega}_k)((i+1)k-0)\|_0.$$

By definition,

$$\nabla\wedge\phi = \nabla\wedge\psi-(\widetilde{u}^*-\widetilde{u}_k)((i+1)k-0).$$

Thus (4.17) and (4.18) yield (4.13).

Lemma 11. *If* $1<s<3/2, \|\widetilde{u}_k(t)\|_{s+1}\le M_2$, *then*

$$\sup_{0\le t\le T}(\|u(t)-u_k(t)\|_1, \|u(t)-\widetilde{u}_k(t)\|_1) \le C_6 k^{\frac{1}{4}-\varepsilon} \tag{4.19}$$

for any $0<\varepsilon<\frac{1}{4}$, *where the constant* C_6 *depends only on the domain* Ω, *constants* $\varepsilon,\nu,s,T,M_2,\alpha_0,\alpha_1,\alpha_2$, *function* f, u_0 *and the solution* u *of* (2.1)–(2.4).

Proof. We denote by C_6 a generic constant which possesses the above property. Taking inner product (4.14) with $(\widetilde{\omega}^*-\widetilde{\omega}_k)$ and noting that

$$((\widetilde{u}_k\cdot\nabla)(\widetilde{\omega}^*-\widetilde{\omega}_k), \widetilde{\omega}^*-\widetilde{\omega}_k)=0$$

we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\widetilde{\omega}^*-\widetilde{\omega}_k\| \le \|u\cdot\nabla(\widetilde{\omega}^*-\omega)-(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k\|_0\cdot\|\widetilde{\omega}^*-\widetilde{\omega}_k\|_0.$$

By (4.16), the right hand side is bounded by

$$\frac{C_6}{2}(\|\widetilde{u}^*-\widetilde{u}_k\|_1^2 + k^{2(\frac{1}{4}-\varepsilon)}) + \frac{1}{2}\|\widetilde{\omega}^*-\widetilde{\omega}_k\|_0^2.$$

Since $\tilde{\omega}^* - \tilde{\omega}_k$ is the vorticity corresponding to velocity $\tilde{u}^* - \tilde{u}_k$, as in the proof of the last lemma, we get

$$\|\tilde{u}^* - \tilde{u}_k\|_1 \le C\|\tilde{\omega}^* - \tilde{\omega}_k\|_0. \tag{4.20}$$

Hence

$$\frac{d}{dt}\|\tilde{\omega}^* - \tilde{\omega}_k\|_0^2 \le C_6(\|\tilde{\omega}^* - \tilde{\omega}_k\|_0^2 + k^{2(\frac{1}{4}-\epsilon)}).$$

By the Gronwall inequality,

$$\|(\tilde{\omega}^* - \tilde{\omega}_k)(t)\|_0^2 \le e^{C_6 k}(\|(\tilde{\omega}^* - \tilde{\omega}_k)(ik)\|_0^2 + k^{1+2(\frac{1}{4}-\epsilon)}). \tag{4.21}$$

Let $t = (i+1)k - 0$. Then

$$\|(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k - 0)\|_0^2 \le e^{C_6 k}(\|(\tilde{\omega}^* - \tilde{\omega}_k)(ik)\|_0^2 + k^{1+2(\frac{1}{4}-\epsilon)}). \tag{4.22}$$

Applying Lemma 1 to problem (2.9)–(2.12), we have

$$\frac{d}{dt}\|\nabla\wedge(u^* - u_k - \nabla\wedge u_1)\|_0^2 \le C\|\frac{\partial u_1}{\partial t}\|_{\frac{1}{2},\partial\Omega}^2 \tag{4.23}$$

where u_1 is the solution of stationary Stokes equation (3.3) with boundary condition

$$u_1(t)|_{x\in\partial\Omega} = g(\frac{(i+1)k - t}{k})(\tilde{u}^* - \tilde{u}_k)((i+1)k - 0)|_{x\in\partial\Omega}. \tag{4.24}$$

Then we have

$$\begin{aligned}&\|(\omega^* - \omega_k)((i+1)k - 0) - \nabla\wedge u_1((i+1)k - 0)\|_0^2 \\ &\le \|(\omega^* - \omega_k)(ik) - \nabla\wedge u_1(ik)\|_0^2 + C\int_{ik}^{(i+1)k}\|\frac{\partial u_1(\tau)}{\partial\tau}\|_{\frac{1}{2},\partial\Omega}^2 d\tau.\end{aligned} \tag{4.25}$$

By uniqueness, $u_1((i+1)k - 0) = 0$. Then by the estimate for the solution of the stationary Stokes problem with boundary value (4.24) and by Lemma 10,

$$\begin{aligned}\|u_1(ik)\|_{1,\Omega} &\le C\|(\tilde{u}^* - \tilde{u}_k)((i+1)k - 0)\|_{x\in\partial\Omega} \\ &\le C_6 k\Big(\sup_{ik\le\tau<(i+1)k}\|(\tilde{u}^* - \tilde{u}_k)(\tau)\|_{1,\Omega} + k^{\frac{1}{4}-\epsilon}\Big).\end{aligned}$$

Besides, we have

$$\frac{\partial u_1}{\partial t}\Big|_{x\in\partial\Omega} = \frac{1}{k}g'\Big(\frac{(i+1)k - t}{k}\Big)(\tilde{u}^* - \tilde{u}_k)((i+1)k - 0)\Big|_{x\in\partial\Omega}.$$

Hence

$$\begin{aligned}\Big\|\frac{\partial u_1}{\partial t}\Big\|_{\frac{1}{2},\partial\Omega} &\le \frac{C}{k}\|(\tilde{u}^* - \tilde{u}_k)((i+1)k - 0)\|_{x\in\partial\Omega} \\ &\le C_6\Big(\sup_{ik\le\tau<(i+1)k}\|(\tilde{u}^* - \tilde{u}_k)(\tau)\|_{1,\Omega} + k^{\frac{1}{4}-\epsilon}\Big).\end{aligned} \tag{4.26}$$

Substituting the above inequalities into (4.25), we obtain

$$\begin{aligned}\|(\omega^* - \omega_k)((i+1)k - 0)\|_0^2 &\le \|(\omega^* - \omega_k)(ik)\|_0^2 + 2\|(\omega^* - \omega_k)(ik)\|_0 \\ &\times C_6 k\Big(\sup_{ik\le\tau<(i+1)k}\|(\tilde{u}^* - \tilde{u}_k)(\tau)\|_1 + k^{\frac{1}{4}-\epsilon}\Big)\end{aligned}$$

$$+ C_6 k\Big(\sup_{ik\le\tau<(i+1)k} \|(\tilde u^* - \tilde u_k)(\tau)\|_1^2 + k^{2(\frac14-\varepsilon)}\Big).$$

By (4.20),

$$\|(\omega^* - \omega_k)((i+1)k-0)\|_0^2 \le (1+C_6k)\|(\omega^*-\omega_k)(ik)\|_0^2$$
$$+ C_6 k\Big(\sup_{ik\le\tau<(i+1)k} \|(\tilde u^* - \tilde u_k)(\tau)\|_1^2 + k^{2(\frac14-\varepsilon)}\Big).$$

Substituting (4.22) in it and noting the initial condition (2.12), we get

$$\|(\omega^* - \omega_k)((i+1)k-0)\|_0^2 \le e^{C_6k}(1+C_6k)\|(\omega^*-\omega_k)(ik)\|_0^2$$
$$+ C_6 k(\|(\tilde\omega^* - \tilde\omega_k)(ik)\|_0^2 + k^{1+2(\frac14-\varepsilon)}).$$

By the initial condition (2.8),

$$\|(\omega^* - \omega_k)((i+1)k-0)\|_0^2 \le (1+C_6k)\|(\omega^*-\omega_k)(ik)\|_0^2 + C_6k^{1+2(\frac14-\varepsilon)}.$$

We get by induction that

$$\begin{aligned}
&\|(\omega^* - \omega_k)(ik)\|_0^2 \le C_6 e^{C_6T}k^{2(\frac14-\varepsilon)} \quad \text{(by (4.21))},\\
&\|(\tilde\omega^* - \tilde\omega_k)(t)\|_0^2 \le C_6 k^{2(\frac14-\varepsilon)} \quad 0\le t\le T \quad \text{(by (4.20))},\\
&\|(\tilde u^* - \tilde u_k)(t)\|_1^2 \le C_6 k^{2(\frac14-\varepsilon)} \quad 0\le t\le T \quad \text{(by (4.23))},\\
&\|(u^* - u_k)(t)\|_1^2 \le C_6 k^{2(\frac14-\varepsilon)}, \quad 0\le t\le T.
\end{aligned} \tag{4.27}$$

By (4.12) and the triangle inequality we obtain (4.19).

Lemma 12. *If $i>0, 0\le s<3/2$, and $\|\tilde u_k(t)\|_{s+1}\le M_2$, for $ik\le t<(i+1)k$, then $\|u_k(t)\|_{s+1}\le M_3$ on the same interval, where the constant M_3 depends only on the domain Ω, the constants ε,ν,s,T,M_2, the function f,u_0, and the solution u of problem* (2.1)–(2.4).

Proof. Let $v = u^* - u_k - u_1$, where u_1 is the solution of the stationary Stokes equation (3.3) with boundary condition (4.24). Then v is the solution of

$$\frac{\partial v}{\partial t} + \frac1\rho \nabla(P^* - P_k - P_1) = \nu\,\Delta v - \frac{\partial u_1}{\partial t}, \quad \nabla\cdot v = 0, \quad v|_{x\in\partial\Omega} = 0,$$
$$v(ik) = u^*(ik) - u_k(ik) - u_1(ik).$$

By (3.5), (4.26), (4.27),

$$\Big\|\frac{\partial u_1}{\partial t}\Big\|_{1,\Omega} \le C_7 k^{\frac14-\varepsilon}$$

where the constant C_7 has the same property as M_3. By (4.24) and the estimate for the Stokes problem, we get

$$\begin{aligned}
\|u_1\|_{s+1,\Omega} &\le C\|u_1\|_{s+\frac12,\partial\Omega} \le C\|(\tilde u^* - \tilde u_k)((i+1)k-0)\|_{s+\frac12,\partial\Omega}\\
&\le C\|(\tilde u^* - \tilde u_k)((i+1)k-0)\|_{s+1,\Omega}.
\end{aligned}\tag{4.28}$$

By Lemma 2,

$$\|v(t)\|_{s+1} \le C\Big(\|u^*(ik) - u_k(ik)\|_{s+1} + \|u_1(ik)\|_{s+1} + \sup_{ik\le\tau<(i+1)k}\Big\|\frac{\partial u_1}{\partial t}\Big\|_{\tau}\Big)$$
$$\le C\|(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\|_{s+1} + \|u_1(ik)\|_1 + \sup_{ik\le\tau<(i+1)k}\Big\|\frac{\partial u_1}{\partial t}\Big\|_1$$
$$\le C\|(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\|_{s+1} + C_7 k^{\frac{1}{4}-\epsilon}. \tag{4.29}$$

By the triangle inequality,

$$\|u_k\|_{s+1} \le \|v\|_{s+1} + \|u^*\|_{s+1} + \|u_1\|_{s+1}.$$

We apply (4.12) to get the estimate of $\|u^*\|_{s+1}$ and $\|\tilde{u}^*\|_{s+1}$. Then the estimate for $\|u_k\|_{s+1}$ follows from (4.28) and (4.29).

§5. Proof of the Theorem

We may assume that $1 < s < 3/2$. Let $M_0 = 2\max_{0\le t\le T}\|u(t)\|_1$. We set $m = 4$ and $\sigma = 4$ or $\sigma \neq s+1$ in Lemma 5. Then, take the large C_1. Taking $\sigma = 4$, we determine constant C_3 in Lemma 8, and constant M_2 in Lemma 9. Raise M_2 if necessary, such that

$$M_2 \ge C_1(\|u_0\|_{s+1} + 1). \tag{5.1}$$

Then we determine constants C_5, C_6, M_3 in Lemmas 10, 11 and 12 respectively.

Let $m = 4$. From (3.24) and Lemma 8, we solve k_0 by

$$k_0 = \frac{1}{C[C_3 k_0^{\frac{s-3}{2}}(M_2+1) + \sup\|f\|_4 + 1]}, \tag{5.2}$$

that is

$$C[C_3 k_0^{\frac{s-1}{2}}(M_2+1) + k_0 \sup_{0\le t\le T}\|f(t)\|_4 + k_0] = 1.$$

Since the left-hand side is monotone from zero to infinity on the interval $k_0 \in [0,\infty)$, (5.2) admits a uniqe solution $k_0 > 0$. Then we take constant k_1 in Lemma 8, and reduce k_1, if necessary, such that

$$\|u_0\|_4 \le C_3 k_1^{\frac{s-3}{2}}(M_2+1), \tag{5.3}$$

$$C_6 k_1^{\frac{1}{4}-\epsilon} \le \frac{M_0}{2}. \tag{5.4}$$

With these determined constants, we prove by induction that if $0 \le k < k_1$, then

$$\|\tilde{u}_k(t)\|_1 \le M_0, \quad \|u_k(t)\|_1 \le M_0, \quad \|\tilde{u}_k(t)\|_{s+1} \le M_2,$$
$$\|u(t) - u_k(t)\|_0 \le C_6 k^{\frac{1}{4}-\epsilon}, \quad \|u(t) - \tilde{u}_k(t)\|_0 \le C_6 k^{\frac{1}{4}-\epsilon}. \tag{5.5}$$

Two cases are considered simultaneously: (a) $j = 0$, (b) $j > 0$, and the above assertion is valid for $0 \le t < jk$. By Lemma 8 and (5.3),

$$\|u_k(ik-0)\|_4 \le C_3 k^{\frac{s-3}{2}}(M_2+1)$$

for $0 \le i \le j, j > 0$, or $j = 0$. Then we set $m = 4$ in Lemma 5. Because k_0 satisfies (5.2) and $k_1 \le k_0$, the conditions of Lemma 5 are fulfilled provided we take the initial value to be $u_k(ik-0)$, $0 \le i \le j$. If $j > 0$, from (3.25) we know the conditions of Lemma 9 are fulfilled, we get (4.10). If $j = 0$, by (5.1) and Lemma 5, (4.10) is also true, by Lemma 11, (5.5) holds for $0 \le t < (j+1)k$. By (5.4), $\|\tilde{u}_k\|_1 \le M_0$, $\|u_k(t)\|_1 \le M_0$ on the same interval. Thus the induction is complete.

Applying Lemma 12 we obtain the upper bound of $\|u_k(t)\|_{s+1}$. Thus (2.13) and (2.14) are verified for $k \le k_1$. If $k > k_1$, there are at most $1 + [\frac{T}{k_1}]$ steps, and the upper bounds of (2.13) and (2.14) are easily obtained.

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